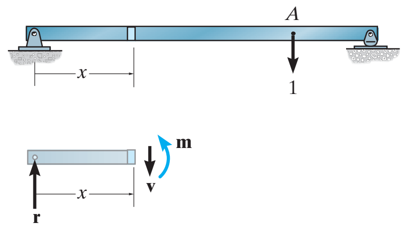
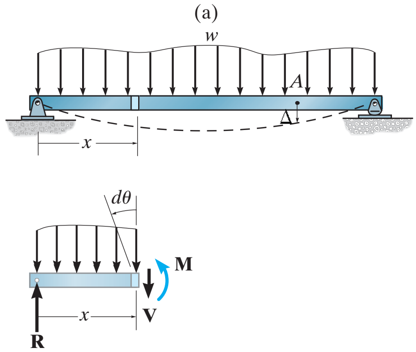


Apply virtual unit load to point A

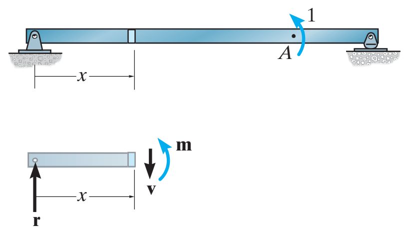
(a)



Apply real load w

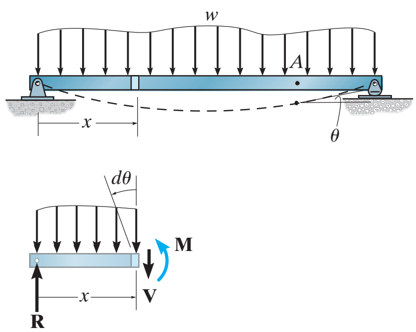
(b)

Fig. 9–14



Apply virtual unit couple moment to point A

(a)



Apply real load w

Fig. 9–15

9.7 Method of Virtual Work: Beams and Frames

The method of virtual work can also be applied to deflection problems involving beams and frames. Since strains due to *bending* are the *primary cause* of beam or frame deflections, we will discuss their effects first. Deflections due to shear, axial and torsional loadings, and temperature will be considered in Sec. 9–8.

The principle of virtual work, or more exactly, the method of virtual force, may be formulated for beam and frame deflections by considering the beam shown in Fig. 9–14*b*. Here the displacement Δ of point A is to be determined. To compute Δ a virtual unit load acting in the direction of Δ is placed on the beam at A, and the *internal virtual moment* **m** is determined by the method of sections at an arbitrary location x from the left support, Fig. 9–14*a*. When the real loads act on the beam, Fig. 9–14*b*, point A is displaced Δ. Provided these loads cause *linear elastic material response,* then from Eq. 8–2, the element dx deforms or rotates $d\theta = (M/EI)\,dx$.* Here M is the internal moment at x caused by the real loads. Consequently, the *external virtual work* done by the unit load is $1 \cdot \Delta$, and the *internal virtual work* done by the moment **m** is $m\,d\theta = m(M/EI)\,dx$. Summing the effects on all the elements dx along the beam requires an integration, and therefore Eq. 9–13 becomes

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI}\,dx \qquad (9\text{–}22)$$

where

1 = external virtual unit load acting on the beam or frame in the direction of Δ.
m = internal virtual moment in the beam or frame, expressed as a function of x and caused by the external virtual unit load.
Δ = external displacement of the point caused by the real loads acting on the beam or frame.
M = internal moment in the beam or frame, expressed as a function of x and caused by the real loads.
E = modulus of elasticity of the material.
I = moment of inertia of cross-sectional area, computed about the neutral axis.

In a similar manner, if the tangent rotation or slope angle θ at a point A on the beam's elastic curve is to be determined, Fig. 9–15, a unit couple moment is first applied at the point, and the corresponding internal moments m_θ have to be determined. Since the work of the unit couple is $1 \cdot \theta$, then

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI}\,dx \qquad (9\text{–}23)$$

*Recall that if the material is strained beyond its elastic limit, the principle of virtual work can still be applied, although in this case a nonlinear or plastic analysis must be used.

9

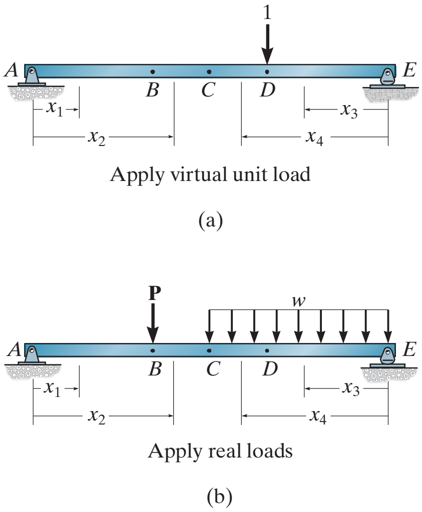


Fig. 9–16

When applying Eqs. 9–22 and 9–23, it is important to realize that the definite integrals on the right side actually represent the amount of virtual strain energy that is *stored* in the beam. If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration cannot be performed across the beam's entire length. Instead, separate x coordinates will have to be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each x have the same origin; however, the x selected for determining the real moment M in a particular region must be the *same* x as that selected for determining the virtual moment m or m_θ within the same region. For example, consider the beam shown in Fig. 9–16. In order to determine the displacement of D, four regions of the beam must be considered, and therefore four integrals having the form $\int (mM/EI)\,dx$ must be evaluated. We can use x_1 to determine the strain energy in region AB, x_2 for region BC, x_3 for region DE, and x_4 for region DC. In any case, each x coordinate should be selected so that both M and m (or m_θ) can be easily formulated.

Integration Using Tables. When the structure is subjected to a relatively simple loading, and yet the solution for a displacement requires several integrations, a *tabular method* may be used to perform these integrations. To do so the moment diagrams for each member are drawn first for both the real and virtual loadings. By matching these diagrams for m and M with those given in the table on the inside front cover, the integral $\int mM\,dx$ can be determined from the appropriate formula. Examples 9–8 and 9–10 illustrate the application of this method.

9

Procedure for Analysis

The following procedure may be used to determine the displacement and/or the slope at a point on the elastic curve of a beam or frame using the method of virtual work.

Virtual Moments m or m_θ

- Place a *unit load* on the beam or frame at the point and in the direction of the desired *displacement*.
- If the *slope* is to be determined, place a *unit couple moment* at the point.
- Establish appropriate x coordinates that are valid within regions of the beam or frame where there is no discontinuity of real or virtual load.
- With the virtual load in place, and all the real loads *removed* from the beam or frame, calculate the internal moment m or m_θ as a function of each x coordinate.
- Assume m or m_θ acts in the conventional positive direction as indicated in Fig. 4–1.

Real Moments

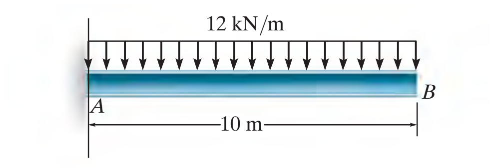
- Using the *same* x coordinates as those established for m or m_θ, determine the internal moments M caused only by the real loads.
- Since m or m_θ was assumed to act in the conventional positive direction, *it is important that positive M* acts in this *same* direction. This is necessary since positive or negative internal work depends upon the directional sense of load (defined by $\pm m$ or $\pm m_\theta$) and displacement (defined by $\pm M\,dx/EI$).

Virtual-Work Equation

- Apply the equation of virtual work to determine the desired displacement Δ or rotation θ. It is important to retain the algebraic sign of each integral calculated within its specified region.
- If the algebraic sum of all the integrals for the entire beam or frame is positive, Δ or θ is in the same direction as the virtual unit load or unit couple moment, respectively. If a negative value results, the direction of Δ or θ is opposite to that of the unit load or unit couple moment.

EXAMPLE 9.7

Determine the displacement of point B of the steel beam shown in Fig. 9–17*a*. Take $E = 200$ GPa, $I = 500(10^6)\ \text{mm}^4$.



(a)

SOLUTION

Virtual Moment *m*. The vertical displacement of point B is obtained by placing a virtual unit load of 1 kN at B, Fig. 9–17*b*. By inspection there are no discontinuities of loading on the beam for *both* the real and virtual loads. Thus, a *single* x coordinate can be used to determine the virtual strain energy. This coordinate will be selected with its origin at B, since then the reactions at A do not have to be determined in order to find the internal moments m and M. Using the method of sections, the internal moment m is formulated as shown in Fig. 9–17*b*.

Real Moment *M*. Using the *same* x coordinate, the internal moment M is formulated as shown in Fig. 9–17*c*.

Virtual-Work Equation. The vertical displacement of B is thus

$$1\ \text{kN} \cdot \Delta_B = \int_0^L \frac{mM}{EI}\,dx = \int_0^{10} \frac{(-1x)(-6x^2)\,dx}{EI}$$

$$1\ \text{kN} \cdot \Delta_B = \frac{15(10^3)\ \text{kN}^2 \cdot \text{m}^3}{EI}$$

or

$$\Delta_B = \frac{15(10^3)\ \text{kN} \cdot \text{m}^3}{200(10^6)\ \text{kN/m}^2(500(10^6)\ \text{mm}^4)(10^{-12}\ \text{m}^4/\text{mm}^4)}$$

$$= 0.150\ \text{m} = 150\ \text{mm} \qquad \textit{Ans.}$$

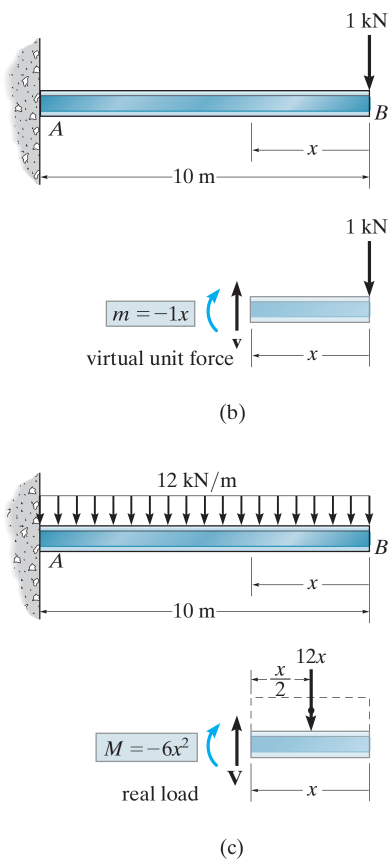


Fig. 9–17

EXAMPLE 9.8

Determine the slope θ at point B of the steel beam shown in Fig. 9–18a. Take $E = 200$ GPa, $I = 60(10^6)$ mm^4.

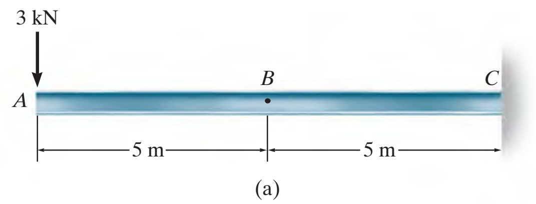


Fig. 9–18

SOLUTION

Virtual Moment m_θ. The slope at B is determined by placing a virtual unit couple moment of 1 kN · m at B, Fig. 9–18b. Here two x coordinates must be selected in order to determine the total virtual strain energy in the beam. Coordinate x_1 accounts for the strain energy within segment AB and coordinate x_2 accounts for that in segment BC. The internal moments m_θ within each of these segments are computed using the method of sections as shown in Fig. 9–18b.

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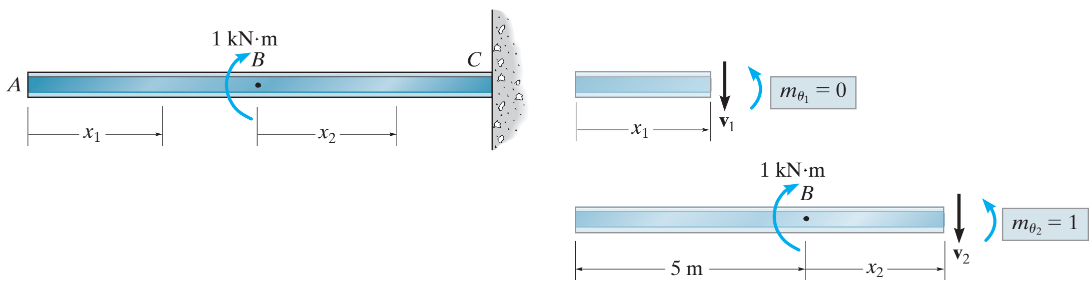
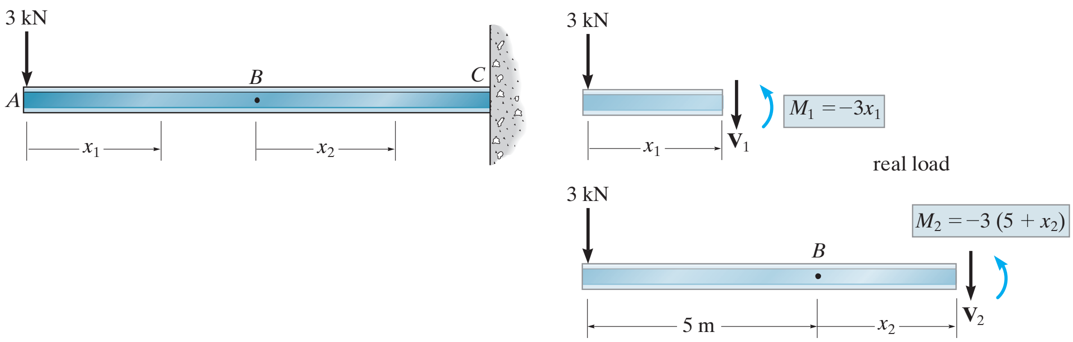


virtual unit couple
(b)



(c)

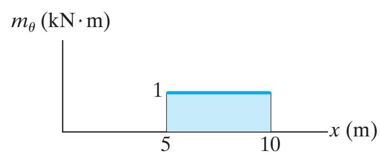
Real Moments *M*. Using the *same* coordinates x_1 and x_2, the internal moments M are computed as shown in Fig. 9–18*c*.

Virtual-Work Equation. The slope at B is thus given by

$$1 \cdot \theta_B = \int_0^L \frac{m_\theta M}{EI} dx$$

$$= \int_0^5 \frac{(0)(-3x_1)\,dx_1}{EI} + \int_0^5 \frac{(1)[-3(5 + x_2)]\,dx_2}{EI}$$

$$\theta_B = \frac{-112.5 \text{ kN} \cdot \text{m}^2}{EI} \qquad (1)$$



(d)

We can also evaluate the integrals $\int m_\theta M\,dx$ graphically, using the table given on the inside front cover of the book. To do so it is first necessary to draw the moment diagrams for the beams in Figs. 9–18*b* and 9–18*c*. These are shown in Figs. 9–18*d* and 9–18*e*, respectively. Since there is no moment m for $0 \le x < 5$ m, we use only the shaded rectangular and trapezoidal areas to evaluate the integral. Finding these shapes in the appropriate row and column of the table, we have

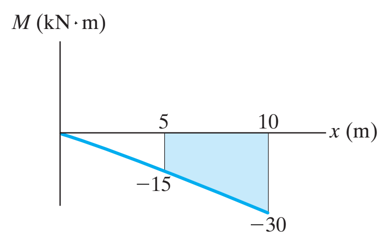
$$\int_5^{10} m_\theta M\,dx = \tfrac{1}{2} m_\theta (M_1 + M_2) L = \tfrac{1}{2}(1)(-15 - 30)5$$

$$= -112.5 \text{ kN}^2 \cdot \text{m}^3$$

This is the same value as that determined in Eq. 1. Thus,

$$(1 \text{ kN} \cdot \text{m}) \cdot \theta_B = \frac{-112.5 \text{ kN}^2 \cdot \text{m}^3}{200(10^6) \text{ kN/m}^2[60(10^6) \text{ mm}^4](10^{-12} \text{ m}^4/\text{mm}^4)}$$

$$\theta_B = -0.00938 \text{ rad} \qquad \textit{Ans.}$$



(e)

The *negative sign* indicates θ_B is *opposite* to the direction of the virtual couple moment shown in Fig. 9–18*b*.

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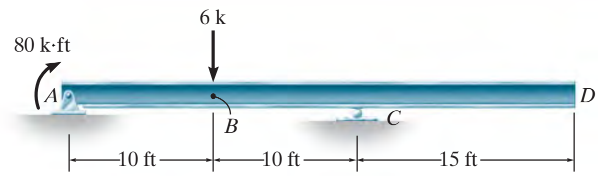
EXAMPLE 9.9

Determine the displacement at D of the steel beam in Fig. 9–19a. Take $E = 29(10^3)$ ksi, $I = 800$ in^4.

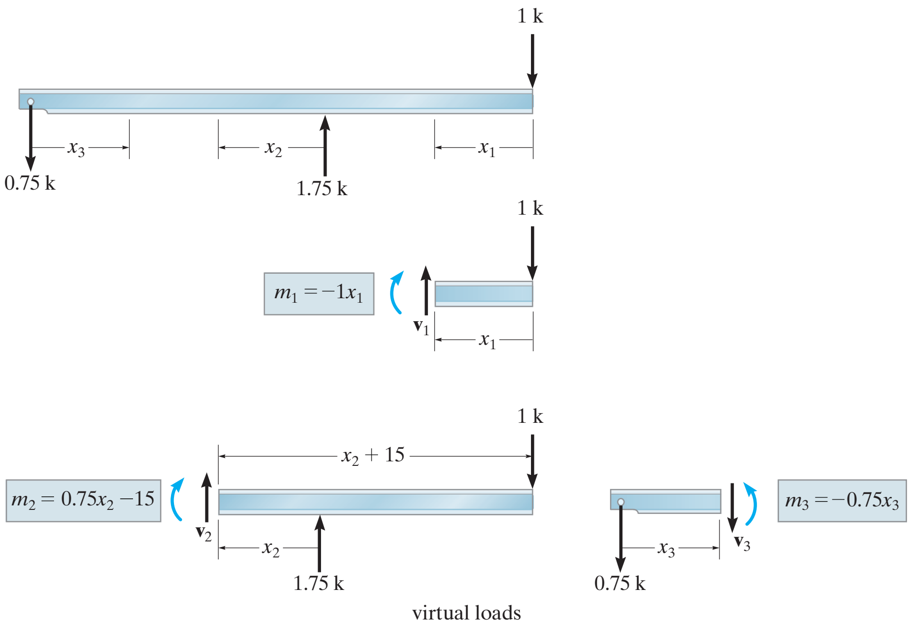


(a)

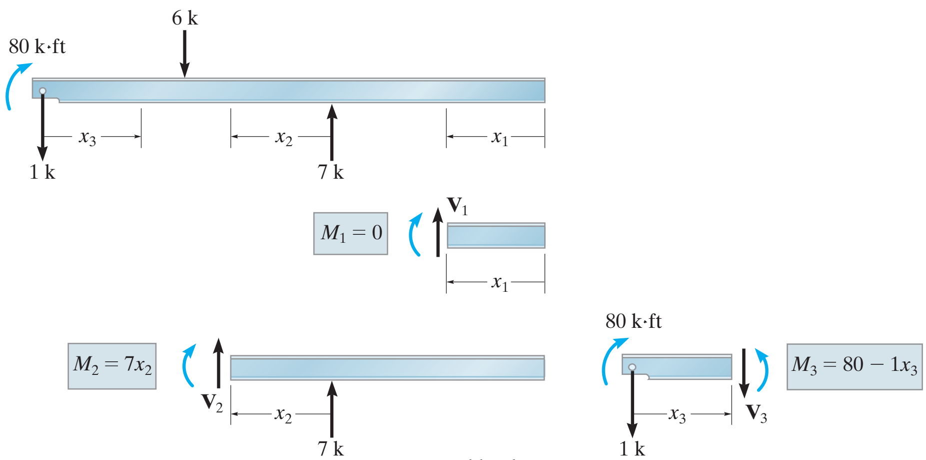
Fig. 9–19

SOLUTION

Virtual Moments *m*. The beam is subjected to a virtual unit load at D as shown in Fig. 9–19b. By inspection, *three coordinates,* such as x_1, x_2, and x_3, must be used to cover all the regions of the beam. Notice that these coordinates cover regions where no discontinuities in either real or virtual load occur. The internal moments m have been computed in Fig. 9–19b using the method of sections.



(b)



real loads
(c)

Real Moments *M*. The reactions on the beam are computed first; then, using the *same* x coordinates as those used for m, the internal moments M are determined as shown in Fig. 9–19*c*.

Virtual-Work Equation. Applying the equation of virtual work to the beam using the data in Figs. 9–19*b* and 9–19*c*, we have

$$1 \cdot \Delta_D = \int_0^L \frac{mM}{EI} dx$$

$$= \int_0^{15} \frac{(-1x_1)(0)\, dx_1}{EI} + \int_0^{10} \frac{(0.75x_2 - 15)(7x_2)\, dx_2}{EI}$$

$$+ \int_0^{10} \frac{(-0.75x_3)(80 - 1x_3)\, dx_3}{EI}$$

$$\Delta_D = \frac{0}{EI} - \frac{3500}{EI} - \frac{2750}{EI} = -\frac{6250 \text{ k} \cdot \text{ft}^3}{EI}$$

or

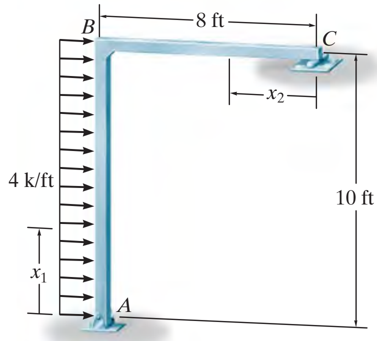
$$\Delta_D = \frac{-6250 \text{ k} \cdot \text{ft}^3 (12)^3 \text{ in}^3/\text{ft}^3}{29(10^3) \text{ k/in}^2 (800 \text{ in}^4)}$$

$$= -0.466 \text{ in.} \qquad \textit{Ans.}$$

The negative sign indicates the displacement is upward, opposite to the downward unit load, Fig. 9–19*b*. Also note that m_1 did not actually have to be calculated since $M_1 = 0$.

EXAMPLE 9.10

Determine the horizontal displacement of point C on the frame shown in Fig. 9–20*a*. Take $E = 29(10^3)$ ksi and $I = 600 \text{ in}^4$ for both members.

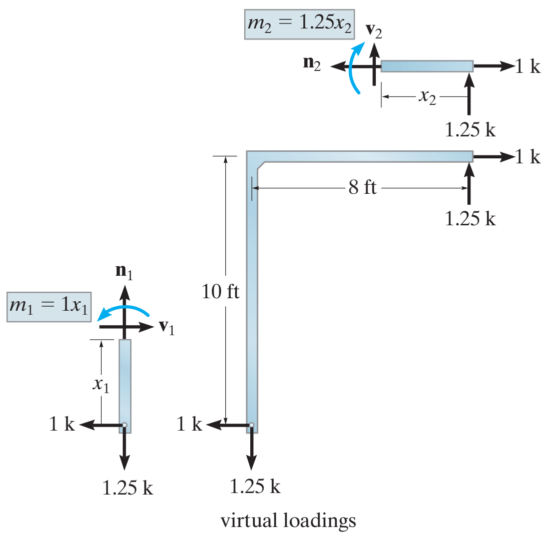


(a)

Fig. 9–20

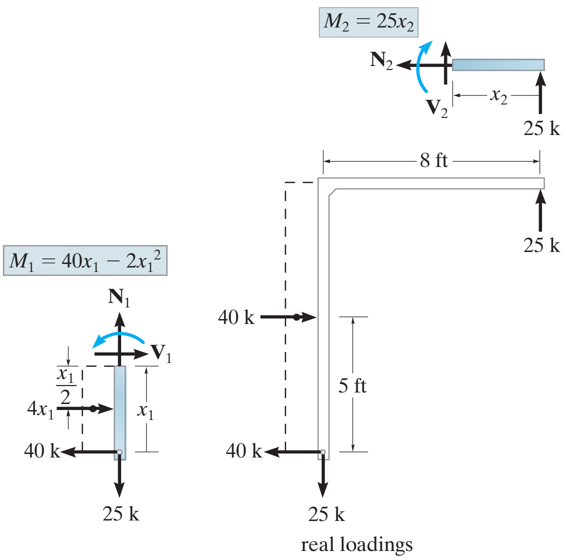
SOLUTION

Virtual Moments *m*. For convenience, the coordinates x_1 and x_2 in Fig. 9–20*a* will be used. A *horizontal* unit load is applied at C, Fig. 9–20*b*. Why? The support reactions and internal virtual moments are computed as shown.



(b)

9



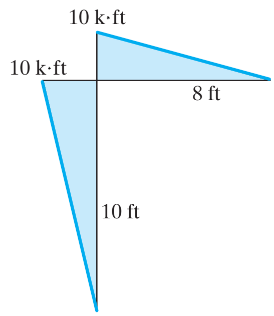
(c)

Real Moments *M*. In a similar manner the support reactions and real moments are computed as shown in Fig. 9–20*c*.

Virtual-Work Equation. Using the data in Figs. 9–20*b* and 9–20*c*, we have

$$1 \cdot \Delta_{C_h} = \int_0^L \frac{mM}{EI} dx = \int_0^{10} \frac{(1x_1)\left(40x_1 - 2x_1^2\right) dx_1}{EI} + \int_0^8 \frac{(1.25x_2)(25x_2)\, dx_2}{EI}$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13\,666.7 \text{ k}\cdot\text{ft}^3}{EI} \qquad (1)$$



(d)

If desired, the integrals $\int mM/dx$ can also be evaluated graphically using the table on the inside front cover. The moment diagrams for the frame in Figs. 9–20*b* and 9–20*c* are shown in Figs. 9–20*d* and 9–20*e*, respectively. Thus, using the formulas for similar shapes in the table yields

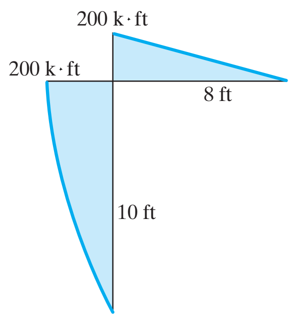
$$\int mM\, dx = \tfrac{5}{12}(10)(200)(10) + \tfrac{1}{3}(10)(200)(8)$$

$$= 8333.3 + 5333.3 = 13\,666.7 \text{ k}^2\cdot\text{ft}^3$$

This is the same as that calculated in Eq. 1. Thus

$$\Delta_{C_h} = \frac{13\,666.7 \text{ k}\cdot\text{ft}^3}{[29(10^3)\text{ k/in}^2((12)^2\text{ in}^2/\text{ft}^2)][600\text{ in}^4(\text{ft}^4/(12)^4\text{ in}^4)]}$$

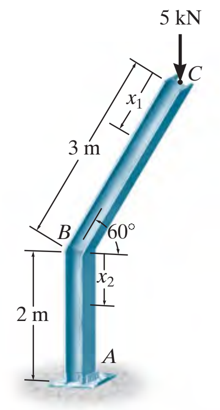
$$= 0.113 \text{ ft} = 1.36 \text{ in.}$$ *Ans.*



(e)

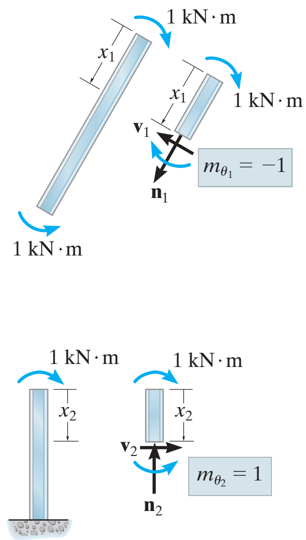
EXAMPLE 9.11

Determine the tangential rotation at point C of the frame shown in Fig. 9–21a. Take $E = 200$ GPa, $I = 15(10^6)$ mm^4.



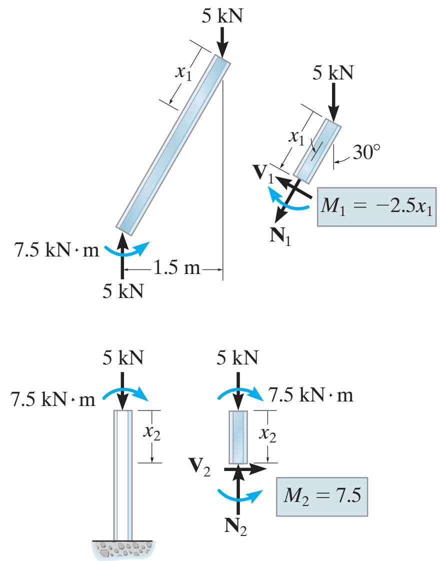
(a)

Fig. 9–21



virtual loads

(b)



real loads

(c)

SOLUTION

Virtual Moments m_θ. The coordinates x_1 and x_2 shown in Fig. 9–21a will be used. A unit couple moment is applied at C and the internal moments m_θ are calculated, Fig. 9–21b.

Real Moments M. In a similar manner, the real moments M are calculated as shown in Fig. 9–21c.

Virtual-Work Equation. Using the data in Figs. 9–21b and 9–21c, we have

$$1 \cdot \theta_C = \int_0^L \frac{m_\theta M}{EI} dx = \int_0^3 \frac{(-1)(-2.5x_1)\,dx_1}{EI} + \int_0^2 \frac{(1)(7.5)\,dx_2}{EI}$$

$$\theta_C = \frac{11.25}{EI} + \frac{15}{EI} = \frac{26.25 \text{ kN}\cdot\text{m}^2}{EI}$$

or

$$\theta_C = \frac{26.25 \text{ kN}\cdot\text{m}^2}{200(10^6)\text{ kN/m}^2[15(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$
$$= 0.00875 \text{ rad} \qquad \textit{Ans.}$$