University of Engineering and Technology Peshawar, Pakistan

CE-409: Introduction to Structural Dynamics and Earthquake Engineering

MODULE 3:

FUNDAMENTALS OF DYNAMIC ANALYSIS FOR S.D.O.F SYSTEMS

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Structural Degrees of Freedom

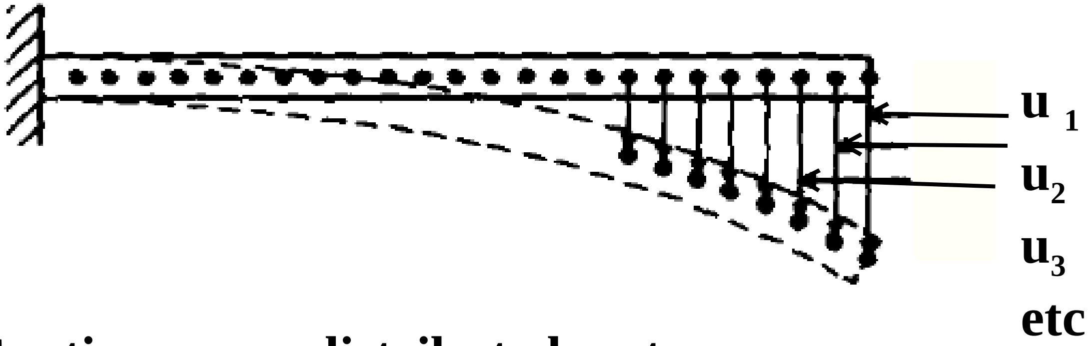
Degrees of freedom (DOF) of a system is defined as the number of ***independent*** variables required to completely determine the positions of all parts of a system at any instant of time.

Discrete vs. Continuous systems

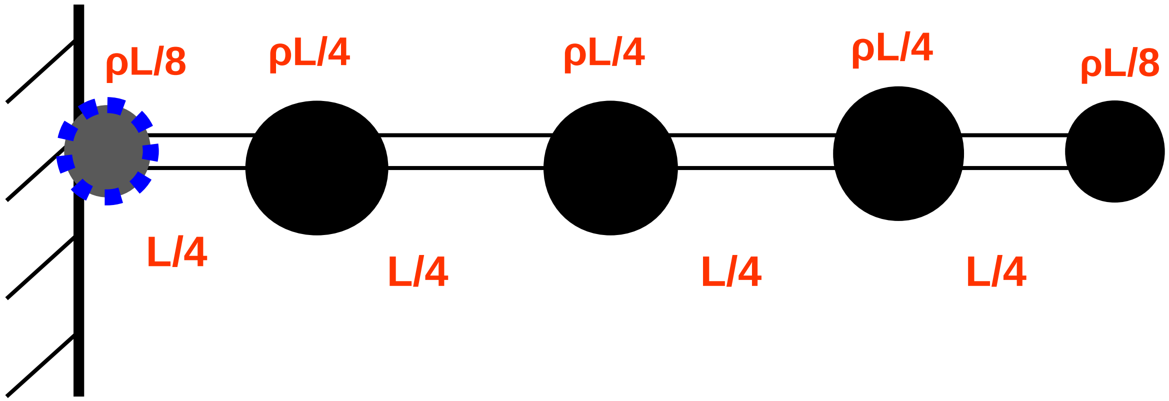
- Some systems, specially those involving continuous elastic members, have an infinite number of DOF. As an example of this is a cantilever beam with self weight only (*see next slide*). This beam has infinites mass points and need infinites number of displacements to draw its deflected shape and thus has an infinite DOF. Systems with infinite DOF are called ***Continuous or Distributed systems***.

- Systems with a finite number of degree of freedom are called ***Discrete or Lumped mass parameter systems***.

Discrete vs. Continuous systems



Continuous or distributed system



Corresponding lumped mass system of the above given cantilever beam with DOF= 4 (*How? there are 5 lumped masses.*)

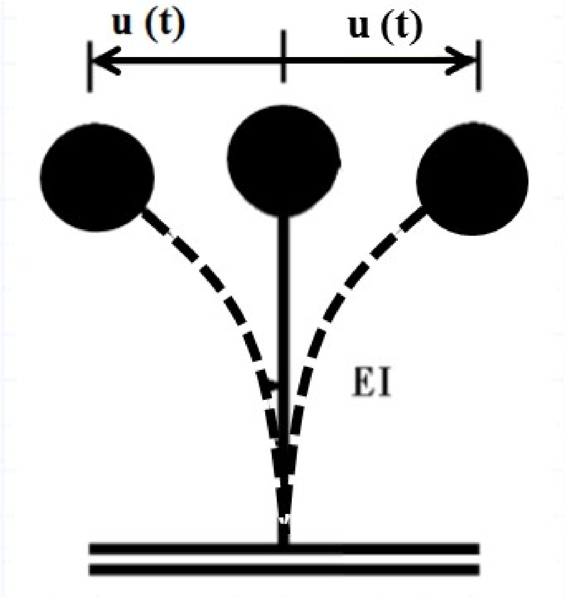
ρ = Mass per unit length

Single Degree-of-Freedom (SDOF) System

- In a single degree of freedom system, the deformation of the entire structure can be described by a single number equal to the displacement of a point from an **at-rest position**.
- Single Degree of freedom systems do not normally exist in real life. We live in a three-dimensional world and all mass is distributed resulting in systems that have an infinite number of degrees of freedom. There are, however, instances where a structure may be approximated as a single degree of freedom system.
- The study of SDOF systems is an integral step in understanding the responses of more complicated and realistic systems.

Idealization of a structural system as SDOF system

- This 3-dimensional water tower may be considered as a single degree of freedom system when one considers vibration in **one horizontal direction only**.



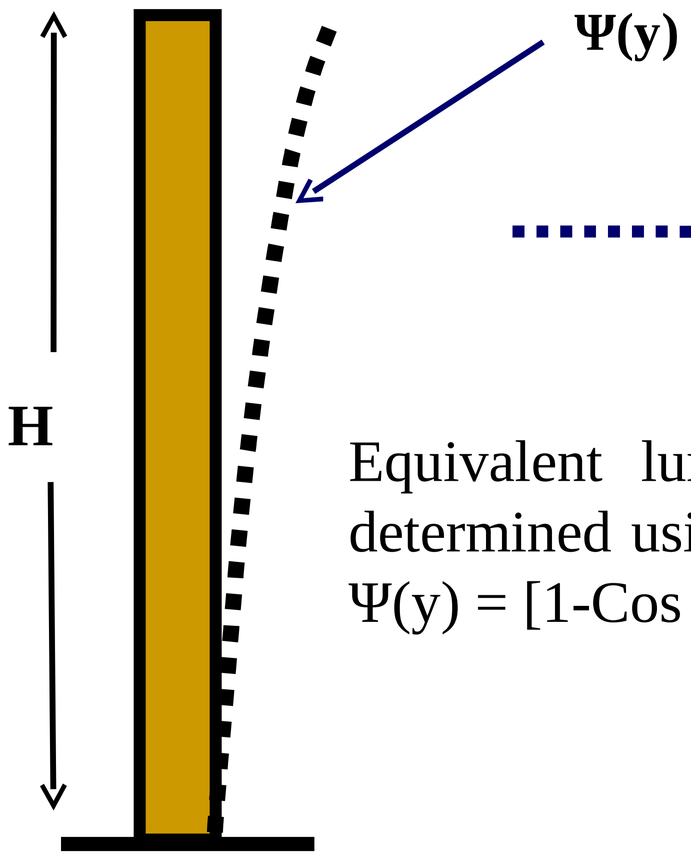
SDOF model of water tank

Idealization of a structural system as SDOF system

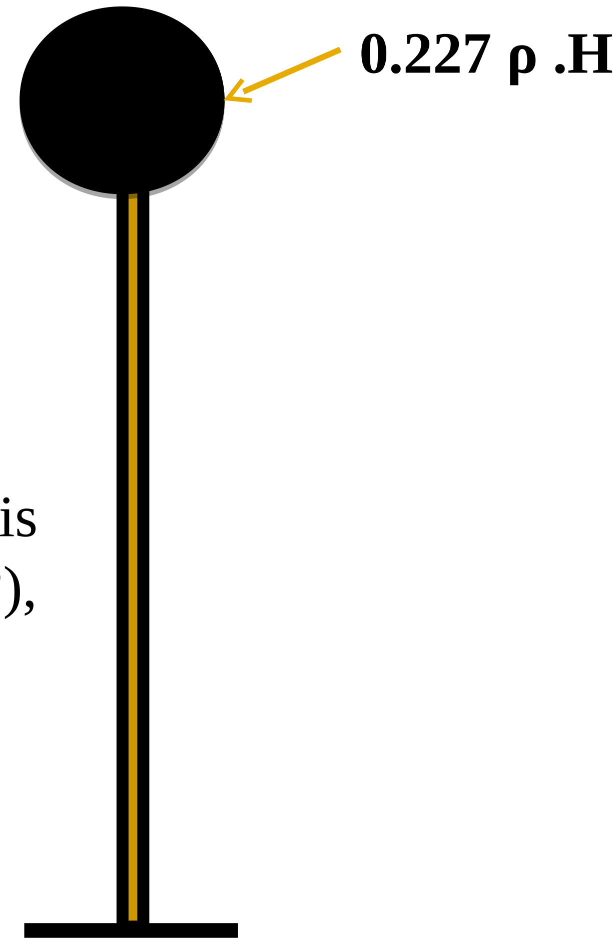
- The structural system of water tank may be simplified by assuming that the column has negligible mass along its length. This is reasonable, assuming that the tube is hollow and that the mass of the tube is insignificant when compared with the mass of the water tank and water at the top.

- This means that we can consider that the tank is a point mass

Equivalent lumped mass SDOF system of a cantilever wall with uniform x-sectional area



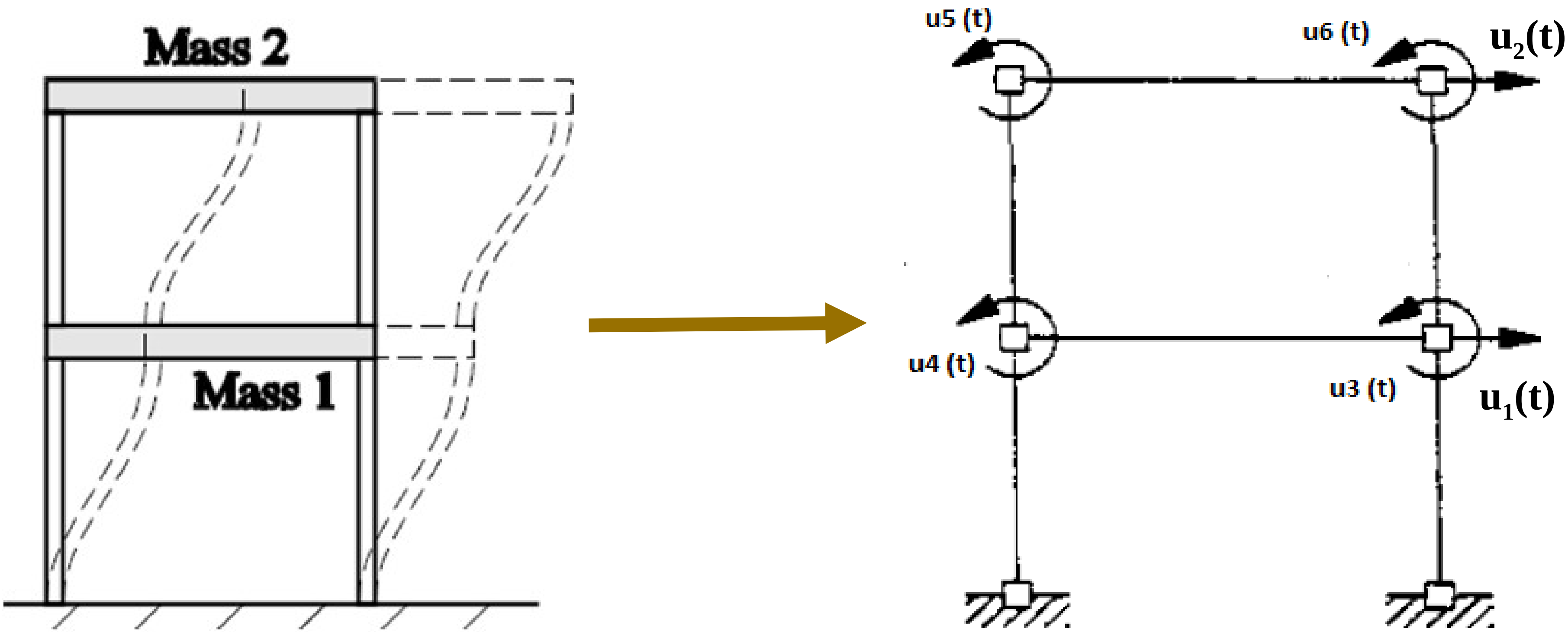
Equivalent lumped mass model is determined using a shape function (?), $\Psi(y) = [1\text{-Cos}\,(\pi y/2H)]$.



ρ = Mass per unit height, H= total height, y= Any distance along height and k = lateral stiffness of cantilever member = EI/H^3

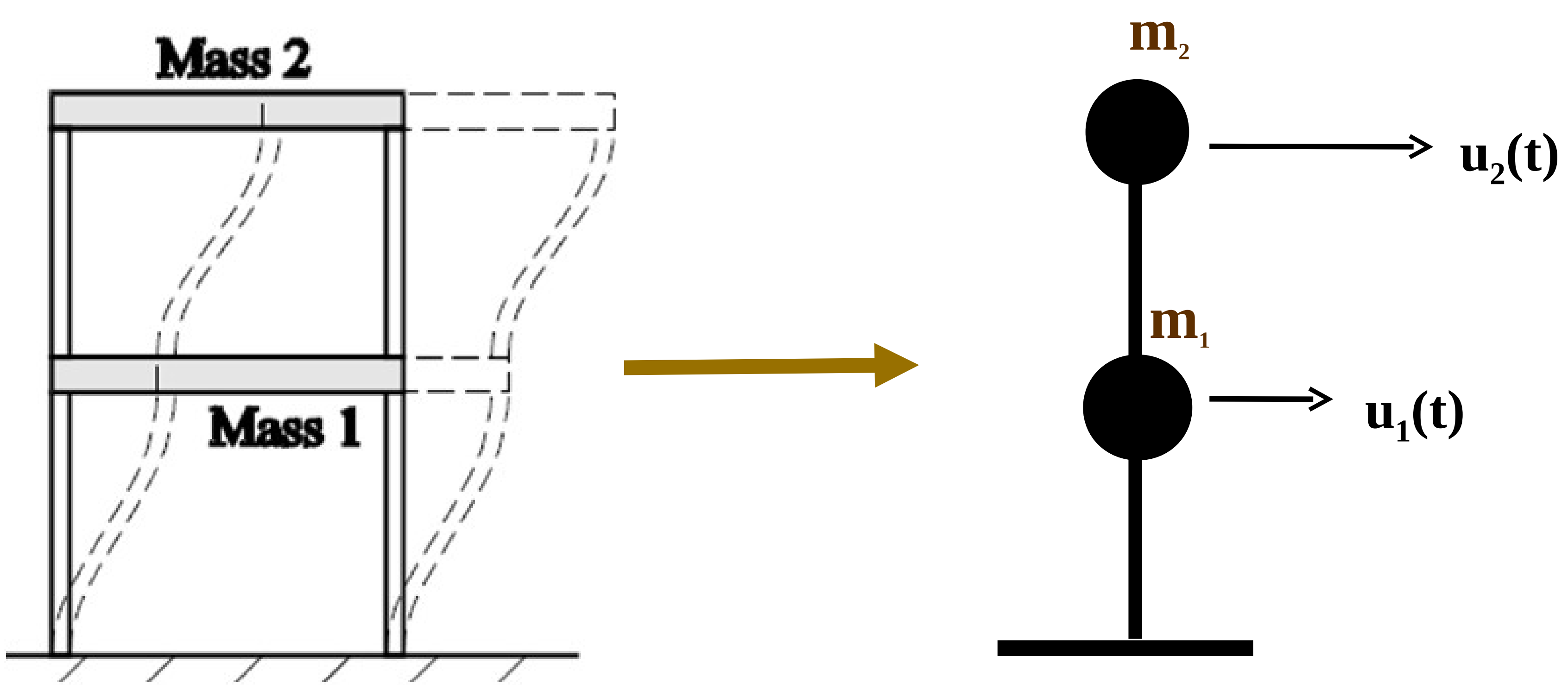
Multiple Degree-of-Freedom (MDOF) System

In a Multi degree of freedom system, the deformation of the entire structure cannot be described by a single displacement. More than one displacement coordinates are required to completely specify the displaced shape.



Considering all DOFs

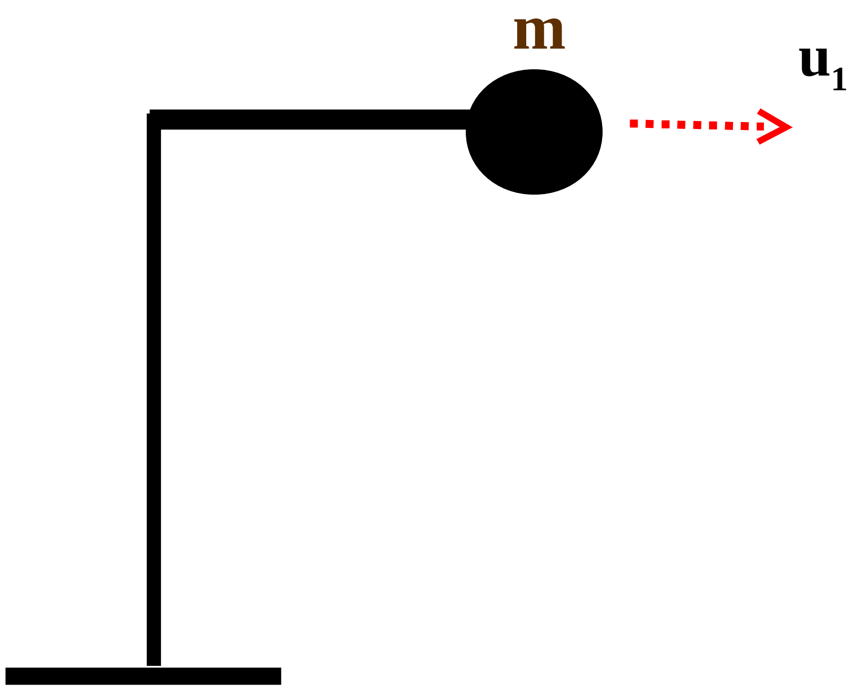
Multiple Degree-of-Freedom (MDOF) System

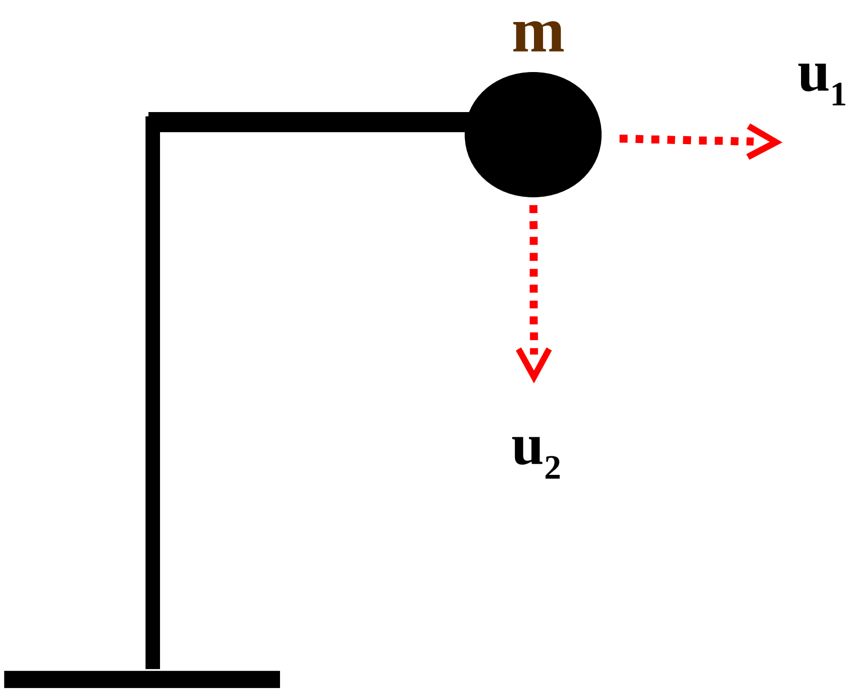


Lumped mass model of building (DOF=2). $u_3(t)$ to $u_6(t)$ (shown on previous slide) got eliminated by lumping the masses at mid length of beam.

Multiple Degree-of-Freedom (MDOF) System

What is the DOF for this system…?



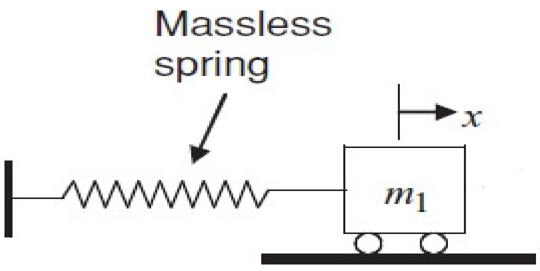


DOF may be taken 1 when flexural stiffness of beam is taken infinite/ too high

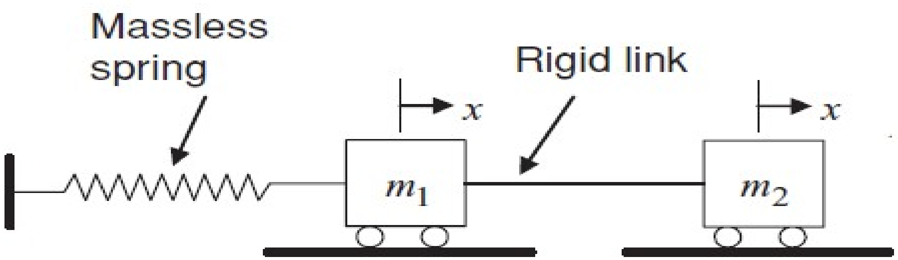
DOF is 2 when we have a flexible beam

Home Assignment No. M3H1

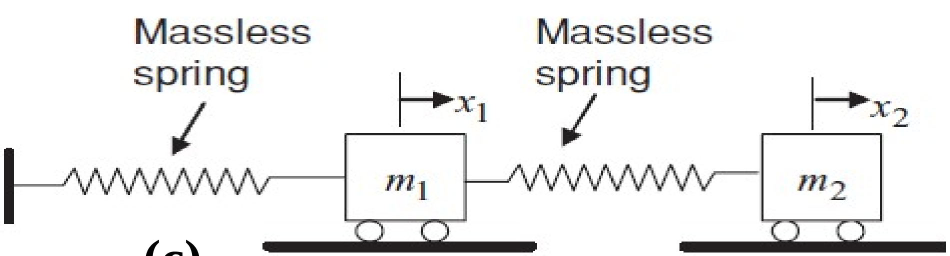
Determine the DOF of systems shown in given figures. Support you answer with argument(s)



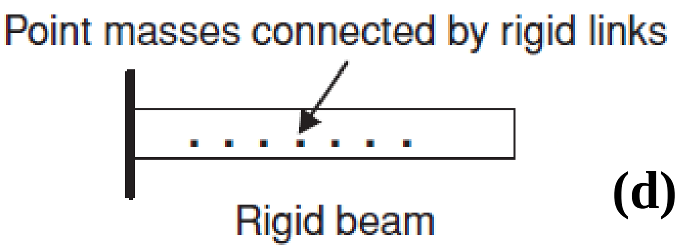
(a)



(b)

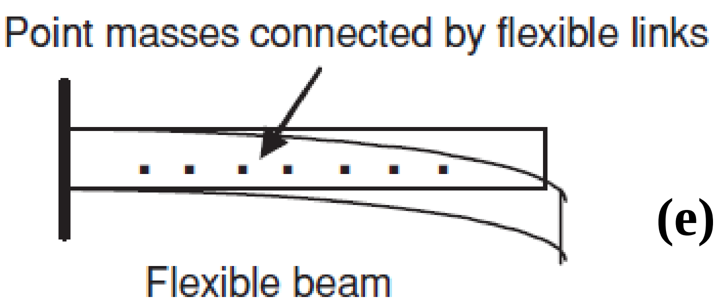


(c)



(d)

A rigid beam fixed at one end



(e)

A flexible beam fixed at one end

Vibrations vs. Oscillations

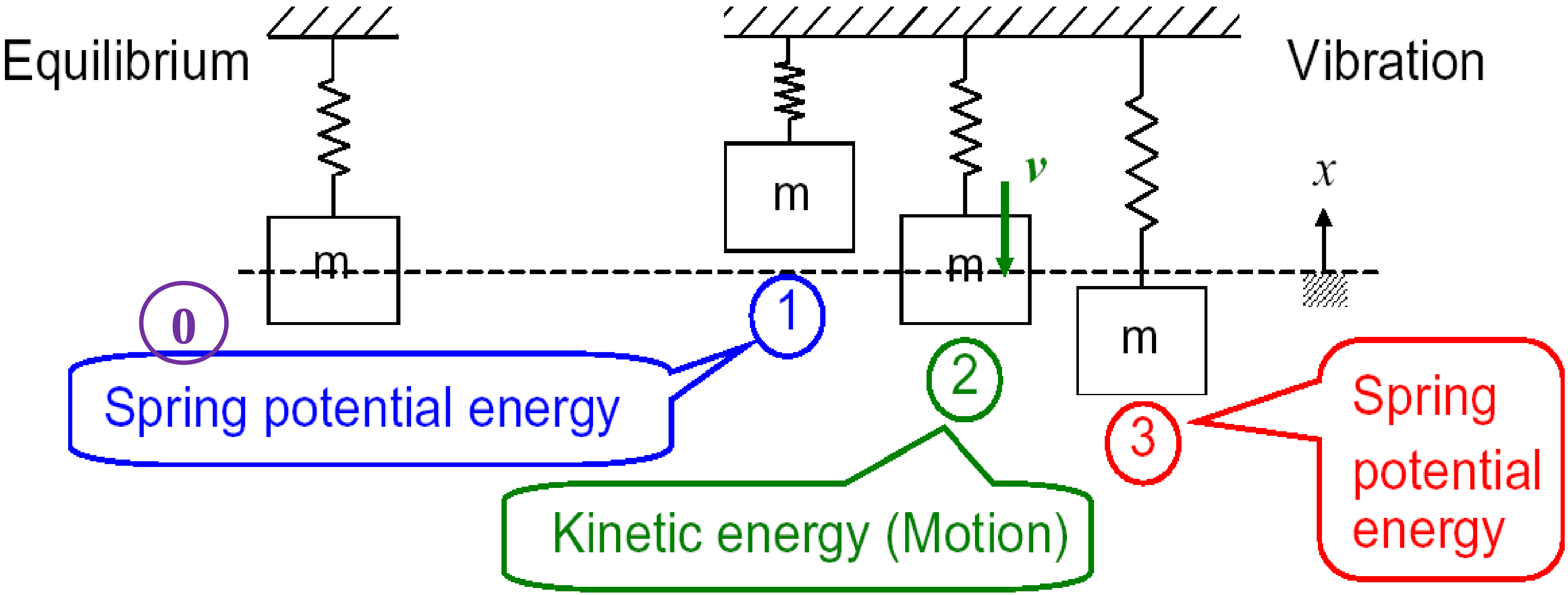
- **Vibration is "the rapid to and fro motion of an elastic /inelastic system whose equilibrium is disturbed"**

- *Vibrations are oscillations due to an elastic restoring force.*

- A flexible beam or string ***vibrates*** while a pendulum ***oscillates***.

- In most of the text books written on the subject, vibration and oscillations are interchangeably used

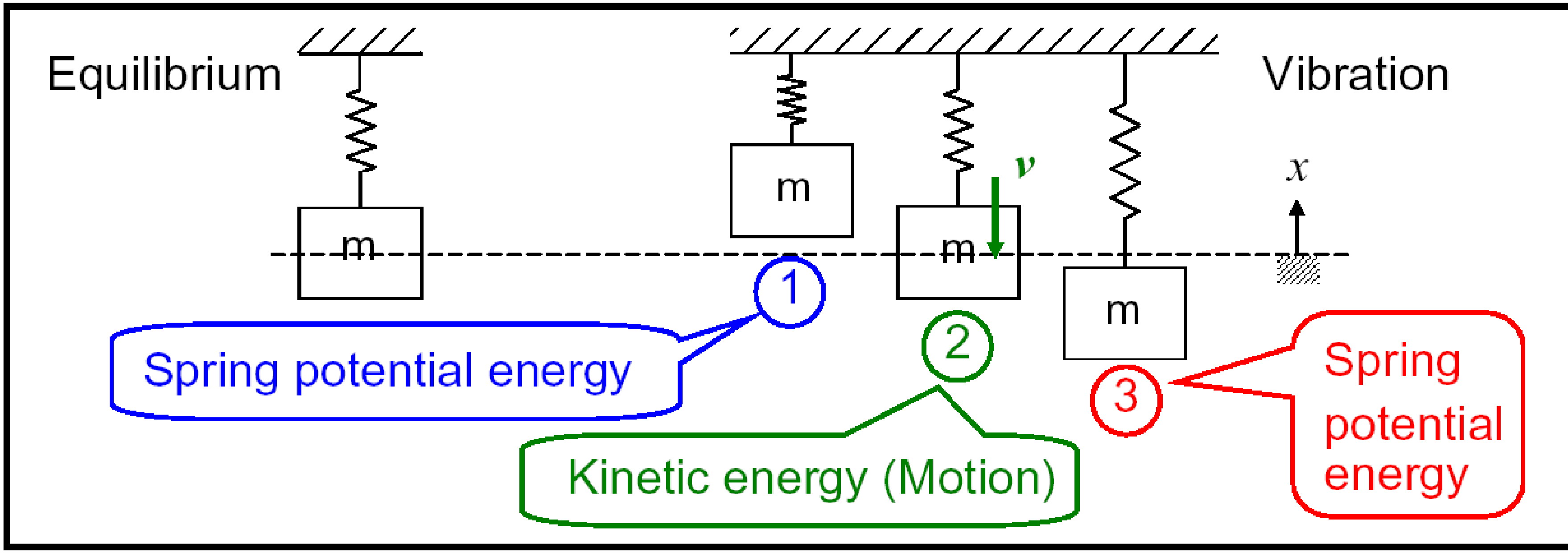
Physical Explanation of Vibration

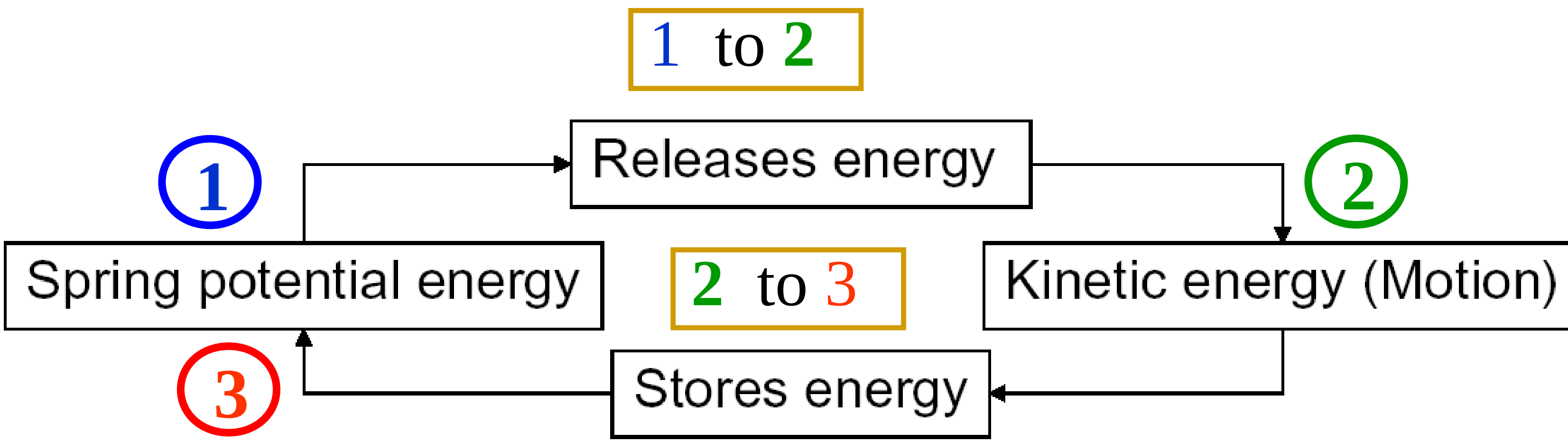


Activity….

Graphically represent the position of mass for positions 0,1,2, 3

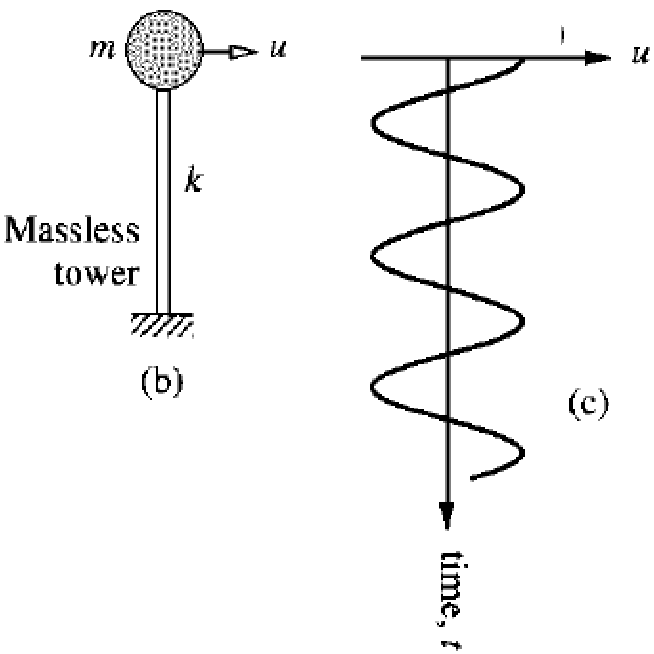
Physical Explanation of Vibration



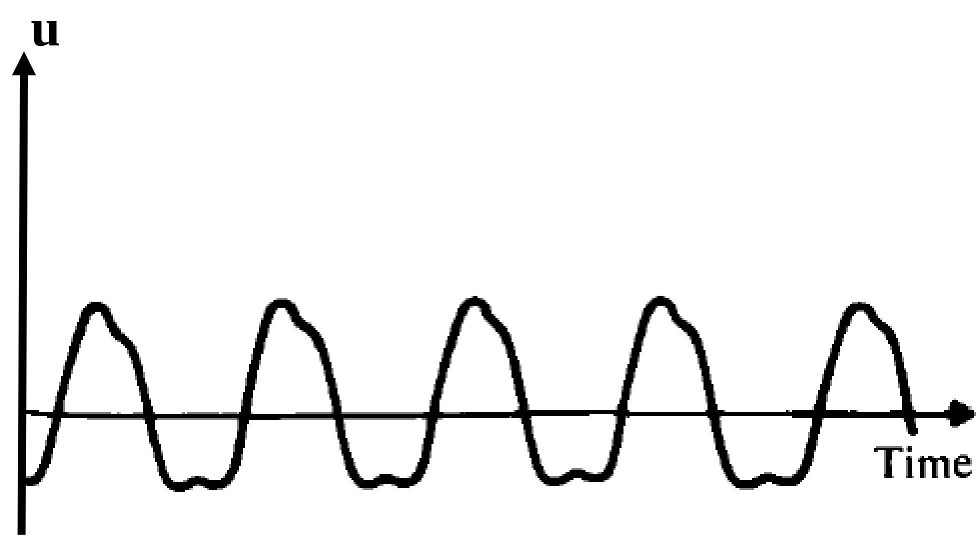


Periodic and Random vibrations

- The vibration can be Periodic (cyclic) or Random (arbitrary).
- If the motion is repeated after equal intervals of time, it is called ***Periodic motion***.
- The simplest type of periodic motion is Harmonic motion.

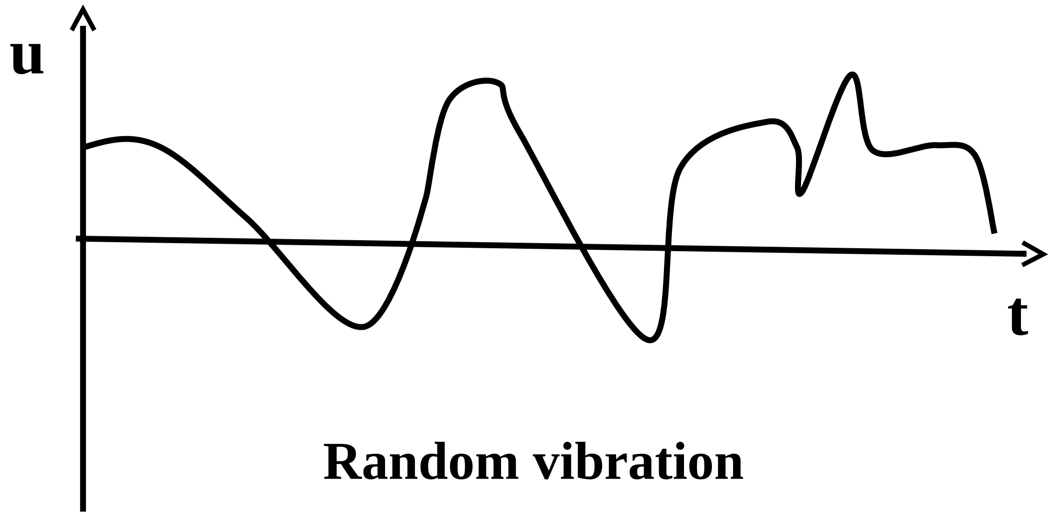


Periodic vibration (Harmonic vibration)



Periodic vibration (Non-harmonic vibration)

Periodic and Random vibrations



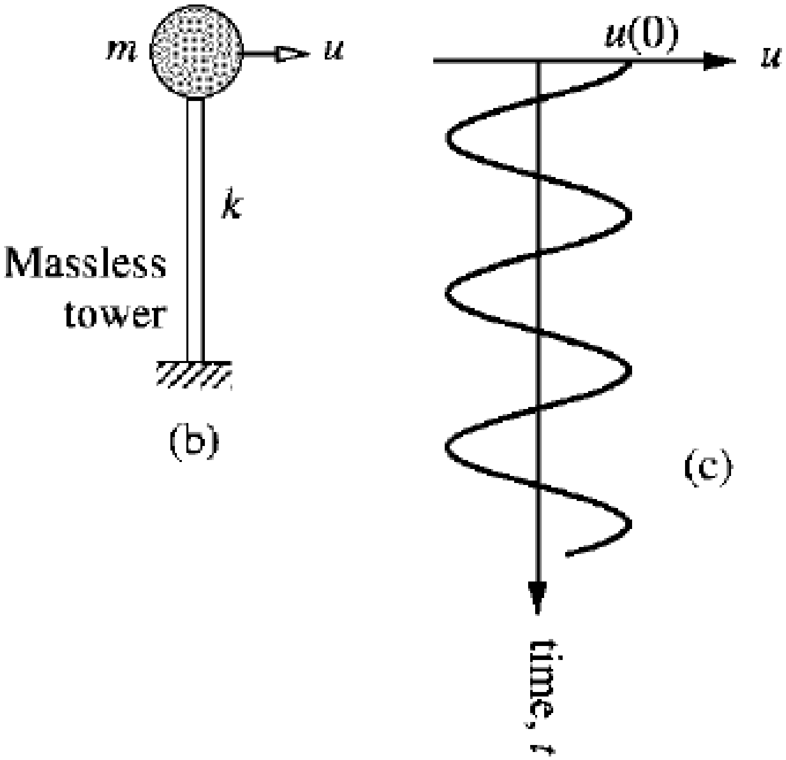
Random vibration

- Such type of vibrations are produced in a system due to wind, earthquake, traffic etc

Free vibrations vs. Forced vibrations

- When a structure vibrates without any externally applied forces, such as when it is pulled out of position, and then released.

- The vibration of strings on a musical instrument after they are struck is a common example of free vibration.



Free vibration of a SDOF lumped mass system when released after being stretched by a displacement u(0) at the top end .

Free vibrations vs. Forced vibrations

- Vibration of a system subjected to an external force is known is known as Forced vibration.
- The vibration that arises in machine such as diesel engines is an example of Forced vibration.
- As stated above vibration of a system in the absence of external force is known is known as Free vibration. Free vibration continues to occur after forced vibration. e.g., vibration of rotating machines continues to occur for some time after power supply is switched off. Similarly, a structure subjected to earthquake continues to vibrate for some time after there are no seismic waves to impart en

Undamped free vibration

If no energy is lost or dissipated in friction or other resistance during vibration, the vibration is known as ***Undamped vibration***

Undamped vibration is a hypothetical phenomena which help in providing an understanding of the Damped vibration.

Damped free vibration

In actual system the energy is always lost due to a number of mechanisms. Such type of vibration is known as ***Damped vibrations***

Damping

- Any energy that is dissipated during motion will reduce the kinetic and potential (or strain) energy available in the system and eventually bring the system to rest unless additional energy is supplied by external sources.

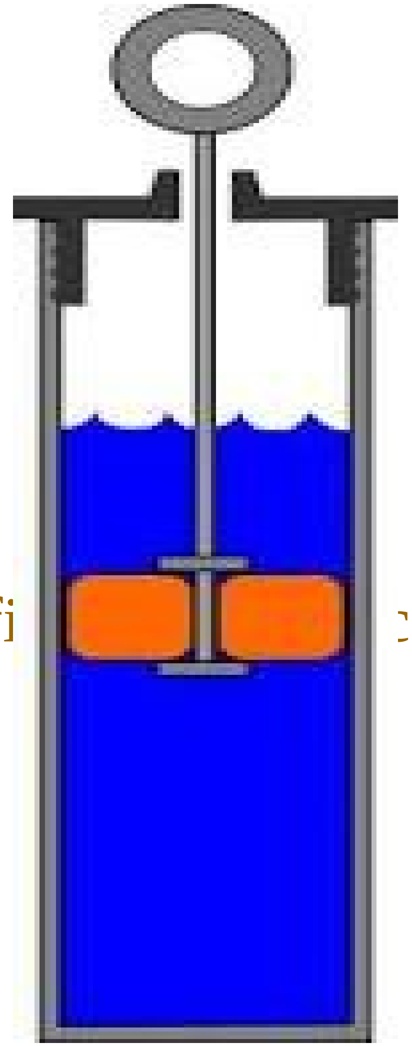
- The term ***Damping*** is used to described all types of energy dissipating mechanisms.

Damping

- In structures many mechanism contributes to the damping. In a vibrating building these include friction at steel connections, opening and closing of microcracks in concrete, and friction between the structures itself and nonstructural elements such as partition walls.

- Since there is considerable uncertainty regarding the exact nature and magnitude of energy dissipating mechanisms in most structural systems, the simple model of a ***dashpot*** is often used to quantify damping.

Damping

- The Dashpot or viscous damper is a ***'device'*** that limit or retard vibrations.
- Dashpot can be imagined as a cylinder fi:ous fluid ar
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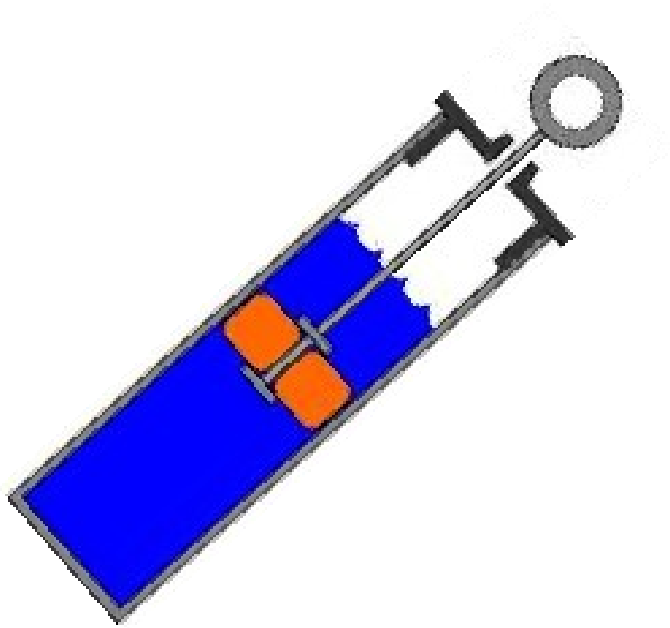
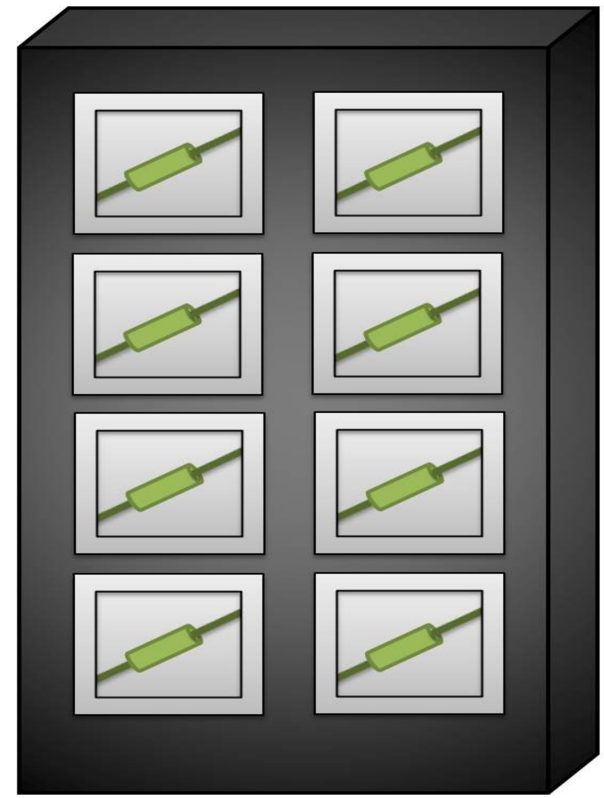
Simplified diagram of linear dashpot

Damping





Dash pots are ***imagined*** to exist diagonally in a building for dissipating energy .

Damping

- Simple dashpots as shown schematically in below given figure exert a force $\mathbf{f_D}$ whose magnitude is proportional to the velocity of the vibrating mass

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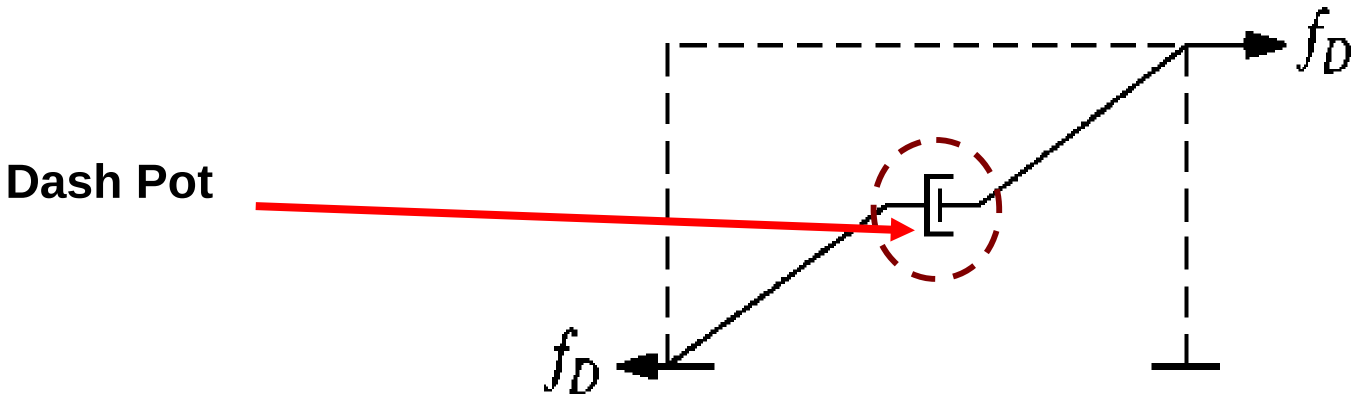
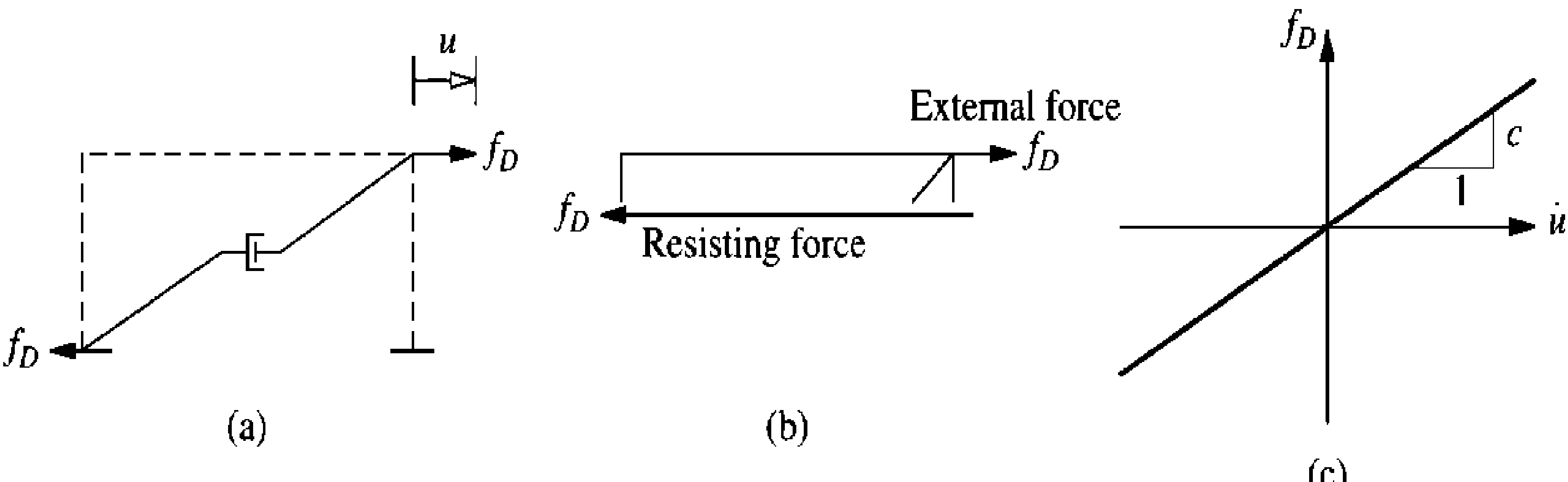


Damping

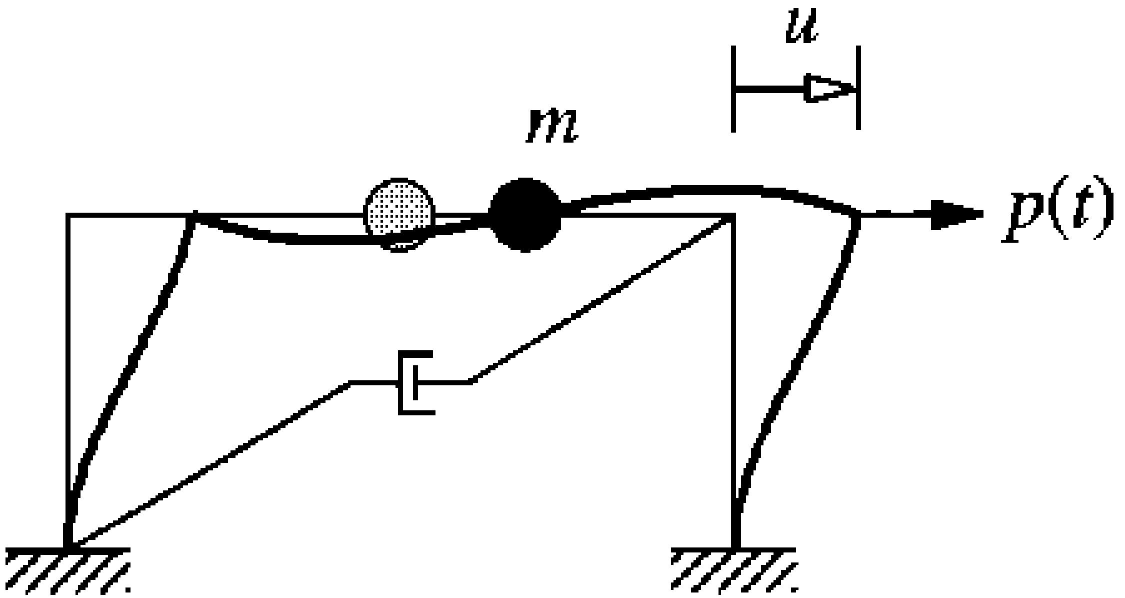
Figure **a** shows a linear viscous damper subjected to a force $\mathbf{f_D}$ along the DOF **u**. The internal force in the damper is equal and apposite to the external force $\mathbf{f_D}$ (Figure **b**). The damping force $\mathbf{f_D}$ is related to the velocity $\dot{u}$ across the linear viscous damper by:

$$f_D = c\dot{u}$$

Where the constant **c** is the *viscous damping coefficient*



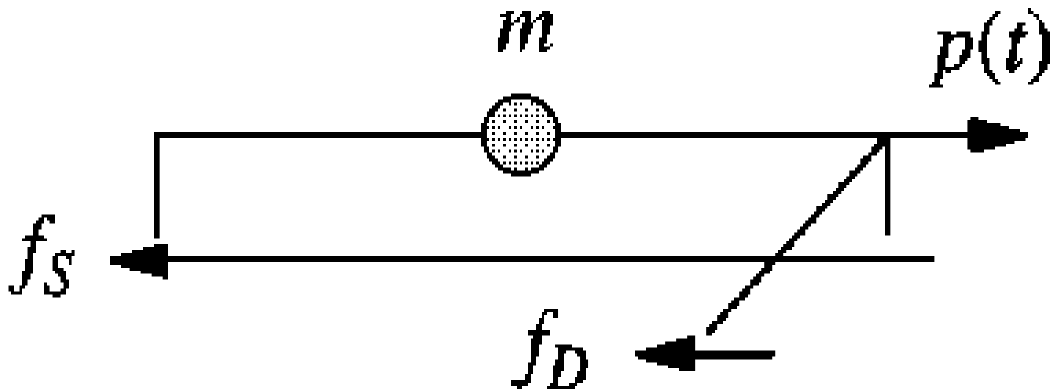
EQUATION OF MOTION (E.O.M) OF A SINGLE STORY FRAME UNDER EXTERNAL DYNAMIC FORCE



Two commonly used vector mechanics based approaches are:

1. **NEWTON'S SECOND LAW OF MOTION**
2. **D'ALEMBERT PRINCIPLE OF DYNAMIC EQUILIBRIUM**

E.O.M USING NEWTON'S SECOND LAW OF MOTION



- The Resultant force along x-axis = p (t) – f_S – f_D

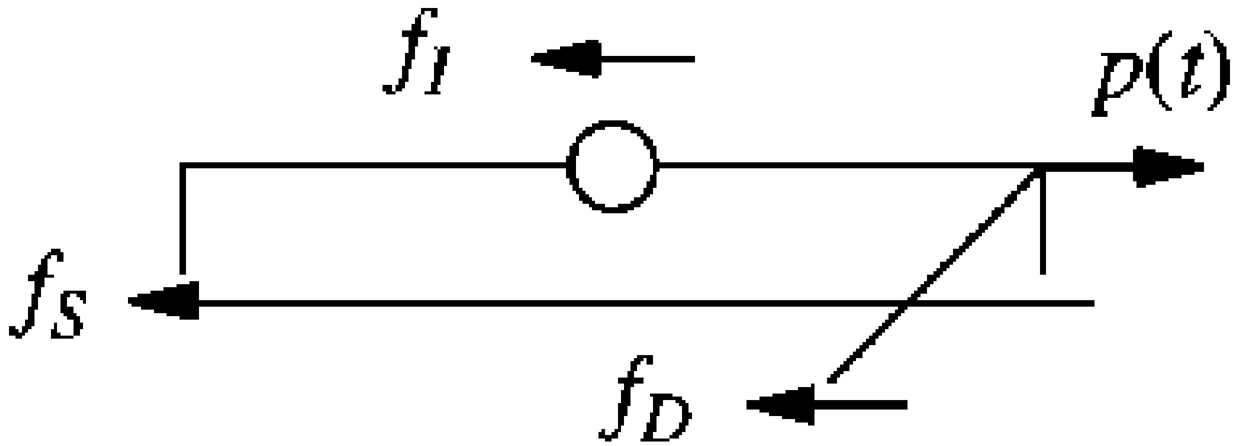
Where f_S = Elastic resisting force; (also known elastic restoring force), f_D= Damping resisting force

- According to Newton's second law, the resulting force causing acceleration = p(t)– f_S – f_D = mü or;

$f_S + f_D + m\ddot{u} = p(t)$; or $$\mathbf{ku + c\dot{u} + m\ddot{u} = p(t)}$$

E.O.M USING DYNAMIC EQUILIBRIUM

Using D'Alembert's Principle, a state of *dynamic equilibrium* can be defined by assuming that a fictitious ***inertial force*** f_I acts on the mass during motion.



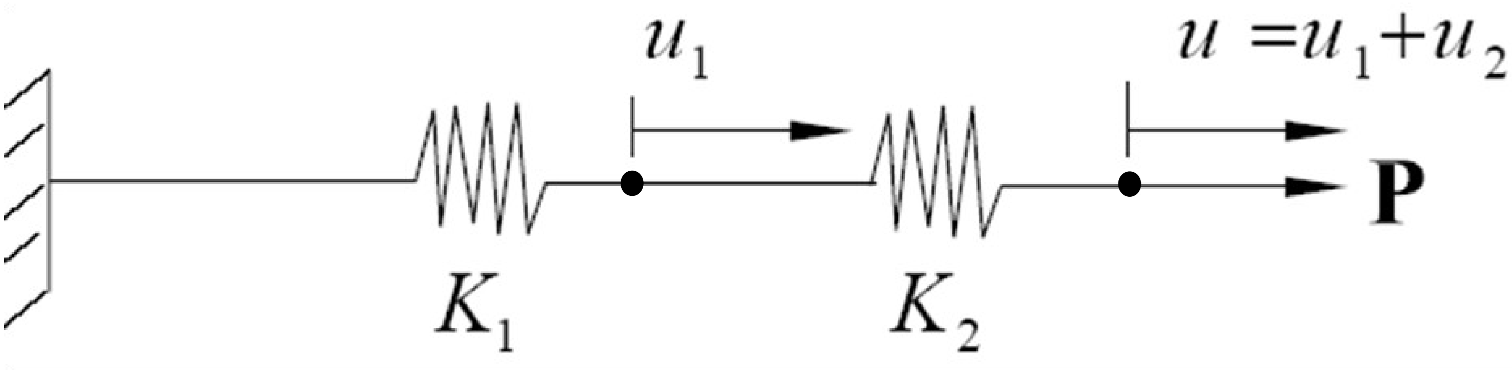
Equilibrium along x-axis requires that:

$-f_S - f_D - f_I + p(t) = 0$ or;

$f_S + f_D + f_I = p(t)$ or;

$$\mathbf{ku + c\dot{u} + m\ddot{u} = p(t)}$$

STIFNESS OF SPRINGS IN SERIES



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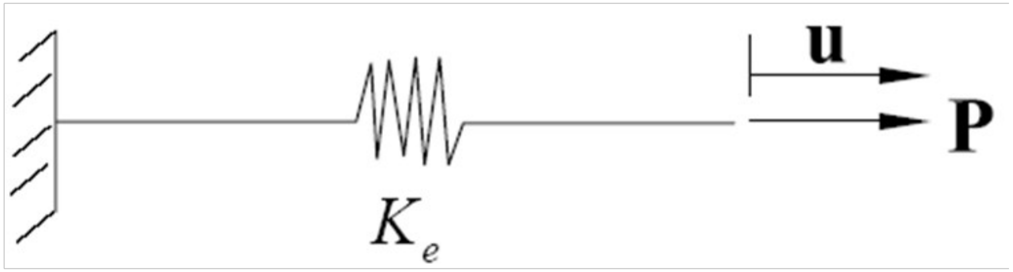
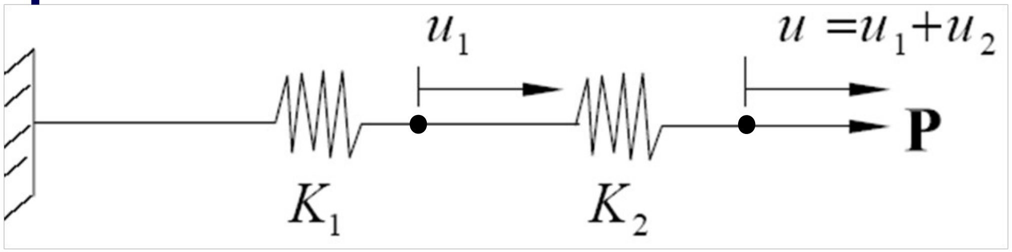


STIFNESS OF SPRINGS IN SERIES



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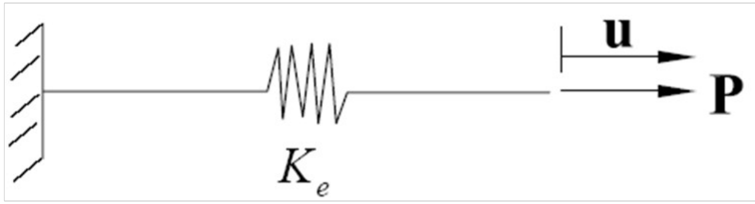


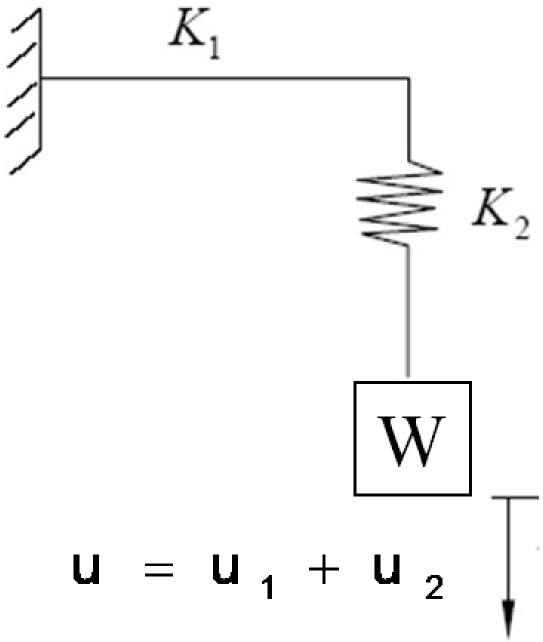
$$u = u_1 + u_2 \qquad \Rightarrow \frac{p}{k_e} = \frac{p_1}{k_1} + \frac{p_2}{k_2}$$

$$\text{Since } p_1 = p_2 = p \quad \Rightarrow \frac{p}{k_e} = \frac{p}{k_1} + \frac{p}{k_2}$$

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

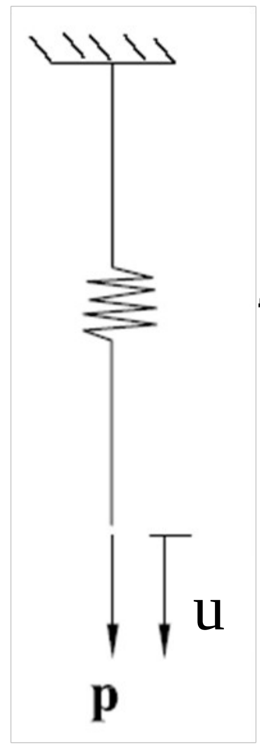
STIFNESS OF SPRINGS IN SERIES

EAXMPLE



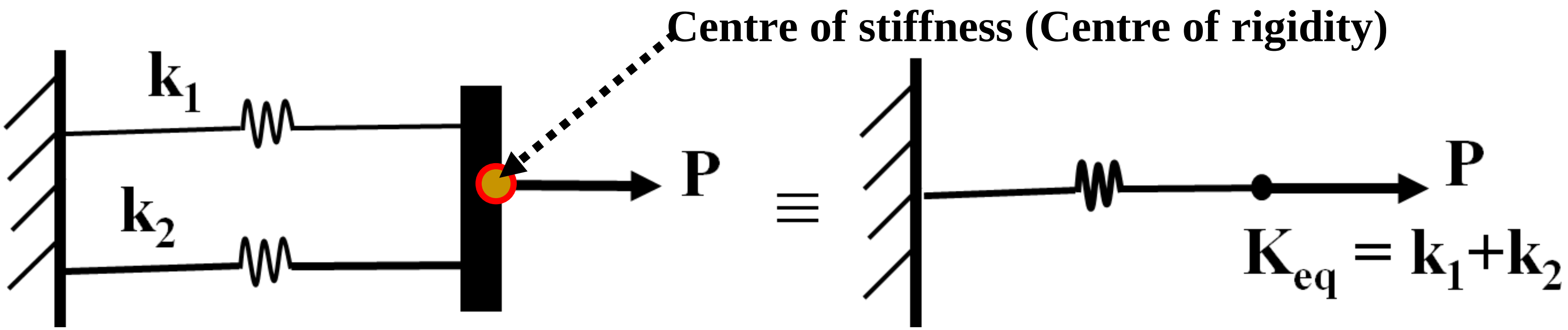
$$u = u_1 + u_2$$

=



$$k_e = \frac{k_1 k_2}{k_1 + k_2}$$

STIFNESS OF SPRINGS IN PARALLEL



$$p = p_1 + p_2$$

$$\Rightarrow k_e u = k_1 u_1 + k_2 u_2$$

$$\text{Since } u_1 = u_2 = u$$

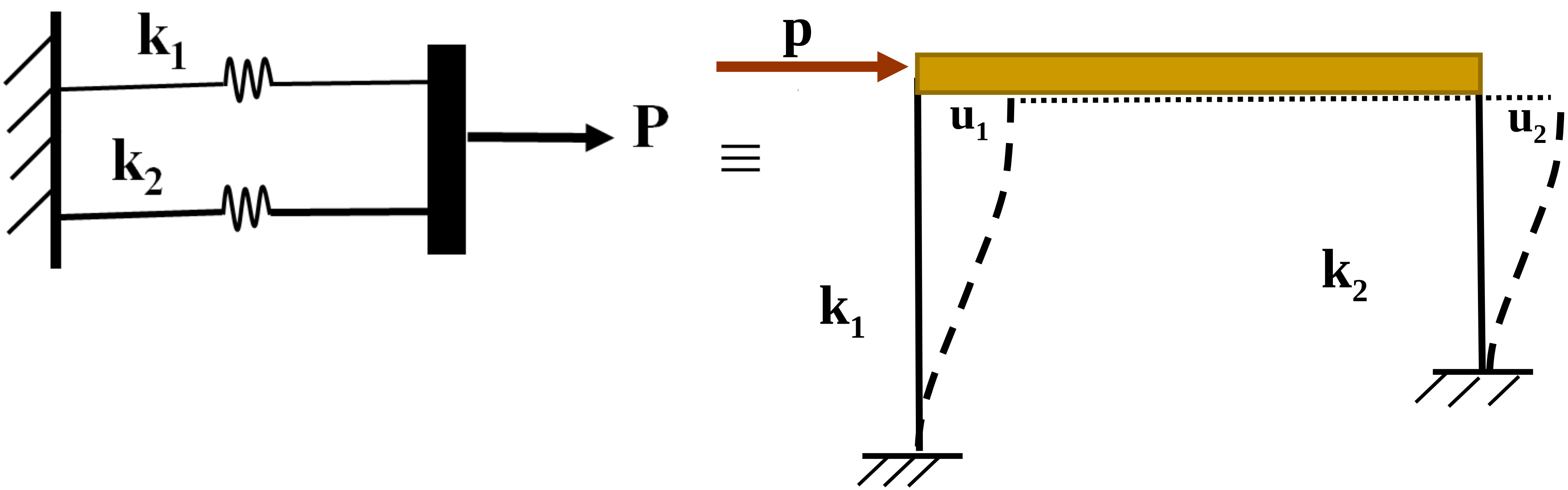
$$\Rightarrow k_e u = k_1 u + k_2 u$$

$$\Rightarrow k_e = k_1 + k_2$$

STIFNESS OF SPRINGS IN PARALLEL

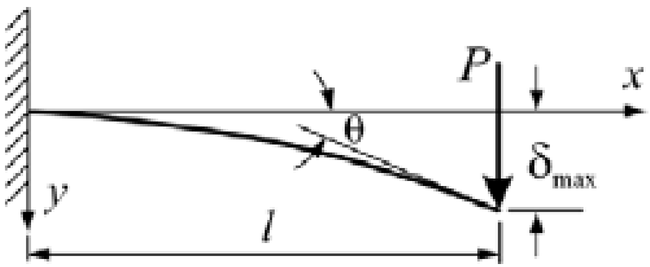
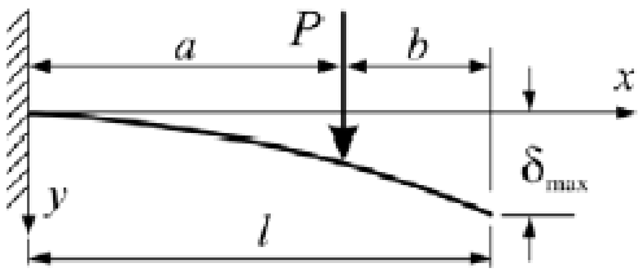
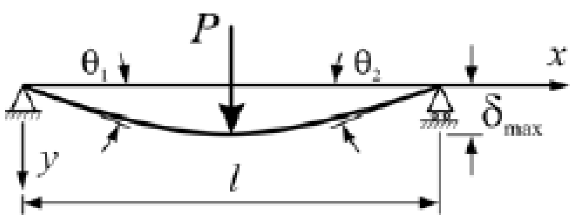
$$k_e = k_1 + k_2$$

EAXMPLE

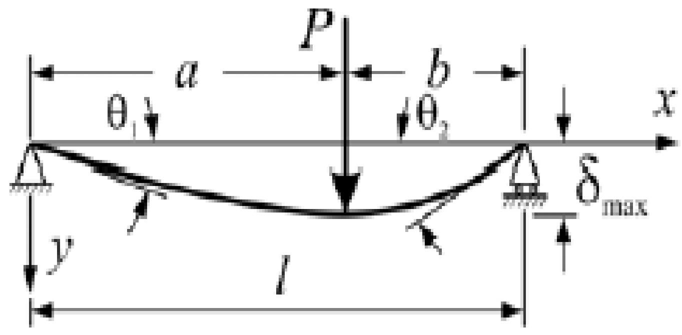


$u_1 = u_2$ provided the change in axial length of beam is neglected (a reasonable assumption).

Deflection in beam and their stiffness

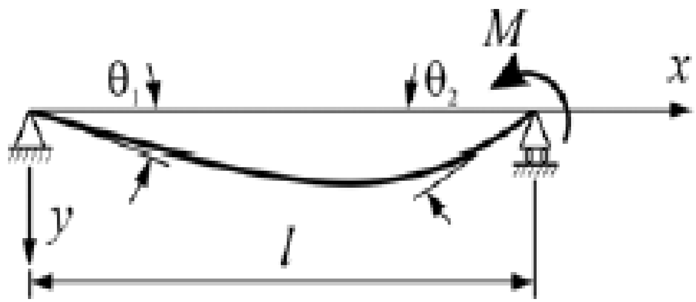
BEAM TYPE	MAXIMUM DEFLECTION	
1. Cantilever Bea		
	$\delta_{max} = \dfrac{Pl^3}{3EI}$	$k = P/\delta_{max} = 3EI/l^3$
2. Cantilever Bea		
	$\delta_{max} = \dfrac{Pa^2}{6EI}(3l - a)$	$k = 6EI/\left\{a^2(3l - a)\right\}$
	$\delta_{max} = \dfrac{Pl^3}{48EI}$	$k = P/\delta_{max} = 48EI/l^3$

Deflection in beam and their stiffness



$$\delta_{max} = \frac{Pb\left(l^2 - b^2\right)^{3/2}}{9\sqrt{3}\, lEI} \text{ at } x = \sqrt{\left(l^2 - b^2\right)/3}$$

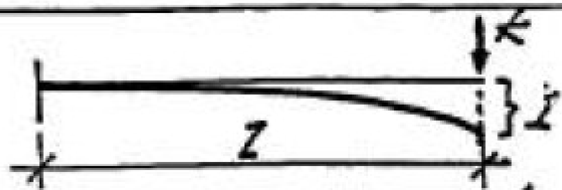
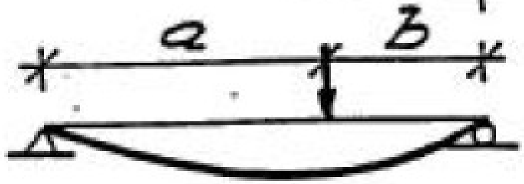
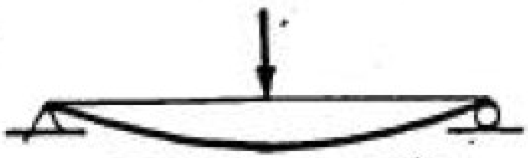
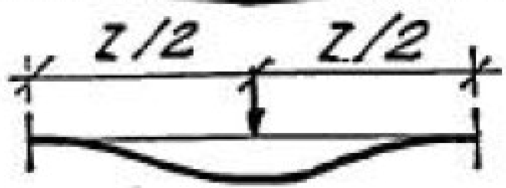
$$\delta = \frac{Pb}{48EI}\left(3l^2 - 4b^2\right) \text{ at the center, if } a > b$$



$$\delta_{max} = \frac{Ml^2}{9\sqrt{3}\, EI} \text{ at } x = \frac{l}{\sqrt{3}}$$

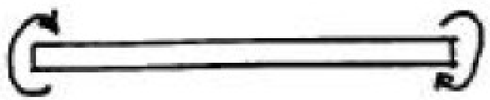
$$\delta = \frac{Ml^2}{16EI} \text{ at the center}$$

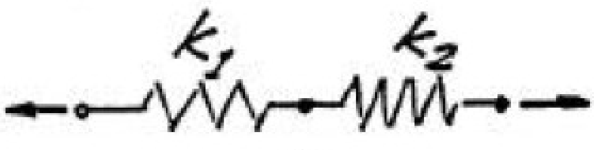
STIFFNESS CONSTANTS OF SOME STRUCTURAL ELEMENTS

	Structure	k =	Comment
1.		$\frac{3EI}{\ell^3}$	Fixed-Pinned
2.		$\frac{12EI}{\ell^3}$	Fixed-Fixed
3.		$\frac{3EI\ell}{a^2b^2}$	Pinned-Pinned
4.		$\frac{48EI}{\ell^3}$	Pinned-Pinned
5.		$\frac{192EI}{\ell^3}$	Fixed-Fixed

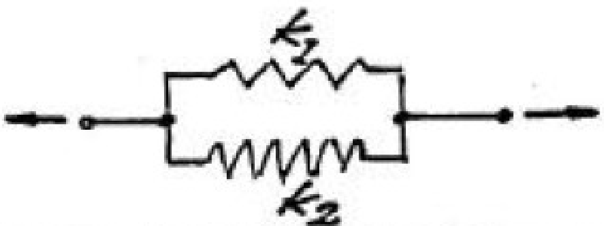
STIFFNESS CONSTANTS OF SOME STRUCTURAL ELEMENTS

7.  $\frac{EA}{\ell}$ Axial

8.  $\frac{GJ}{\ell}$ Torsion

9. 

$\frac{1}{1/k_1+1/k_2}$ Springs in Series

10. 

k_1+k_2 Parallel Springs

I = moment of inertia of cross-sectional area

A = cross-sectional area

J = torsional constant of cross section

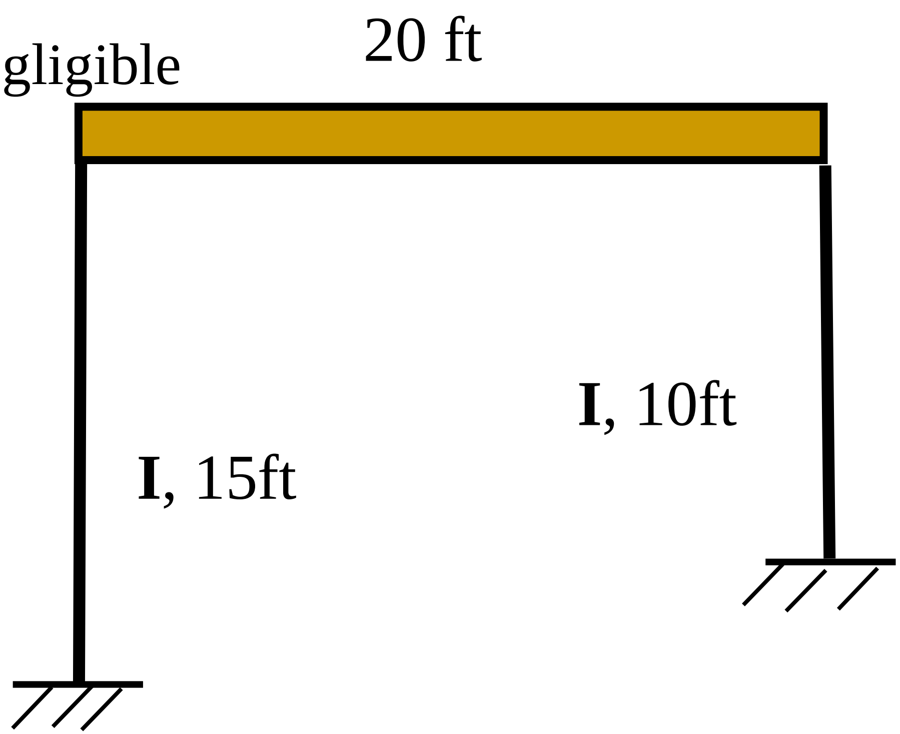
ℓ = length of element

Problem M 3.1

Determine lateral stiffness of the frame if a lateral load is applied at beam level. Assume:

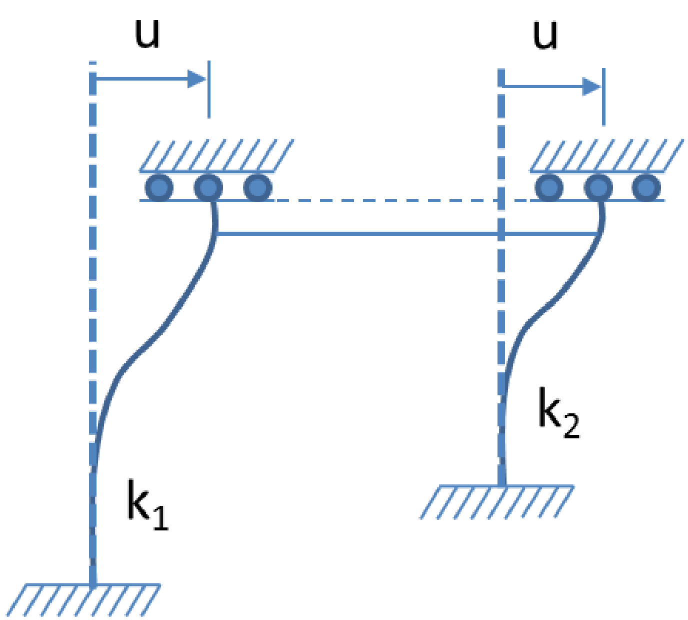
1. The flexural stiffness of beam is too high as compared to that of connected columns.
2. Axial deformations in beam is negligible

Take **E** = 29,000 ksi, **I** = 1200 in^4



Solution M3.1

$$k_{eq} = k_1 + k_2$$



$$k = \frac{12EI}{h_1^3} + \frac{12EI}{h_2^3}$$

$$= 12EI\left[\frac{1}{h_1^3} + \frac{1}{h_2^3}\right]$$

$$= 12 \times (29000\ k/in^2) \times (1200 in^4)\left[\frac{1}{(15 \times 12\ in)^3} + \frac{1}{(10 \times 12\ in)^3}\right]$$

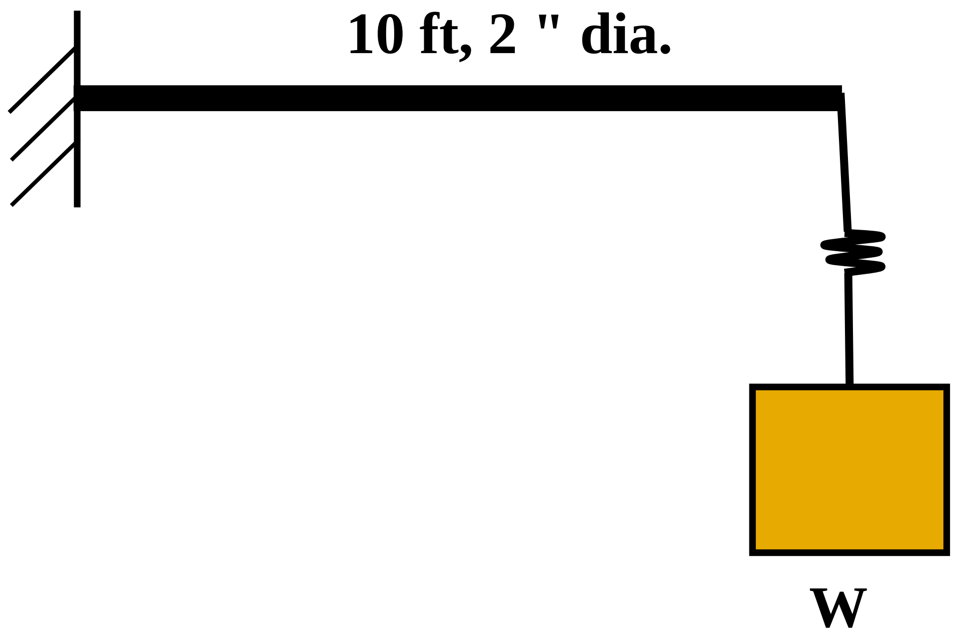
$$= 313.29\, k/in$$

$$= 3759\, k/ft$$

LATERAL STIFFNESS OF A CANTILEVER BEAM CONNECTED TO A SPRING SYSTEM

Problem M 3.2

Determine the stiffness of cantilever beam by assuming that the self weight of beam is negligible Take **E** = 29,000 ksi, $\mathbf{k}_{spring}$ = 200 lb/ft.



Solution M3.2

$$k_1 = 200\, lb/ft$$

$$k_2 = \frac{3EI}{l^3} = \frac{3 \times \left(29000\, {}^{k}/_{in^2}\right) \times \left(\frac{\pi}{64} \times (2\ in)^2\right)}{(10 \times 12\ in)^3}$$

$$= 0.0396\, k/in^2 = 474.7\, lb/ft$$

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2} = \frac{200 \times 474.7}{200 + 474.7}$$

$$k_{eq} = 140.7\, lb/ft$$

Home Assignment No. M3H2

1. Determine the equivalent stiffness of system shown in Figure **a** **(Answer** = 2.6k)

2. Determine the equivalent stiffness of system shown in Figure **b** **(Answer=** 7.03 * 10^7 N/m)

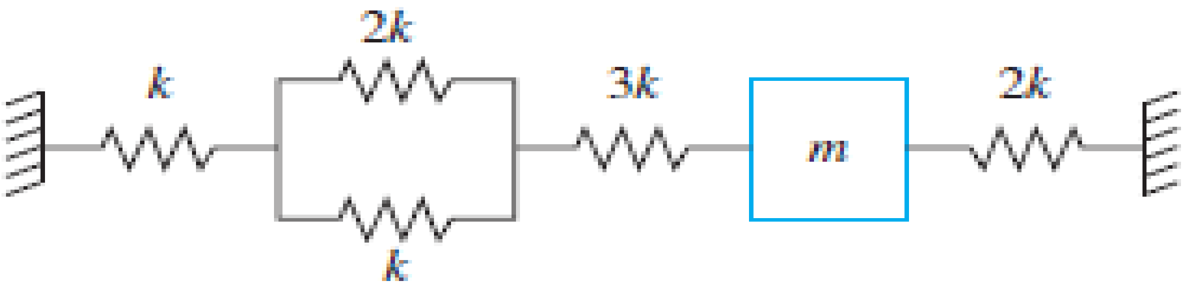
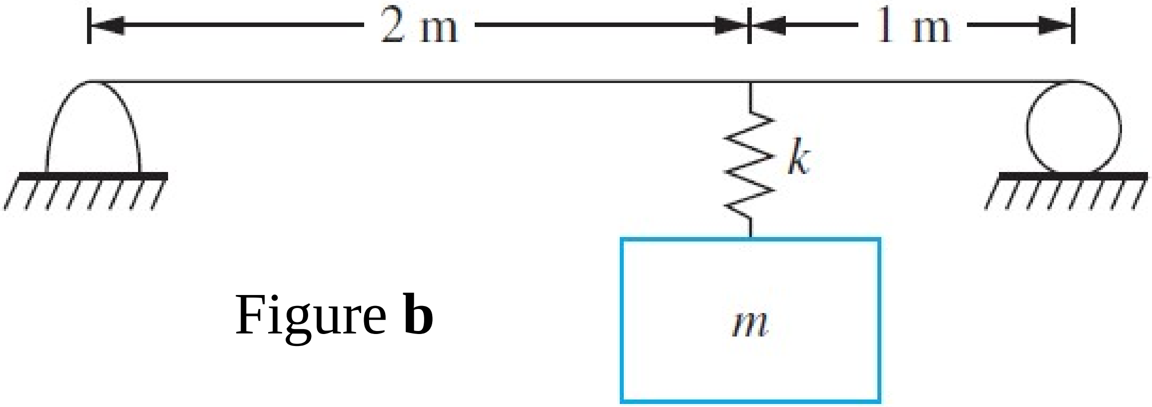


Figure **a**



$E = 210 \times 10^9$ N/m^2
$I = 5 \times 10^{-4}$ m^4
$k = 1 \times 10^8$ N/m

Figure **b**

Home Assignment No. M3H2

3. Determine the equivalent stiffness of the systems (in vertical direction when vertical force is applied at point A) shown in Figures **c & d**. Take $\boldsymbol{k_1}$= 1 kN/m and $\boldsymbol{k_{beam}}$ = 5 kN/m

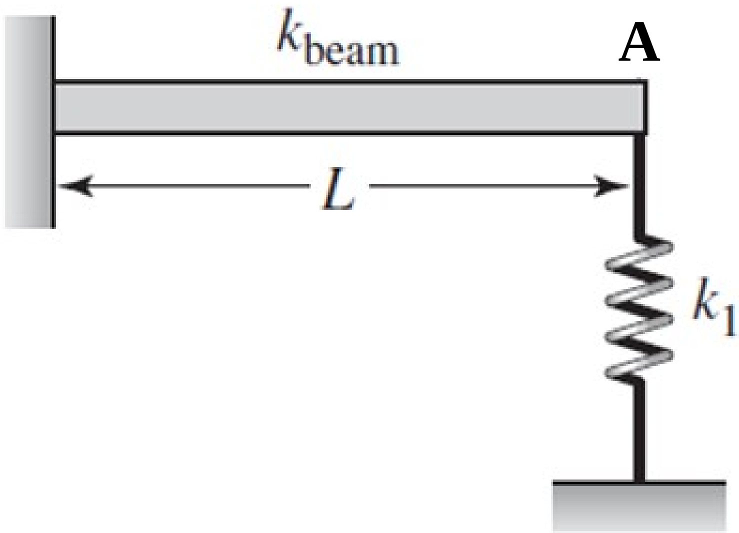


Fig. c

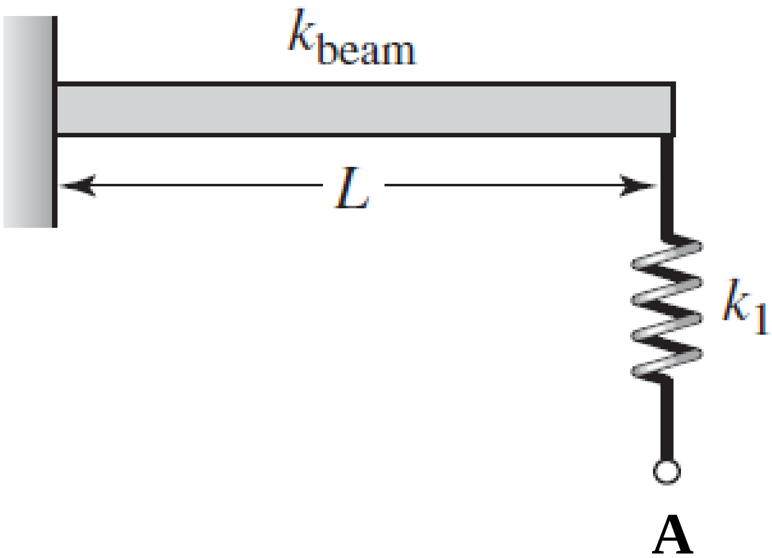


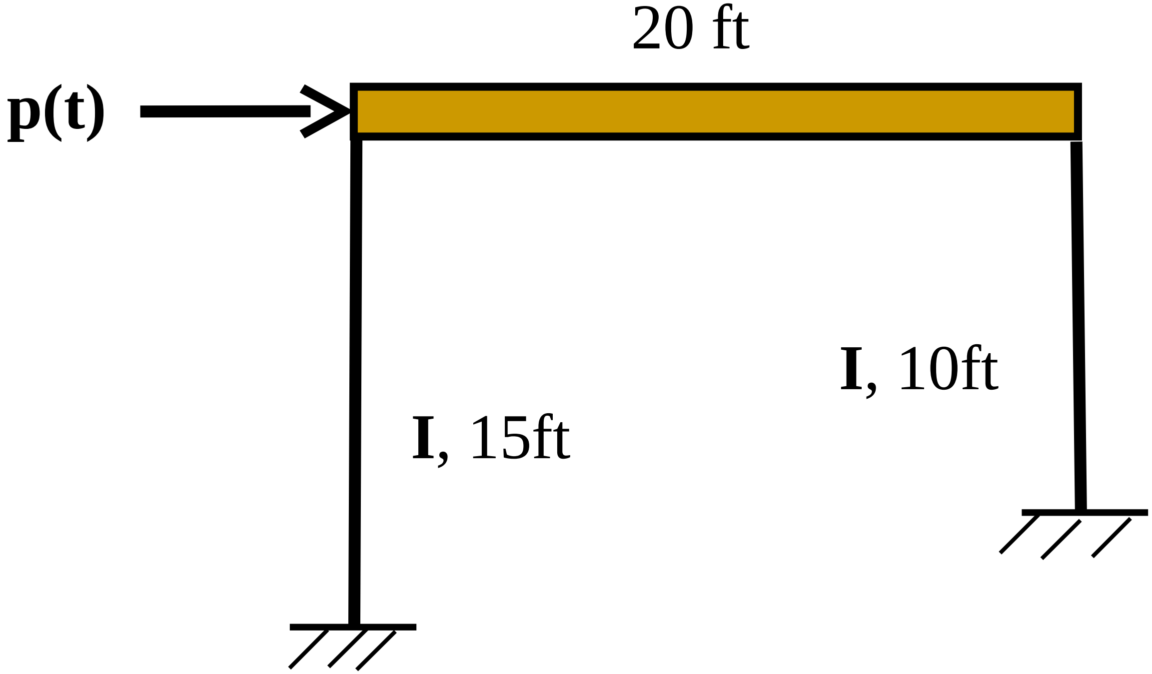
Fig. d

E.O.M FOR A SINGLE STORY FRAME UNDER LATERAL DYNAMIC FORCE

Problem M 3.3

Develop the equation of motion of the frame shown in problem **M 3.1** under the action of a lateral dynamic force **p(t)**. Consider a uniformly Distributed gravity load of 5 k/ft acting on the beam.

Neglect damping effect



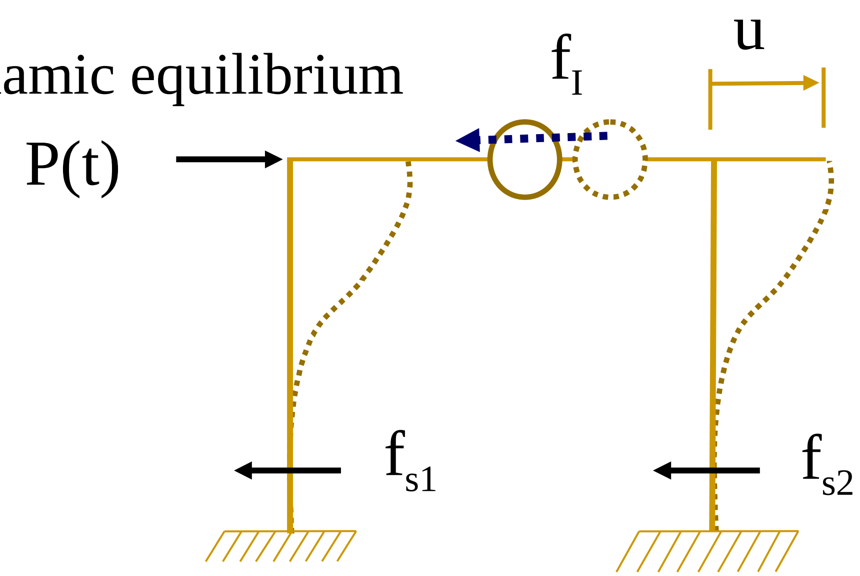
Solution M3.3

$$m = \frac{w}{g} = \frac{5 \times 20k}{32.2\, ft/sec^2}$$

$$m = 3.106\, k.sec^2/ft = 3106\ lb.sec^2/ft$$

$$m = 3.106\, slug$$

Using D-Alembert's Principle of dynamic equilibrium



$$P(t) - f_I - f_{s_1} - f_{s_2} = 0$$

$$P(t) - m\ddot{u} - \left(f_{s_1} + f_{s_2}\right) = 0$$

$$(k_1 u + k_2 u) + m\ddot{u} = P(t)$$

$$(ku) + m\ddot{u} = P(t)$$

As , k= 3759k/ft

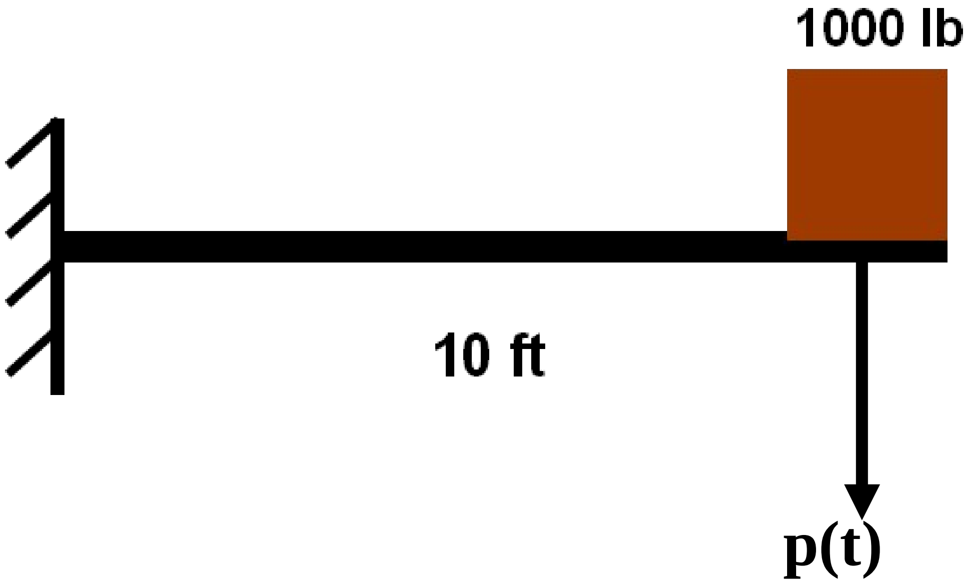
$$3106\ddot{u} + 3.76 * 10^6 u = p(t)$$

Where u & p(t) are in ft and lb, respectively

E.O.M FOR A CANTILVER BEAM UNDER LATERAL DYNAMIC FORCE

Problem M 3.4

Develop the equation of motion for the cantilever beam under the action of **p(t)**. Neglect the self weight of beam as well as damping effect. Take **E**= 29000 ksi & **I** = 150 in^4



M-3.4

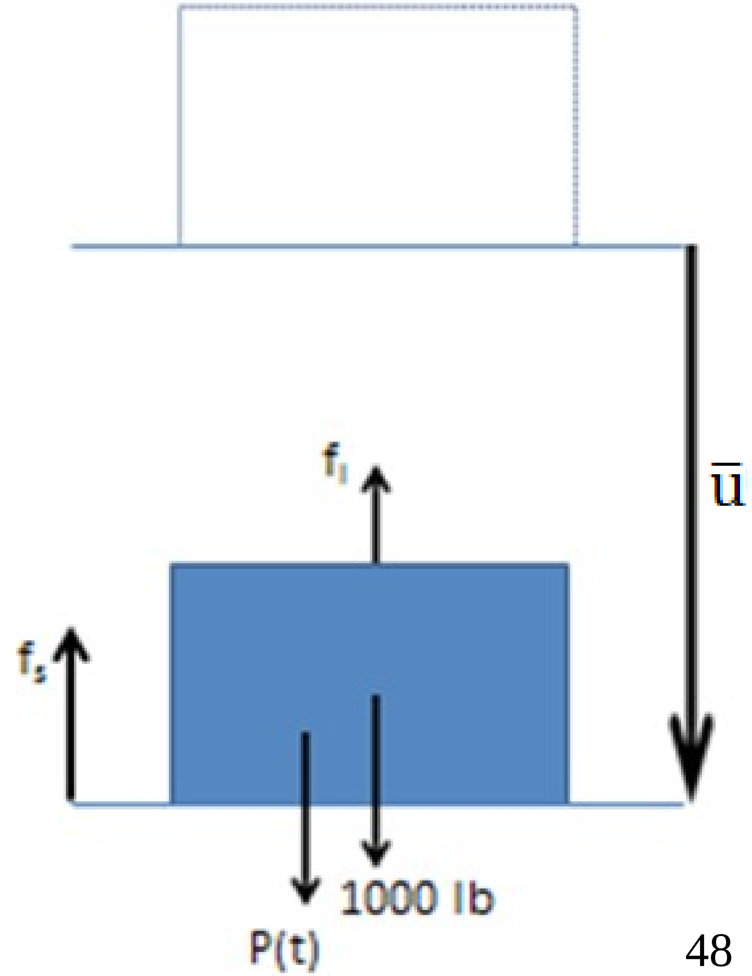
The displacement comprise of two parts.

1.Constant deflection at free end, δ_{st}, due to static load of 100lb.

2.Time dependent displacement [u(t)] at free end due to P(t).

Total Displacement, $\overline{u} = \delta_{st} + u$......... eq (i)

$\sum Fy = 0 \quad +\uparrow \quad -\downarrow$



$f_s + f_I - W - P(t) = 0$

$K.\overline{u} + m\,\ddot{\overline{u}} - W = P(t)$ ....eq (ii)

$\overline{u} = \delta_{st} + u$

$\dot{\overline{u}} = \dot{u}$

$\ddot{\overline{u}} = \ddot{u}$

Inserting values in eq (ii)

$K(\delta_{st} + u) + m\ddot{u} - W = P(t)$

$K\delta_{st} + Ku + m\ddot{u} - K\delta_{st} = P(t)$

$Ku + m\ddot{u} = P(t)$

the above equation indicate that EOM is unaffected by static displacements. As considering dynamic displacements from static equilibrium position result in same EOM.

However, the gravity loads must be considered if they act as either restoring forces (e.g., Pendulum) or as destabilizing forces (inverted pendulum).

Now

$$K = \frac{3EI}{L^3} = \frac{3 * (29000\ ^k/_{in^2}) * (150\,in^4)}{(10 * 12in)^3}$$

K = 7.552 K/in = 90625 lb/ft

$$m = w/g = \frac{1000\,lb}{32.2\frac{ft}{sec^2}} = 31.06 \text{ slug}$$

substituting in eq (iii)

$90625u + 31.06\ddot{u} = P(t)$ Where u & p(t) are in ft and lb, respectively

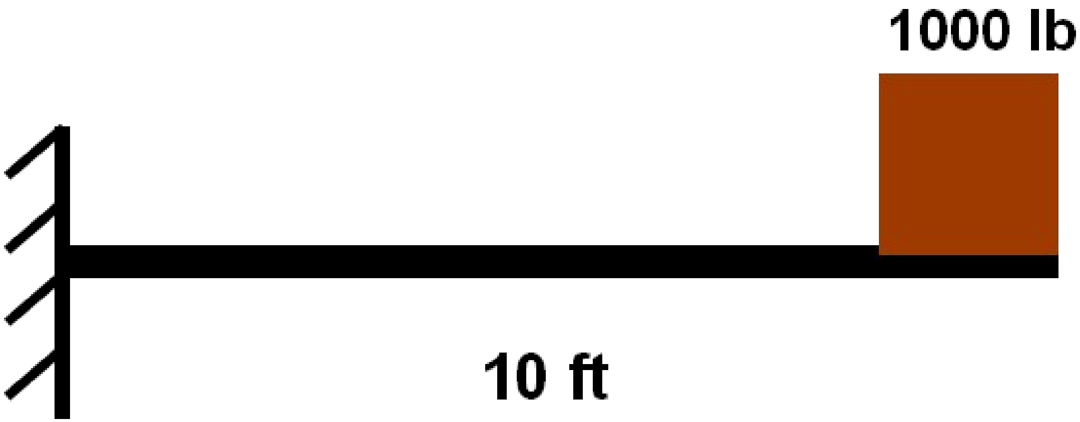
E.O.M FOR A CANTILVER BEAM UNDER THE GRAVITY LOAD (SUDDENLY PLACED)

Problem M 3.5

Develop the equation of motion for the cantilever beam under the action 1000 lb weight. Assume that time required to place the weight on beam is Insignificant as compared to natural time period of beam (?).

Neglect the self weight of beam as well as damping effect.

Take **E** = 29000 ksi & **I** = 150 in^4



M-3.5

The gravity load in given case acts as dynamic load (as per statement of problem)

$\sum Fy = 0 \quad + \uparrow \quad - \downarrow$

$f_s + f_I - 1000 = 0$

$K.u + m\ddot{u} - 1000 = 0$

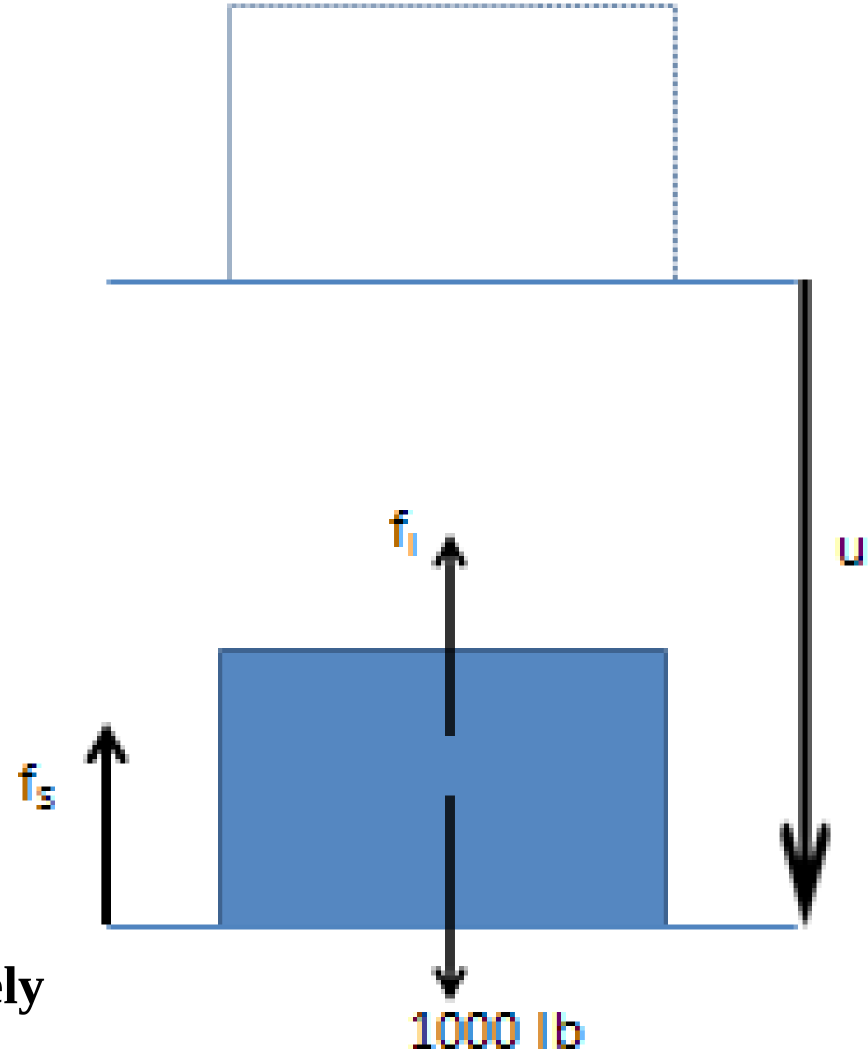
From problem 3.4

K=90625 lb/ft

And m=31.06 slug

So,

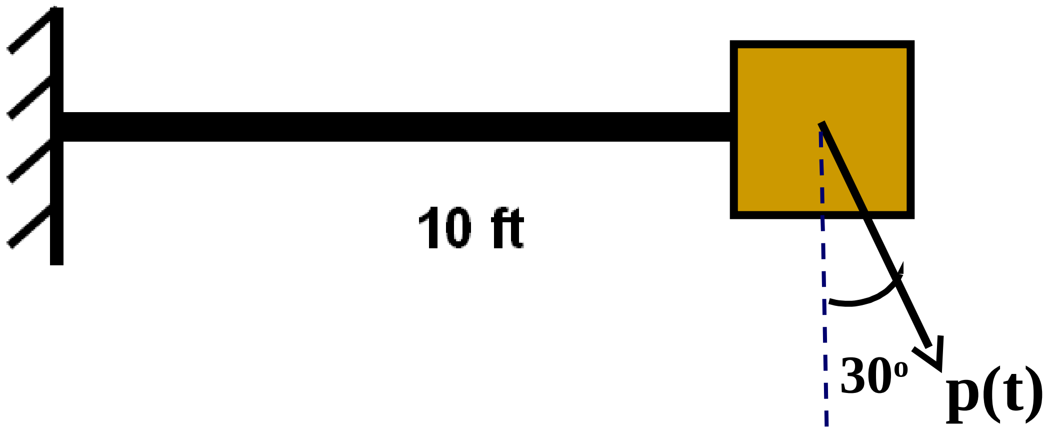
$90625u + 31.06\,\ddot{u} = 1000$



Where u & p(t) are in ft and lb, respectively

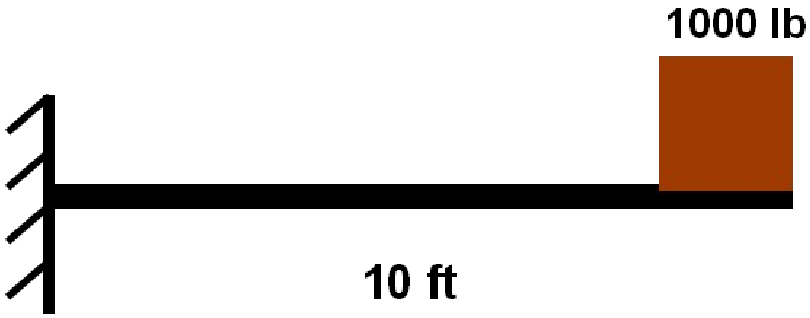
Home Assignment No. M3H3

Q1: Develop the EOMs for the cantilever beam along axes which are parallel and perpendicular to the beam, under the action of **p(t)**. Neglect the self weight of beam as well as damping effect. Take **E**= 29000 ksi and **I** = 200 in^4 , **A**= 50 in^2. Ignore the dimensions of 1000 lb weight



Home Assignment No. M3H3

Q 2. Develop the equation of motion for the cantilever beam under the action 1000 lb weight. A weight of 500 lb is placed on already acting weight of 1000 lb. Assume that the time required to place the 500 lb weight on beam is insignificant as compared to natural time period of beam. Neglect the self weight of beam as well as damping effect. Take **E** = 29000 ksi & **I** = 150 in^4



Q 3. Develop the EOMs for the beams mentioned in Figures **c** and **d** (Q 3, M3H2) for a vertical dynamic load **p(t)** acting at **A**. Consider a

METHODS OF SOLUTIONS FOR DIFFRENTIAL EQUATIONS

The differential equation of the type $ku + c\dot{u} + m\ddot{u} = p(t)$

can be solved by any one of the following four methods

1. **Classical mathematical solutions**
2. **Duhamel's Integral**
3. **Frequency- Domain method**
4. **Numerical methods**