Lecture # 11
Discrete Structure

Ordered Pair

- An ordered pair (a, b) consists of two elements a and b
- a is the first element and b is the second element
- Ordered pairs (a, b) and (c, d) are equal iff a = c and b = d
- (a, b) and (b, a) are not equal unless a = b

Example: Find x and y given that (2x, x + y) = (6, 2)

Solution: Ordered pairs are equal iff the corresponding components are equal. Hence, we obtain the equations:

$$2x = 6 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$x + y = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving equation (1) we get x = 3

Substituted x = 3 in equation (2) we get y = -1

Cartesian Product of Two Sets

- Let A and B be sets
- The Cartesian product of A and B, denoted $A \times B$ (read "A cross B")
- $A \times B$ is the set of all ordered pairs (a, b), where a is in set A and b is in set B
- Symbolically: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- If set A has *m* elements and set B has *n* elements then $A \times B$ has $m \times n$ elements i.e. $|A \times B| = |A| \times |B| = m \times n$
- Cartesian Product of Two non-empty and unequal sets A and B is not commutative: $A \times B \neq B \times A$
- $A \times \phi = \phi \times A = \phi$

Cartesian Product of Two Sets (Cont.)

Example: Let A = {1, 2}, B = {a, b, c} then

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2, b), (2, c)\}$

$B \times A = \{(a,1), (a,2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

$A \times A = \{(1, 1), (1,2), (2, 1), (2, 2)\}$

$B \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b),(c, c)\}$

Cartesian Product of More than Two Sets

- Cartesian product of sets A_1, A_2, …, A_n, denoted $A_1 \times A_2 \times \ldots \times A_n$
- It is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$
- $a_1 \in A_1$, $a_2 \in A_2$,…, $a_n \in A_{n.}$
- Symbolically:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{ for I} = 1, 2, \ldots, n\}$$

Cartesian Product of More than Two Sets (Cont.)

Example: Let A = {1, 2}, B = {a, b, c}, C = {x, y}, then

$A \times B \times C$ = {(1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1, c, y),
(2, a, x), (2, a, y), (2, b, x), (2, b, y), (2, c, x), (2, c, y) }

Also $(A \times B) \times C = \{(u, v) \mid u \in A \times B \text{ and } v \in C\}$

Now $A \times B$ = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}

and $(A \times B) \times C$ ={((1, a), x), ((1, a), y), ((1, b), x) ,((1, b), y),
((1, c), x), ((1, c), y), ((2, a), x),((2, a), y), ((2, b), x), ((2, b), y),
((2, c), x), ((2, c), y) }

Note that $(A \times B) \times C \neq (A \times (B \times C)$

Binary Relation

- Let A and B be sets
- A binary relation R from A to B is a subset of $A \times B$
- When $(a, b) \in R$, we write a R b, means a is related to b by R
- If $(a, b) \notin R$, we write $a \not R b$, means a is not related to b by R
- If $|A| = m$, $|B| = n$, and $|A \times B| = m \times n$ then the total number of relations from A to B are $2^{m \times n}$

Example: Let A = {1, 2}, B = {1, 2, 3}

Then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$

Let $R_1=\{(1,1), (1, 3), (2, 2)\}$ $R_2=\{(1, 2), (2, 1), (2, 2), (2, 3)\}$

$R_3=\{(1, 1)\}$ $R_4= A \times B$

$R_5= \varnothing$

All above relations are subsets of $A \times B$

Binary Relation (Cont.)

- Domain of a relation R from A to B is the set of all 1st elements of the ordered pairs which belong to R
- It is denoted by Dom(R)
- Symbolically: $\text{Dom}(R) = \{a \in A \mid (a, b) \in R\}$
- Range of A relation R from A to B is the set of all 2nd elements of the ordered pairs which belong to R
- It is denoted by Ran(R)
- Symbolically: $\text{Ran}(R) = \{b \in B \mid (a, b) \in R\}$
- Domain of a relation from A to B is a subset of A
- Range of a relation from A to B is a subset of B

Binary Relation (Cont.)

Example: Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$

Define a binary relation R from A to B as follows:

$$R = \{(a, b) \in A \times B \mid a < b\}$$

Then

a. Find the ordered pairs in R.

b. Find the Domain and Range of R.

c. Is 1R3, 2R2?

Solution: As $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$

a. $R = \{(a, b) \in A \times B \mid a < b\}$ or $R = \{(1, 2), (1, 3), (2, 3)\}$

b. $Dom(R) = \{1, 2\}$ and $Ran(R) = \{2, 3\}$

c. Since $(1, 3) \in R$ so 1R3 but $(2, 2) \notin R$ so |2R2

Binary Relation (Cont.)

Example: Find all binary relations from $A = \{0, 1\}$ to $B = \{1\}$

Solution: $A \times B = \{(0, 1), (1, 1)\}$

All binary relations from A to B are:

$R_1 = \varnothing$

$R_2 = \{(0, 1)\}$

$R_3 = \{(1, 1)\}$

$R_4 = \{(0, 1), (1, 1)\} = A \times B$

As $|A| = 2$ and $|B| = 1$ then $| A \times B | = 2 \times 1 = 2$ and the total number of relations from A to B are $2^{2 \times 1} = 4$

Relation on a Set

- A relation on a set A is a relation from A to A
- A relation on a set A is a subset of $A \times A$
- $A \times A$ is known as the universal relation
- $\varnothing$ is known as the empty relation

Example: Let A = {1, 2, 3, 4}

Define a relation R on A as

$(a, b) \in R$ iff a divides b {written as $a \mid b$}

Then R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}

Relation on a Set (Cont.)

Example: Define a binary relation E on the set of the integers Z, as follows: for all m, n $\in$ Z, mEn $\Leftrightarrow$ m – n is even

a. (i) Is 0E0? (ii) Is 5E2? (iii) Is (6, 6) $\in$ E? (iv) Is (-1, 7) $\in$ E?

b. Prove that for any even integer n, nE0.

Solution: E = {(m, n) $\in$ Z $\times$ Z | m – n is even}

a. (i) (0, 0) $\in$ Z $\times$ Z and 0 – 0 = 0 is even so 0E0

(ii) (5, 2) $\in$ Z $\times$ Z but 5 – 2 = 3 is not even so 5 $\not E$ 2

(iii) (6, 6) $\in$ E and 6 – 6 = 0 is an even integer

(iv) (-1,7) $\in$E and (-1) – 7 = -8 is an even integer

b. For any even integer, n, we have

n – 0 = n, an even integer

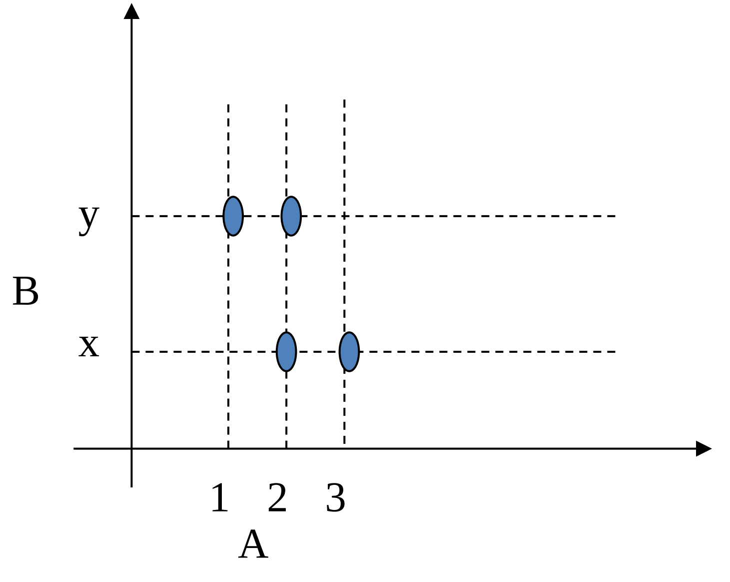
so (n, 0) $\in$ E or equivalently nE0

Graph of a Relation

- Let A = {1, 2, 3} and B = {x, y}
- Let R be a relation from A to B defined as

  R = {(1, y), (2, x), (2, y), (3, x)}

- The relation may be represented in a coordinate diagram as follows:



Graph of a Relation (Cont.)

Example: Draw the graph of the binary relation C from R to R defined as follows:

$$\text{for all } (x, y) \in R \times R, \quad (x, y) \in C \Leftrightarrow x^2 + y^2 = 1$$

Solution: All ordered pairs (x, y) in relation C satisfies the equation $x^2 + y^2 = 1$, which when solved for y gives

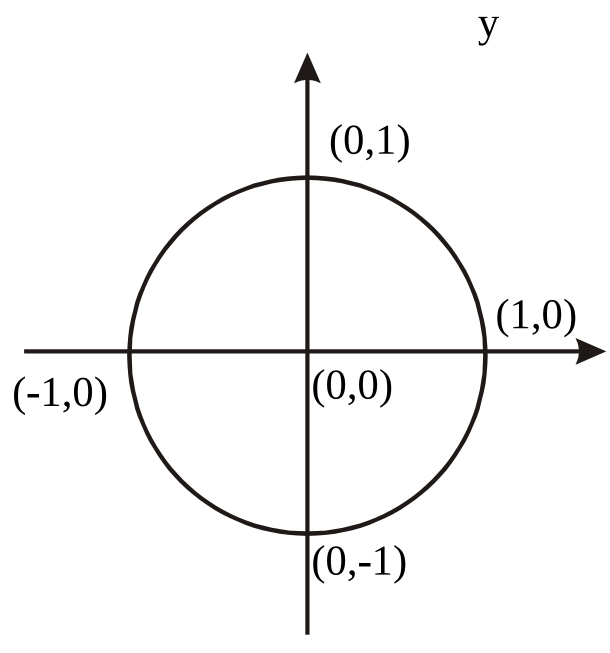
$$y = \pm\sqrt{1 - x^2}$$

Clearly y is real, whenever $-1 \leq x \leq 1$

Similarly x is real, whenever $-1 \leq y \leq 1$

Hence the graph is limited in the range $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

The graph of relation is a circle with center at (0,0) & radius 1.



Arrow Diagram of a Relation

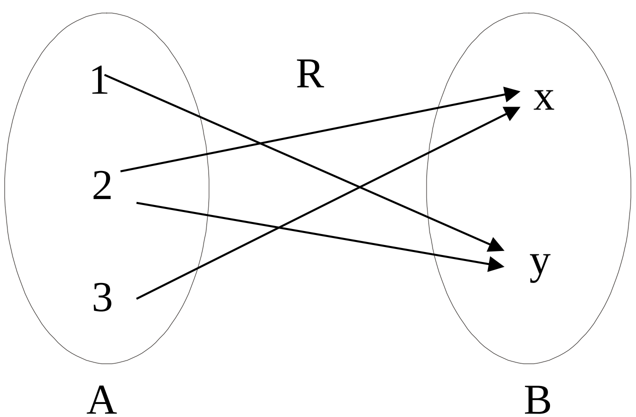
Example: Let A = {1, 2, 3} and B = {x, y}

Let R = {1, y), (2, x), (2, y), (3, x)} be a relation from A to B

The arrow diagram for above relation is given below.

We simply extend an arrow corresponding to each order pair in the relation R from the first element to the second.

For example we have an arrow from 1 to y because we have order pair (1, y) in R.



Directed Graph of a Relation

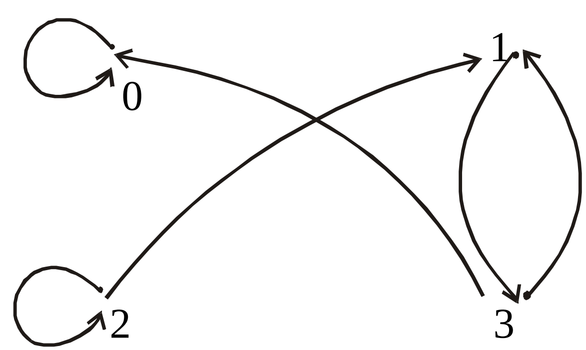
Example: Let A = {0, 1, 2, 3}

Let R = {(0, 0), (1, 3), (2, 1), (2, 2), (3, 0), (3, 1)} be a binary relation on A

The directed graph of R is obtained by representing points of A only once, and drawing an arrow from each point of A to each related point.

If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

The graph for above relation R is shown below.



Matrix Representation of a Relation

- Let A = {a_1, a_2, …, a_n} and B = {b_1, b_2, …, b_m}
- Let R be a relation from A to B
- Define the $n \times m$ order matrix M by

$$m(i, j) = \begin{cases} 1 \text{ if } (a_i, b_i) \in R \\ 0 \text{ if } (a_i, b_i) \notin R \end{cases}$$

for I = 1, 2,…, n and j = 1, 2,…, m

Matrix Representation of a Relation (Cont.)

Example: Let A = {1, 2, 3} and B = {x, y}

Let R ={(1, y), (2, x), (2, y), (3, x)} be a relation from A to B

As |A| = 3 elements and |B| = 2 elements, so it's a 3 ×2 matrix

Write the elements of A corresponding to the 3 rows and elements of B corresponding to the 2 columns of the matrix

If the *ith* element of A is related to *jth* element of B then place 1 on *ijth* position other wise place 0.

Hence we have the following matrix representation for above relation R.

$$M = \begin{matrix} & x & y \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{matrix}_{3\times 2}$$

Directed Graph and Matrix Representation

Example: For the given relation matrix M given below:

1. List the set of ordered pairs represented by M.
2. Draw the directed graph of the relation.

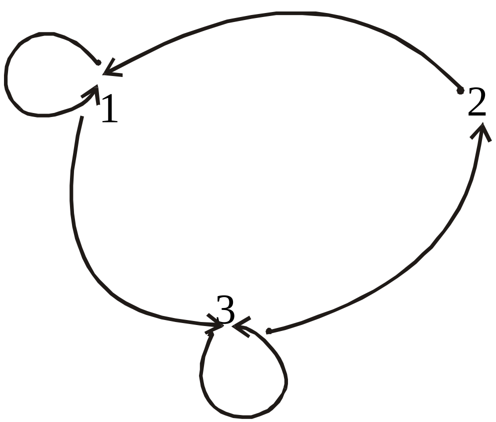
Solution:

1. The relation corresponding to the given Matrix is

   R = {(1, 1), (1, 3), (2, 1), (3, 2), (3, 3)}

2. The directed graph is given below.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$



Directed Graph and Matrix Representation (Cont.)

Example: Let A = {2, 3, 5, 6, 8}

The congruence modulo 3 relation T is defined on A as follows:

for all integers m, n $\in$A, m T n $\Leftrightarrow$ 3 | (m – n)

It means T is set ordered pairs which satisfies:

$m \equiv n \ (modulo \ \ a)$ $m - n$ is a multiple of a or a divides $m - n$

1. Write T as a set of ordered pairs.
2. The directed graph representation.
3. The matrix representation.

Directed Graph and Matrix Representation (Cont.)

Solution:

1. 2T2: 2 – 2 = 0, which is devisable by 3 i.e. 3|0

   2T5: 2 – 5 = -3, which is divisible by 3

   2T8: 2 – 8 = -6, which is divisible by 3

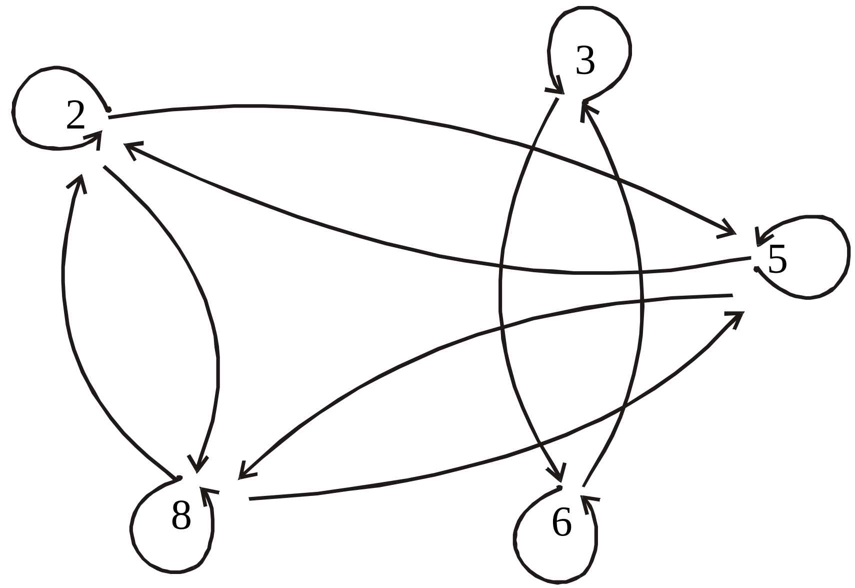
And similarly, 3T3, 3T6, 5T2, 5T5, 5T8, 6T3, 6T6, 8T2, 8T5 and 8T8

Hence

T = {(2, 2), (2, 5), (2, 8), (3, 3), (3, 6), (5, 2), (5, 5), (5, 8), (6, 3), (6, 6), (8, 2), (8, 5), (8, 8)}

Directed Graph and Matrix Representation (Cont.)

2. Directed graph:



3. Matrix representation:

$$
\begin{array}{c} \\ 2 \\ 3 \\ 5 \\ 6 \\ 8 \end{array}
\begin{array}{c} \begin{array}{ccccc} 2 & 3 & 5 & 6 & 8 \end{array} \\
\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{5\times 5} \end{array}
$$

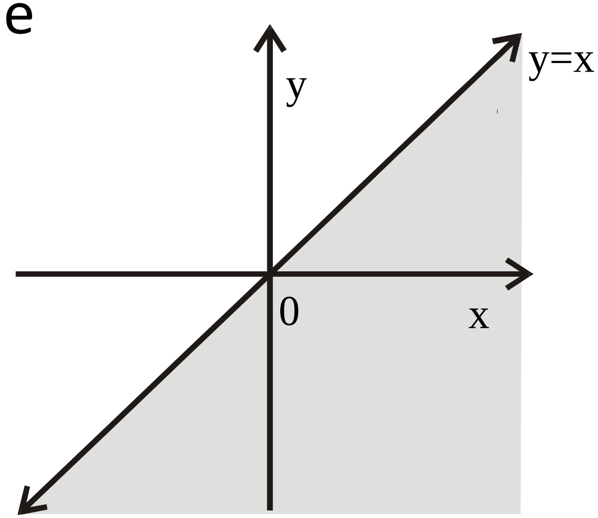
Relations and Graphs

Example: Define a binary relation S from R to R as follows:

for all $(x, y) \in R \times R$, $xSy \Leftrightarrow x \geq y$

a. Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is 2S3? Is (-1)S(-2)?

b. Draw the graph of S in the Cartesian plane



Solution:

a. $(2, 1) \in S$ as $2 \geq 1$

$(2, 2) \in S$ as $2 \geq 2$

2 S 3 as $2 \geq 3$

(-1)S(-2) as $-1 \geq -2$

b. The graph of this relation is given in figure

$S = \{(x, y) \in R \times R \mid \{x \geq y\}$

S consists of all points on and below the line $y = x$

Relations and Graphs (Cont.)

Example: Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$

Let define a relations R and S from A to B as follows:

for all $(x, y) \in A \times B$, $\quad x\ R\ y \Leftrightarrow x \mid y$

for all $(x, y) \in A \times B$, $\quad x\ S\ y \Leftrightarrow y - 4 = x$

State which ordered pairs are in $A \times B$, R, S, $R \cup S$ and $R \cap S$.

Solution:

$A \times B = \{(2, 6), (2, 8), (2, 10), (4, 6), (4, 8), (4, 10)\}$

$R = \{(2, 6), (2, 8), (2, 10), (4, 8)\}$

$S = \{(2, 6), (4, 8)\}$

$R \cup S = \{(2, 6), (2, 8), (2, 10), (4, 8)\} = R$

$R \cap S = \{(2, 6), (4, 8)\} = S$

Relations and Graphs (Cont.)

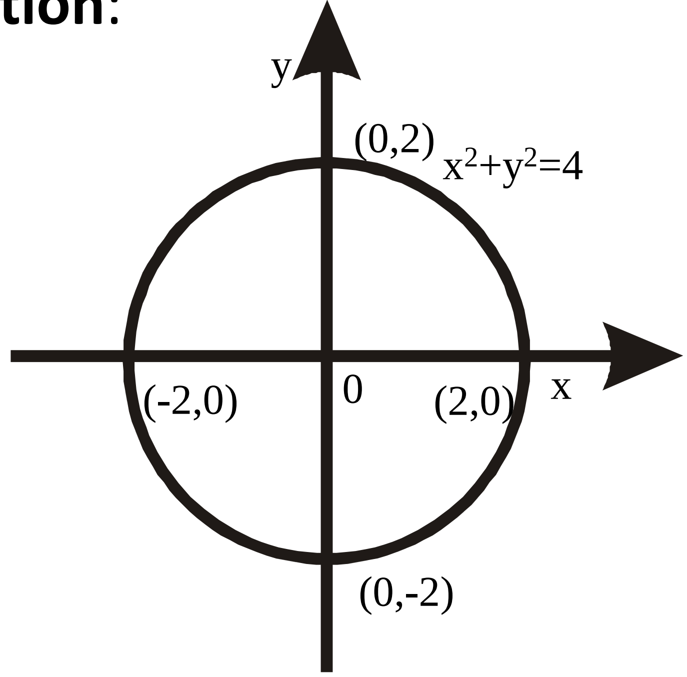
Example: Define binary relations R and S from R to R as follows:

$R = \{(x, y) \in R \times R \mid x^2 + y^2 = 4\}$and

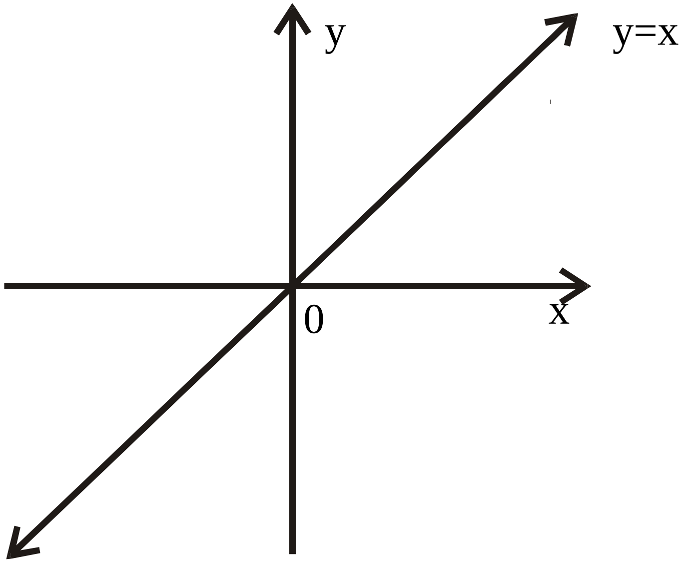
$S = \{(x, y) \in R \times R \mid x = y\}$

Graph R, S, $R \cup S$, and $R \cap S$ in Cartesian plane.

Solution:

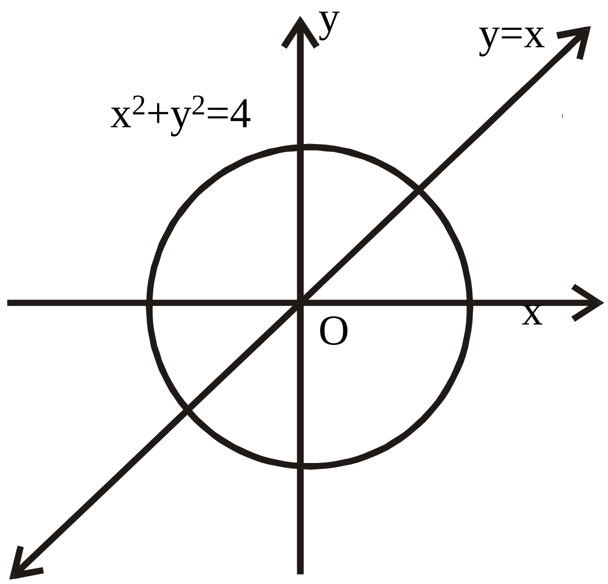


Graph of R

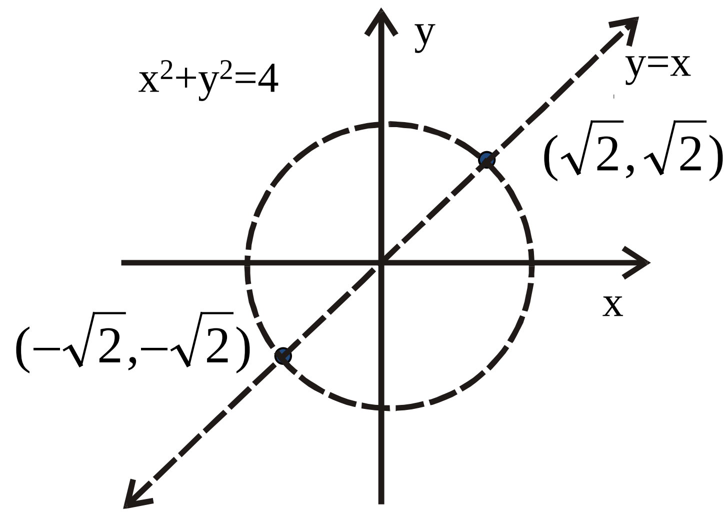


Graph of S

Relations and Graphs (Cont.)



Graph of R ∪ S



Graph of R ∩ S

The points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ are common to $x^2+y^2 = 4$ & $y = x$