Discrete Probability Distributions

Discrete Random Variable :=

A random variable 'X' is defined to be discrete if it can assume values which are finite or countably infinite. When "X" takes on a finite number of values, they may be listed as

$$x_1, x_2, \ldots, x_n$$

but in the countably infinite case, the values may be listed as

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$

Example :=

The number of heads obtained in coin tossing experiment, the number of defective items in a consignment, the number of fatal accidents, the number of bacteria in 1cc of water etc.

Let "X" be a discrete random variable, taking on distinct values $x_1, x_2, \ldots x_n, \ldots$ Then the function, denoted by $P(x)$ or $f(x)$ is

$$f(x_i) = P(X = x_i) \text{ for } i = 1, 2, 3, \ldots n$$

is called the probability function of random variable X.

Find the Probability distribution for the number of heads when 3 balanced coins are tossed.

Solution

The sample space for the above experiment is

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Let "x" be the random variable the denotes the number of heads.

Then $x = 0, 1, 2, 3$ and their probabilities are

$$f(0) = P(X=0) = P\{TTT\} = \frac{1}{8}$$

$$f(1) = P(X=1) = P\{HTT, THT, TTH\} = 3/8$$

$$f(2) = P(X=2) = P\{HHT, HTH, THH\} = 3/8$$

$$f(3) = P(x=3) = P\{HHH\} = \frac{1}{8}$$

No: of heads (x_i)	0	1	2	3
Probability $f(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Binomial Probability Distribution

Many experiments consist of repeated independent trials, each trial having two possible outcomes. For example the two possible outcome of a trial may be head and Tail, Success and Failure, right and wrong etc.

If the probability of each outcome remains the same throughout the Trials then such Trials are called **"Bernoulli trials"** and the experiment having "n" Bernoulli trials is called **"Binomial experiment"**

Formula :=

$P(X=x)$ $f(x) = {}^{n}C_{x}\, P^{x}\, q^{n-x}$ $x = 0,1,2,\dots,n$

where

n = The number of Sample size
x = The number of Success
P = The probability of Success
q = The probability of Failure

Note := $P+q = 1 \Rightarrow q = 1-P \Rightarrow P = 1-q$

Example:= A fair coin is tossed 5 times. Find the probability of obtaining various number of heads

Sol:=

→ Each trial has two possible outcomes, head and tail
→ The probability of head (Success) is $p = \frac{1}{2}$ and remains the same for successive tosses.
→ The Successive Tosses of the coin are independent
→ The coin is tossed 5 times

X denotes the number of heads (Successes)

$P = \frac{1}{2}$ $q = 1-P = 1-\frac{1}{2} = \frac{1}{2}$

$n = 5$

The possible values of $X = 0,1,2,3,4$ and 5

Hence

$$P(X=x) = {}^{n}C_{x}\, P^{x}\, q^{n-x}$$

$P(\text{no head}) = P(X=0) = {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0}$

$= {}^5C_0 (1) \left(\frac{1}{2}\right)^5$

$= \frac{5!}{0!(5-0)!} (1) \left(\frac{1}{2}\right)^5$

$= \frac{5!}{5!} (1) \left(\frac{1}{2}\right)^5$

$= (1) \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

$P(1\text{ head}) = P(X=1) = \binom{5}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{5-1}$

$= (5) \left(\frac{1}{2}\right)^5 = \frac{5}{32}$

$P(2\text{ heads}) = P(X=2) = \binom{5}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2}$

$= (10) \left(\frac{1}{2}\right)^5 = \frac{10}{32}$

$P(3\text{ heads}) = P(X=3) = \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3}$

$= (10) \left(\frac{1}{2}\right)^5 = \frac{10}{32}$

$P(4\text{ heads}) = P(X=4) = \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4}$

$= (5) \left(\frac{1}{2}\right)^5 = \frac{5}{32}$

$P(5\text{ heads}) = P(X=5) = \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0$

$(1) \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

The probabilities can also be obtained by

$$(a+b)^n = a^n + n a^{n-1} \cdot b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \cdots$$

where

$a = \frac{1}{2}$ $b = \frac{1}{2}$

Example := Let "X" have a bi-nomial distribution with $n=4$ and $p = \frac{1}{3}$

Find

$P(X=1)$, $P(X=\frac{3}{2})$, $P(X=3)$, $P(X=6)$ and $P(X \leq 2)$

Sol:=

The bi-nomial probability distribution for $n=4$ $p = \frac{1}{3}$

$p+q=1$ $q = 1-p$

$= 1-\frac{1}{3}$

$q = \frac{2}{3}$

$$f(x) = {}^nC_x \, p^x \, q^{n-x}$$

$${}^4C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x} \qquad x = 0,1,2,3,4$$

Now

$$P(X=1) = ({}^4C_1)\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{4-1}$$

$$= \frac{32}{81}$$

(P.T.O)

$P(X = \frac{3}{2}) = f(\frac{3}{2}) = 0$ because a random variable "X" with a binomial distribution takes only one of the integer values $0, 1, 2, \ldots, n$

$$P(X=3) = f(3) = \binom{4}{3}\left(\frac{1}{3}\right)^{3}\left(\frac{2}{3}\right)^{4-3}$$

$$= \frac{8}{81}$$

$P(X=6) = f(6) = 0$ because X can only take values $0, 1, 2, 3, 4$

and

$$P(X \leq 2) = \sum_{x=0}^{2} f(x) = f(0) + f(1) + f(2)$$

$$= \frac{16}{81} + \frac{32}{81} + \frac{24}{81}$$

$$= \frac{16+32+24}{81}$$

$$= \frac{72}{81} = \frac{8}{9}$$

where

$$f(0) = \binom{4}{0}\left(\frac{1}{3}\right)^{0}\left(\frac{2}{3}\right)^{4} = \frac{16}{81}$$

$$f(1) = \binom{4}{1}\left(\frac{1}{3}\right)^{1}\left(\frac{2}{3}\right)^{3} = \frac{32}{81}$$

$$f(2) = \binom{4}{2}\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{2} = \frac{24}{81}$$

Example:= A and B play a game in which A's probability of wining is $2/3$. In a series of 8 games, what is the probability that "A" will win

i) exactly 4 games

ii) at least 4 games

iii) 6 or more games

iv) from 3 to 6 games

Sol:= Given that $P = \frac{2}{3}$ $n = 8$

$$q = 1 - P$$
$$= 1 - \frac{2}{3}$$
$$q = \frac{1}{3}$$

Let "X" denotes the number of games won by A, Then

i) $P(X=4)$

$$= \binom{8}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^4$$

$$= \frac{1120}{6561}$$

$$= 0.1707$$

ii) $P(X \geq 4)$

$$1 - P(X < 4)$$

$$= 1 - \sum_{x=0}^{3}\binom{8}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{8-x}$$

$$= 1 - \left[\left(\frac{1}{3}\right)^8 + 8\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^7 + 28\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^6 + 56\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^5\right]$$

$$= 1 - \frac{1}{6561}\left[1 + 16 + 112 + 448\right]$$

$$= 1 - \frac{577}{6561}$$

$$= \frac{6561 - 577}{6561}$$

$$= \frac{5984}{6561}$$

$$= 0.9121$$

iii) $P(x \geq 6)$

$$\sum_{x=6}^{8} \binom{8}{x}\left(\frac{2}{3}\right)^{x}\left(\frac{1}{3}\right)^{8-x}$$

$$= \binom{8}{6}\left(\frac{2}{3}\right)^{6}\left(\frac{1}{3}\right)^{2} + \binom{8}{7}\left(\frac{2}{3}\right)^{7}\left(\frac{1}{3}\right) + \binom{8}{8}\left(\frac{2}{3}\right)^{8}\left(\frac{1}{3}\right)^{0}$$

$$= \frac{64}{6561}\left(28 + 16 + 4\right)$$

$$= \frac{64 \times 48}{6561}$$

$$= \frac{1024}{6561}$$

$$= 0.4682$$

iv) $P(3 \leq X \leq 6)$

$$\sum_{x=3}^{6} \binom{8}{x} \left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{8-x}$$

$$= \binom{8}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{5} + \binom{8}{4}\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{4} + \binom{8}{5}\left(\frac{2}{3}\right)^{5}\left(\frac{1}{3}\right)^{3} +$$

$$\binom{8}{6}\left(\frac{2}{3}\right)^{6}\left(\frac{1}{3}\right)^{2}$$

$$= \frac{8}{(3)^8}\left[56 + 140 + 224 + 224\right]$$

$$= \frac{8 \times 644}{6561} = \frac{5152}{6561} = 0.7852$$