Convolution Sum Representation Of LTI Systems: (Proof):

V.V.V Imp

The response of a linear system to a discrete time signal $x[n]$ is the superposition of the scaled responses of the system to each of the shifted impulses.

Consider the response of a linear system to an arbitrary input $x[n]$. Let $h_K[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n-K]$. Then according to the superposition property for a linear system, the response $y[n]$ of a linear system to the input $x[n]$ is simply the weighted linear combination of these basic responses.

i.e

$$y[n] = \sum_{K=-\infty}^{+\infty} x[K]\, h_K[n] \longrightarrow (1)$$

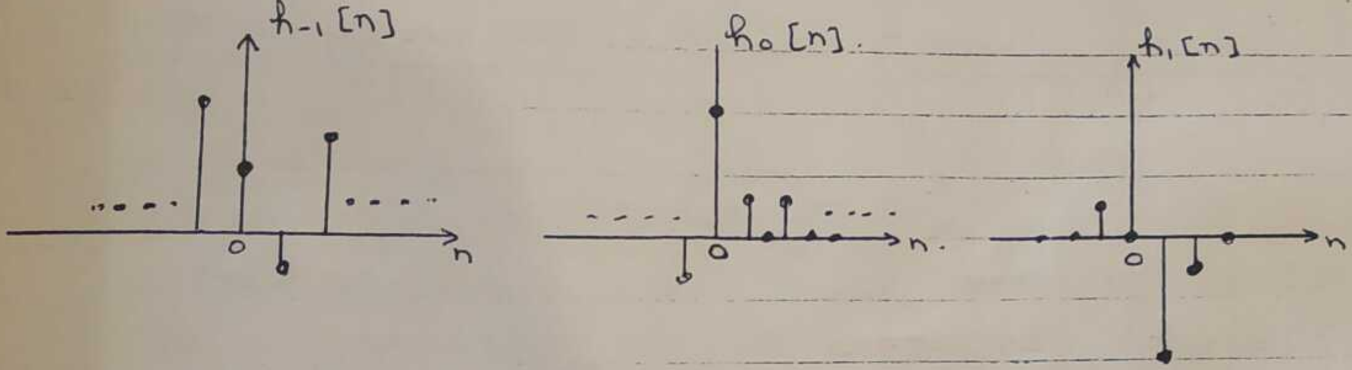
Let a signal $x[n]$ is applied as the input to a linear system whose responses are given by:

$h_{-1}[n]$ is the response of $\delta[n+1]$

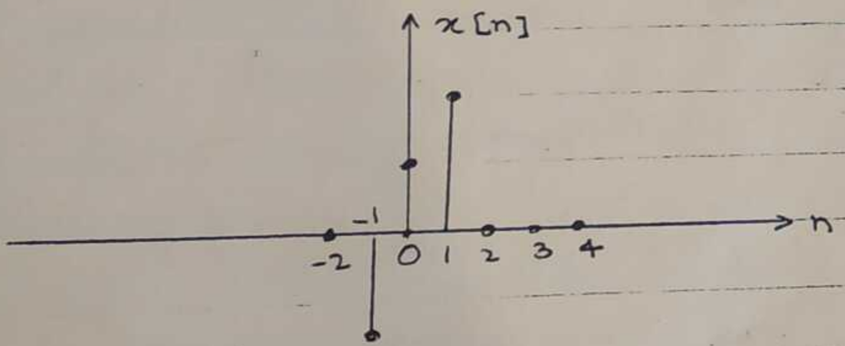
$h_0[n]$ is the response of $\delta[n]$

$h_{+1}[n]$ is the response of $\delta[n-1]$.

Graphically, these responses are given by:



Let, the signal $x[n]$ be given by:



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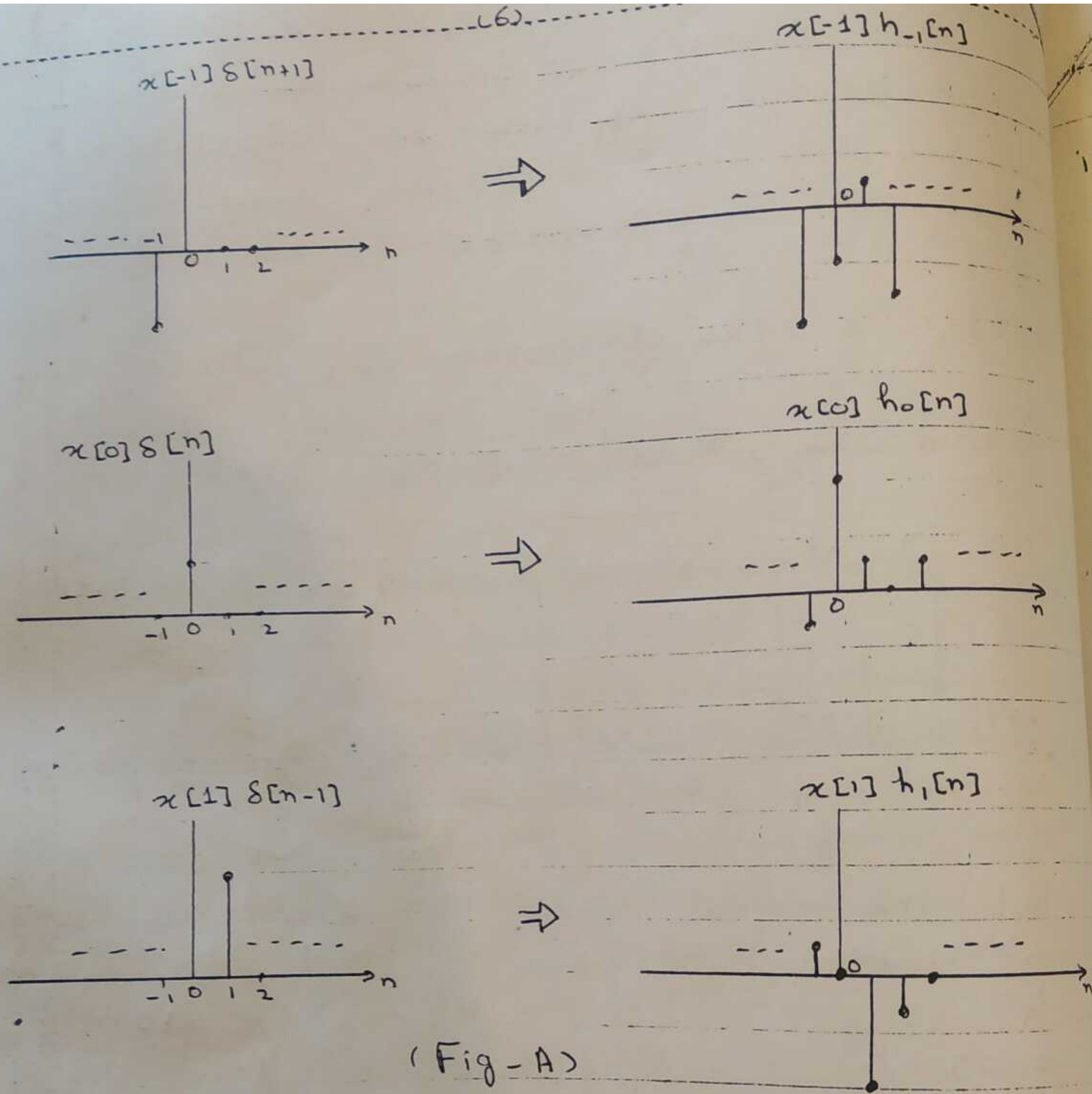
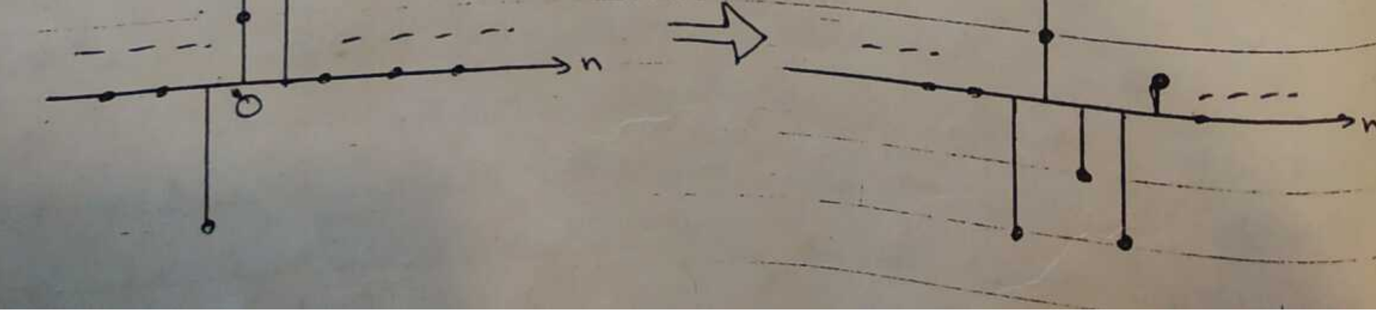
Now, the superposition property allows us to write the response to $x[n]$ as a linear combination of the responses to the individual shifted impulses.

The individual shifted and scaled impulses that constitute $x[n]$ are illustrated on the left hand side of fig (A), while the responses to these component signals are pictured on the right hand side.



(Fig - A)

The actual signal $x[n]$ is the sum of the components on the left hand side while the actual output $y[n]$, according to superposition, is the sum of the components on the right hand side.



If the linear system is also time invariant, these responses to time-shifted unit impulses are all time shifted versions of each other. Since, $\delta[n-k]$ is a time-shifted version of $\delta[n]$, the response $h_k[n]$ is a time-shifted version of $h_0[n]$ i.e.

$$h_k[n] = h_0[n-k].$$

For notational convenience ; $h[n] = h_0[n]$.

Hence, for an LTI system, eq (1) becomes:-

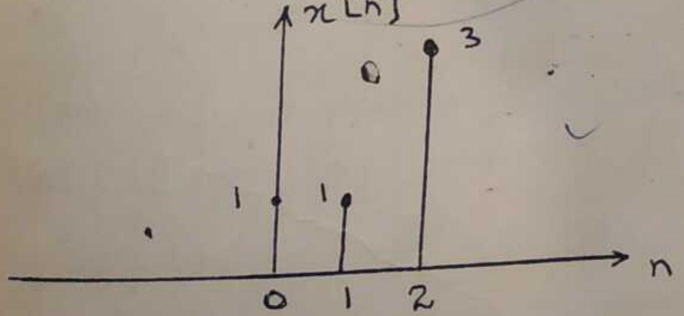
$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]}$$

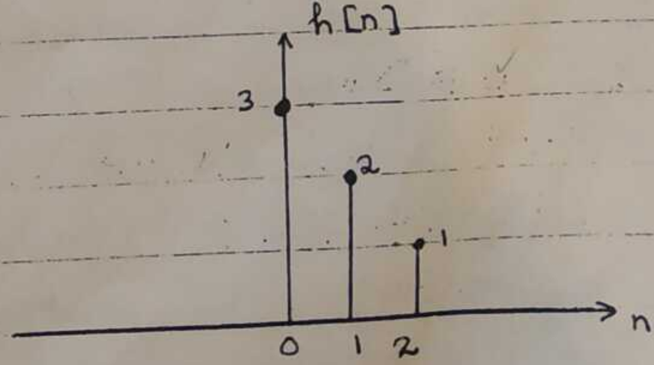
This result is known as the convolution sum or superposition sum & the operation on the right hand side is known as the convolution of the sequences $x[n]$ and $h[n]$. This can also be represented as:

$$\boxed{y[n] = x[n] * h[n].}$$

Example:-

Given that:-





Find $y[n]$ using convolution summation.

Solution:

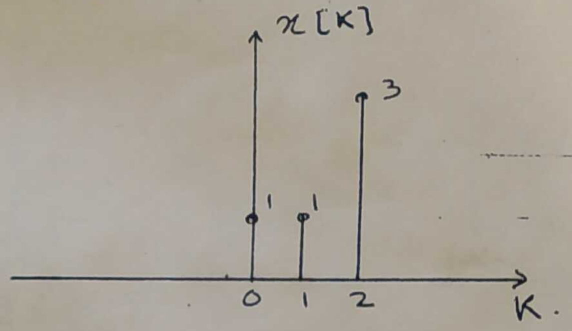
As we know that the formula for convolution summation is given by.

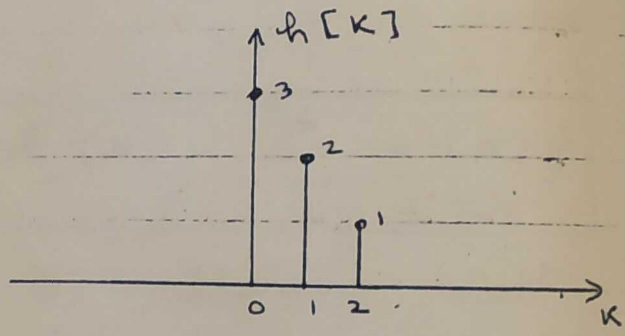
$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$

Step #1:

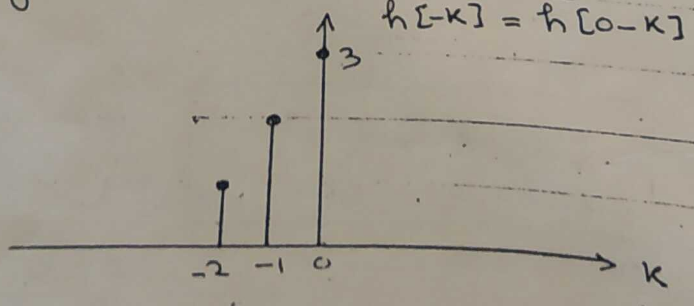
Replace "K" for "n" in the given signal & impulse response.





Step #2:

Reflect the signal (i.e impulse response) h[K] to get h[-K].



Step #3:

For the interval.

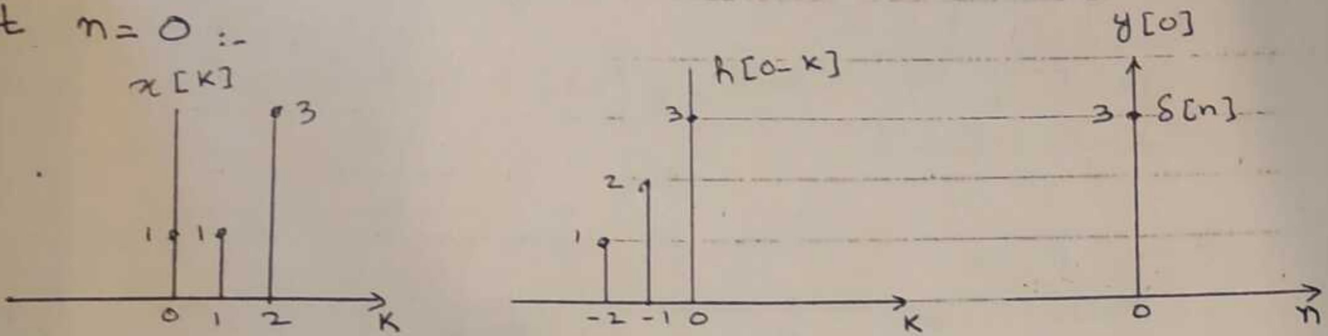
$-\infty < n < 0$

$h(n-K)$ is a value between $-\infty$ and 0. So when $h(n-K)$ is multiplied by $x[K]$, the output is zero. i.e

$$\boxed{y[n] = 0}$$

For $n \geq 0$.

At $n = 0$:-



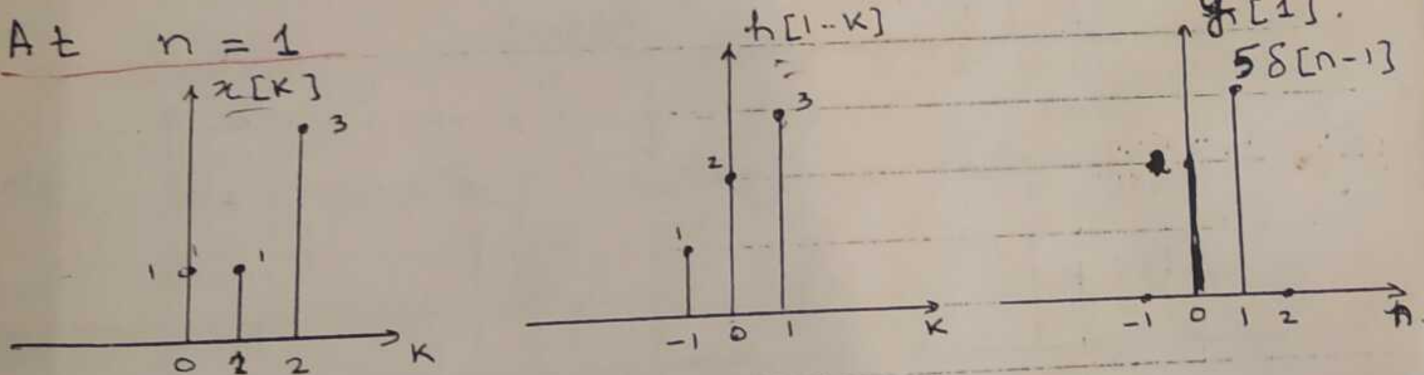
i.e

$$y[0] = \sum_{K=-\infty}^{\infty} x[K]\, h[0-K]$$

$$= (1)(3)$$

$\boxed{y[0] = 3}$ $\Rightarrow$ $y[0] = 3\,\delta[n] \longrightarrow (1)$

At $n = 1$



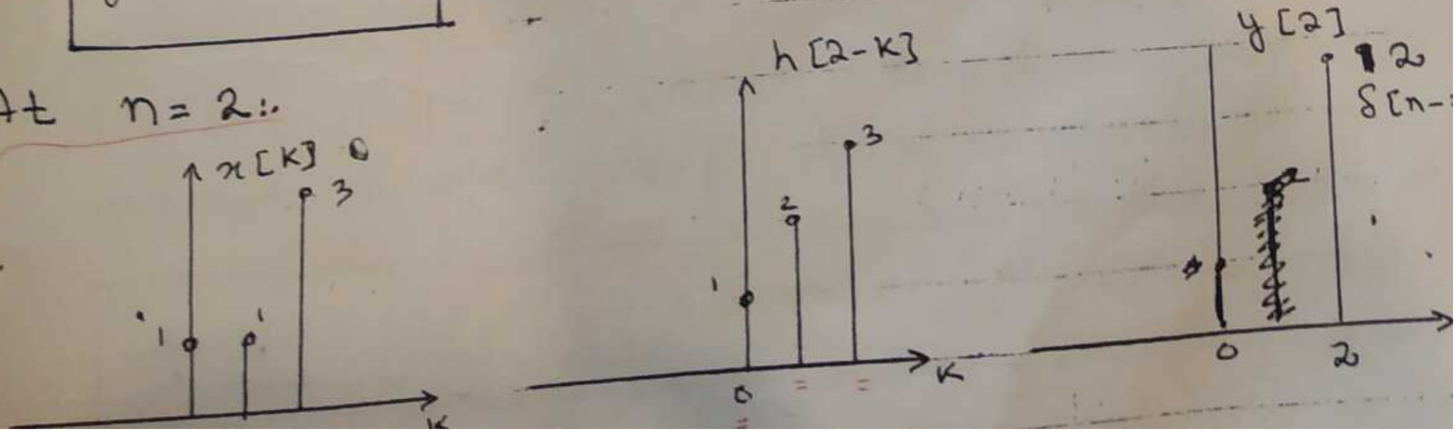
i.e

$$y[1] = \sum_{K=-\infty}^{\infty} x[K]\, h[1-K]$$

$$= (2)(1) + (1)(3)$$

$\boxed{y[1] = 5}$ $\Rightarrow$ $y[1] =$ [illegible] $+ 5\,\delta[n-1] \rightarrow$

At $n = 2$:

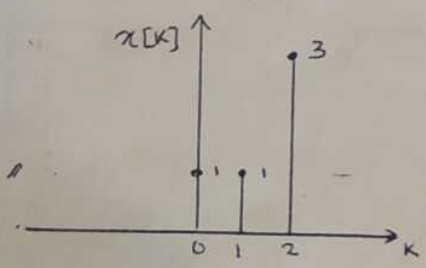


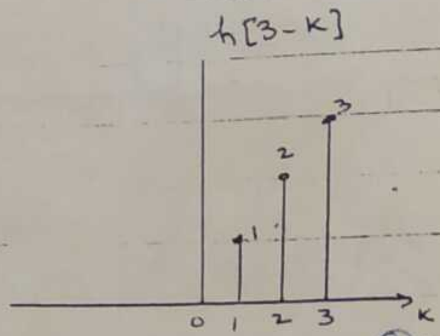
$$y[2] = \sum_{K=-\infty}^{\infty} x[K]\, h[2-K].$$

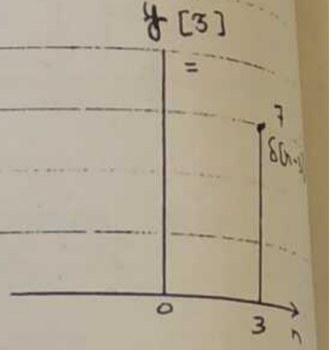
$$y[2] = 1 + 2 + 9$$

$\boxed{y[2] = 12}$ $\Rightarrow$ $y[2] = 12\,\delta[n-2].$

At $n = 3$:







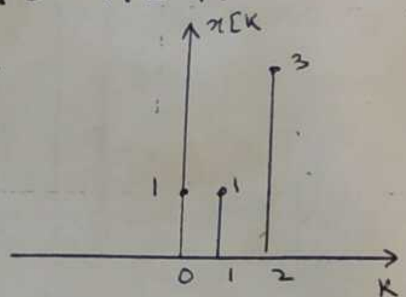
$0 \times 1 + 1 \times 1 + 2 \times 3 + 3 \times 0 = 6 + 1 = 7$

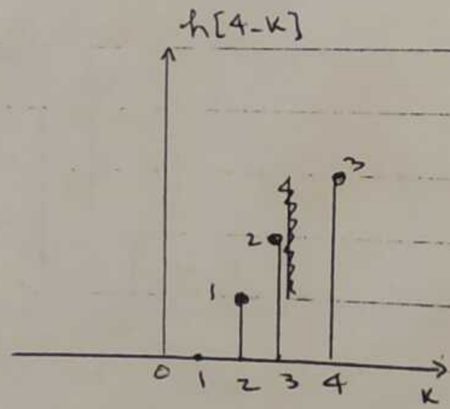
i.e

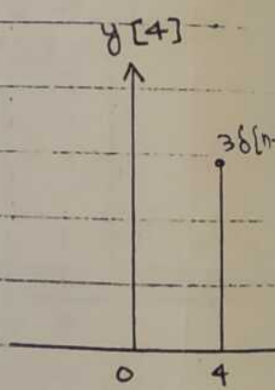
$$y[3] = (1)(1) + (3)(2) = 7$$

$$= 7\,\delta[n-3].$$

At $n = 4$:







$y[4] = 3\,\delta[n-4]$ $\quad 1 \times 0 + 1 \times 0 + 3 \times 1 +$

For $n > 4$

There is no overlaping of the signal $x[K]$ & $h[n-K]$. Hence.

$$y[n] = 0.$$

Overall out put $y[n]$

Overall output $y[n]$ can be written as:

$$\boxed{y[n] = 3\,\delta[n] + 5\,\delta[n-1] + 12\,\delta[n-2] + 7\,\delta[n-3] + 3\,\delta[n-4]}$$

Graphically: