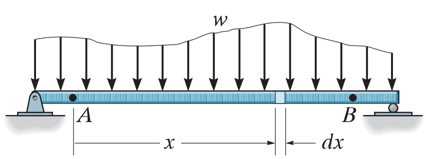
8.4 Moment-Area Theorems

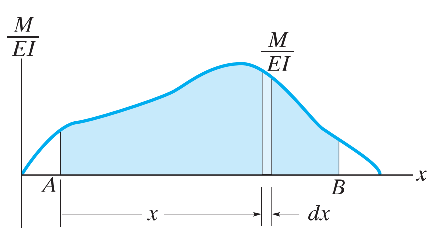
The initial ideas for the two moment-area theorems were developed by Otto Mohr and later stated formally by Charles E. Greene in 1873. These theorems provide a semigraphical technique for determining the slope of the elastic curve and its deflection due to bending. They are particularly advantageous when used to solve problems involving beams, especially those subjected to a series of concentrated loadings or having segments with different moments of inertia.

To develop the theorems, reference is made to the beam in Fig. 8–14*a*. If we draw the moment diagram for the beam and then divide it by the flexural rigidity, EI, the "M/EI diagram" shown in Fig. 8–14*b* results. By Eq. 8–2,

$$d\theta = \left(\frac{M}{EI}\right) dx$$



(a)



(b)

Thus it can be seen that the change $d\theta$ in the slope of the tangents on either side of the element dx is equal to the lighter-shaded *area* under the M/EI diagram. Integrating from point A on the elastic curve to point B, Fig. 8–14*c*, we have

$$\theta_{B/A} = \int_A^B \frac{M}{EI}\,dx \qquad (8\text{–}5)$$

This equation forms the basis for the first moment-area theorem.

Theorem 1: The change in slope between any two points on the elastic curve equals the area of the *M/EI* diagram between these two points.

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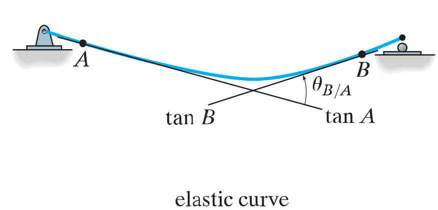
The notation $\theta_{B/A}$ is referred to as the angle of the tangent at B measured with respect to the tangent at A. From the proof it should be evident that this angle is measured *counterclockwise* from tangent A to tangent B if the area of the M/EI diagram is *positive,* Fig. 8–14*c*. Conversely, if this area is *negative,* or below the x axis, the angle $\theta_{B/A}$ is measured *clockwise* from tangent A to tangent B. Furthermore, from the dimensions of Eq. 8–5, $\theta_{B/A}$ is measured in radians.



(c)

Fig. 8–14

The second moment-area theorem is based on the relative deviation of *tangents* to the elastic curve. Shown in Fig. 8–15*c* is a greatly exaggerated view of the *vertical deviation* dt of the tangents on each side of the differential element dx. This deviation is measured along a vertical line passing through point A. Since the slope of the elastic curve and its deflection are assumed to be very small, it is satisfactory to approximate the length of each tangent line by x and the arc ds' by dt. Using the circular-arc formula $s = \theta r$, where r is of length x, we can write $dt = x\,d\theta$. Using Eq. 8–2, $d\theta = (M/EI)\,dx$, the vertical deviation of the tangent at *A with respect to the tangent* at B can be found by integration, in which case

$$t_{A/B} = \int_A^B x \frac{M}{EI}\,dx \qquad (8\text{–}6)$$

Recall from statics that the centroid of an area is determined from $\bar{x}\int dA = \int x\,dA$. Since $\int M/EI\,dx$ represents an area of the M/EI diagram, we can also write

$$t_{A/B} = \bar{x}\int_A^B \frac{M}{EI}\,dx \qquad (8\text{–}7)$$

Here $\bar{x}$ is the distance from the vertical axis through A to the *centroid* of the area between A and B, Fig. 8–15*b*.

The second moment-area theorem can now be stated as follows:

Theorem 2: The vertical deviation of the tangent at a point (A) on the elastic curve with respect to the tangent extended from another point (B) equals the "moment" of the area under the M/EI diagram between the two points (A and B). This moment is computed about point A (the point on the elastic curve), where the deviation $t_{A/B}$ is to be determined.

Provided the moment of a *positive M/EI* area from A to B is computed, as in Fig. 8–15*b*, it indicates that the tangent at point A is *above* the tangent to the curve extended from point B, Fig. 8–15*c*. Similarly, *negative M/EI* areas indicate that the tangent at A is *below* the tangent extended from B. Note that in general $t_{A/B}$ is not equal to $t_{B/A}$, which is shown in Fig. 8–15*d*. Specifically, the moment of the area under the M/EI diagram between A and B is computed about point A to determine $t_{A/B}$, Fig. 8–15*b*, and it is computed about point B to determine $t_{B/A}$.

It is important to realize that the moment-area theorems can only be used to determine the angles or deviations between two tangents on the beam's elastic curve. In general, they *do not* give a direct solution for the slope or displacement at a point on the beam. These unknowns must first be related to the angles or vertical deviations of tangents at points on the elastic curve. Usually the tangents at the supports are drawn in this regard since these points do not undergo displacement and/or have zero slope. Specific cases for establishing these geometric relationships are given in the example problems.

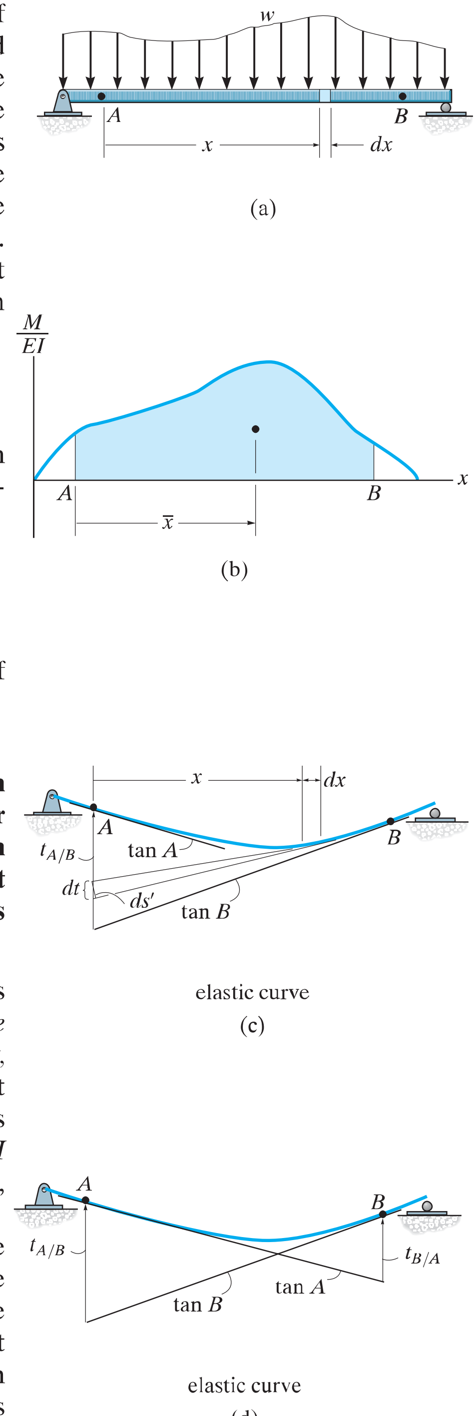


Fig. 8–15

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the moment-area theorems.

M/EI Diagram

- Determine the support reactions and draw the beam's *M/EI* diagram.
- If the beam is loaded with concentrated forces, the *M/EI* diagram will consist of a series of straight line segments, and the areas and their moments required for the moment-area theorems will be relatively easy to compute.
- If the loading consists of a *series* of concentrated forces and distributed loads, it may be simpler to compute the required *M/EI* areas and their moments by drawing the *M/EI* diagram in parts, using the method of superposition as discussed in Sec. 4–5. In any case, the *M/EI* diagram will consist of parabolic or perhaps higher-order curves, and it is suggested that the table on the inside back cover be used to locate the area and centroid under each curve.

Elastic Curve

- Draw an exaggerated view of the beam's elastic curve. Recall that points of zero slope occur at fixed supports and zero displacement occurs at all fixed, pin, and roller supports.
- If it becomes difficult to draw the general shape of the elastic curve, use the moment (or *M/EI*) diagram. Realize that when the beam is subjected to a *positive moment* the beam bends *concave up,* whereas *negative moment* bends the beam *concave down*. Furthermore, an inflection point or change in curvature occurs where the moment in the beam (or *M/EI*) is zero.
- The displacement and slope to be determined should be indicated on the curve. Since the moment-area theorems apply only between two tangents, attention should be given as to which tangents should be constructed so that the angles or deviations between them will lead to the solution of the problem. In this regard, *the tangents at the points of unknown slope and displacement and at the supports should be considered,* since the beam usually has zero displacement and/or zero slope at the supports.

Moment-Area Theorems

- Apply Theorem 1 to determine the angle between two tangents, and Theorem 2 to determine vertical deviations between these tangents.
- Realize that Theorem 2 in general *will not* yield the displacement of a point on the elastic curve. When applied properly, it will only give the vertical distance or deviation of a tangent at point A on the elastic curve from the tangent at B.
- After applying either Theorem 1 or Theorem 2, the algebraic sign of the answer can be verified from the angle or deviation as indicated on the elastic curve.

EXAMPLE 8.6

Determine the slope at points B and C of the beam shown in Fig. 8–16*a*. Take $E = 29(10^3)$ ksi and $I = 600\,\text{in}^4$.

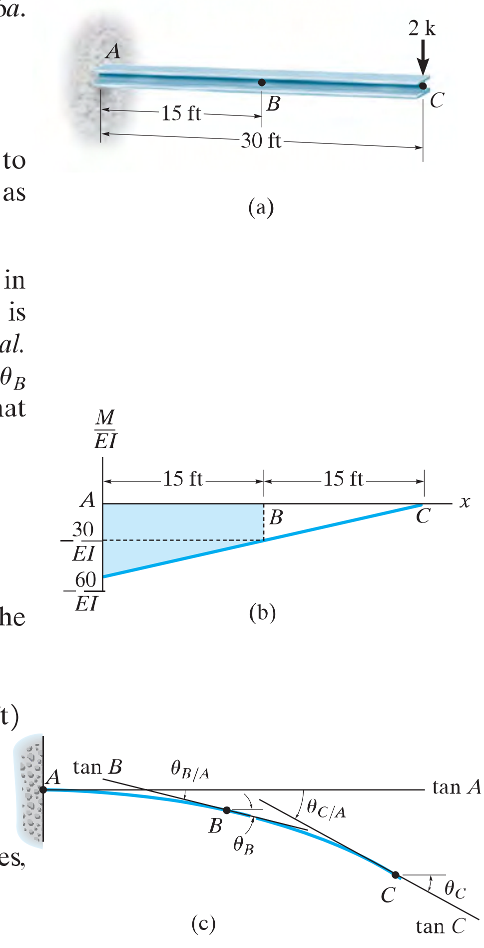


Fig. 8–16

SOLUTION

***M/EI* Diagram.** This diagram is shown in Fig. 8–16*b*. It is easier to solve the problem in terms of EI and substitute the numerical data as a last step.

Elastic Curve. The 2-k load causes the beam to deflect as shown in Fig. 8–16*c*. (The beam is deflected concave down, since M/EI is negative.) Here the tangent at A (the support) is *always horizontal.* The tangents at B and C are also indicated. We are required to find θ_B and θ_C. By the construction, the angle between tan A and tan B, that is, $\theta_{B/A}$, is equivalent to θ_B.

$$\theta_B = \theta_{B/A}$$

Also,

$$\theta_C = \theta_{C/A}$$

Moment-Area Theorem. Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between points A and B; that is,

$$\theta_B = \theta_{B/A} = -\left(\frac{30\text{ k}\cdot\text{ft}}{EI}\right)(15\text{ ft}) - \frac{1}{2}\left(\frac{60\text{ k}\cdot\text{ft}}{EI} - \frac{30\text{ k}\cdot\text{ft}}{EI}\right)(15\text{ ft})$$

$$= -\frac{675\text{ k}\cdot\text{ft}^2}{EI}$$

Substituting numerical data for E and I, and converting feet to inches, we have

$$\theta_B = \frac{-675\text{ k}\cdot\text{ft}^2(144\text{ in}^2/1\text{ ft}^2)}{29(10^3)\text{ k/in}^2(600\text{ in}^4)}$$

$$= -0.00559\text{ rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle is measured clockwise from A, Fig. 8–16*c*.

In a similar manner, the area under the M/EI diagram between points A and C equals $\theta_{C/A}$. We have

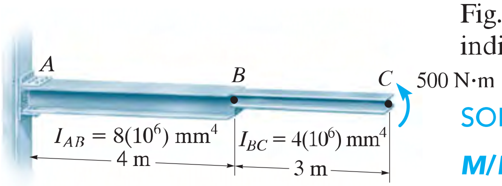
$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{60\text{ k}\cdot\text{ft}}{EI}\right)(30\text{ ft}) = -\frac{900\text{ k}\cdot\text{ft}^2}{EI}$$

Substituting numerical values for EI, we have

$$\theta_C = \frac{-900\text{ k}\cdot\text{ft}^2(144\text{ in}^2/\text{ft}^2)}{29(10^3)\text{ k/in}^2(600\text{ in}^4)}$$

$$= -0.00745\text{ rad} \qquad \textit{Ans.}$$

EXAMPLE 8.7

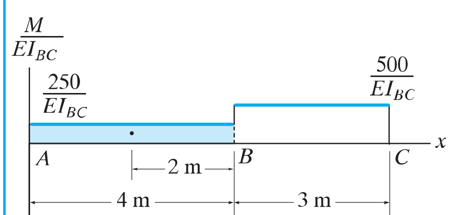


(a)

Determine the deflection at points B and C of the beam shown in Fig. 8–17a. Values for the moment of inertia of each segment are indicated in the figure. Take $E = 200$ GPa.

SOLUTION

***M/EI* Diagram.** By inspection, the moment diagram for the beam is a rectangle. Here we will construct the M/EI diagram relative to I_{BC}, realizing that $I_{AB} = 2I_{BC}$. Fig. 8–17b. Numerical data for EI_{BC} will be substituted as a last step.



(b)

Elastic Curve. The couple moment at C causes the beam to deflect as shown in Fig. 8–17c. The tangents at A (the support), B, and C are indicated. We are required to find Δ_B and Δ_C. These displacements can be related *directly* to the deviations between the tangents, so that from the construction Δ_B is equal to the deviation of tan B relative to tan A; that is,

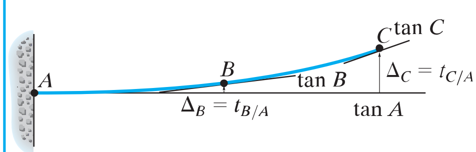
$$\Delta_B = t_{B/A}$$

Also,

$$\Delta_C = t_{C/A}$$

Moment-Area Theorem. Applying Theorem 2, $t_{B/A}$ is equal to the moment of the area under the M/EI_{BC} diagram between A and B computed about point B, since this is the point where the tangential deviation is to be determined. Hence, from Fig. 8–17b,

$$\Delta_B = t_{B/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](2\ \text{m}) = \frac{2000\ \text{N}\cdot\text{m}^3}{EI_{BC}}$$



(c)

Fig. 8–17

Substituting the numerical data yields

$$\Delta_B = \frac{2000\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)\ \text{mm}^4(1\ \text{m}^4/(10^3)^4\ \text{mm}^4)]}$$
$$= 0.0025\ \text{m} = 2.5\ \text{mm.} \qquad \textit{Ans.}$$

Likewise, for $t_{C/A}$ we must compute the moment of the entire M/EI_{BC} diagram from A to C about point C. We have

$$\Delta_C = t_{C/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](5\ \text{m}) + \left[\frac{500\ \text{N}\cdot\text{m}}{EI_{BC}}(3\ \text{m})\right](1.5\ \text{m})$$
$$= \frac{7250\ \text{N}\cdot\text{m}^3}{EI_{BC}} = \frac{7250\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)(10^{-12})\ \text{m}^4]}$$
$$= 0.00906\ \text{m} = 9.06\ \text{mm} \qquad \textit{Ans.}$$

Since both answers are *positive,* they indicate that points B and C lie *above* the tangent at A.

8

EXAMPLE 8.8

Determine the slope at point C of the beam in Fig. 8–18*a*. $E = 200$ GPa, $I = 6(10^6)$ mm^4.

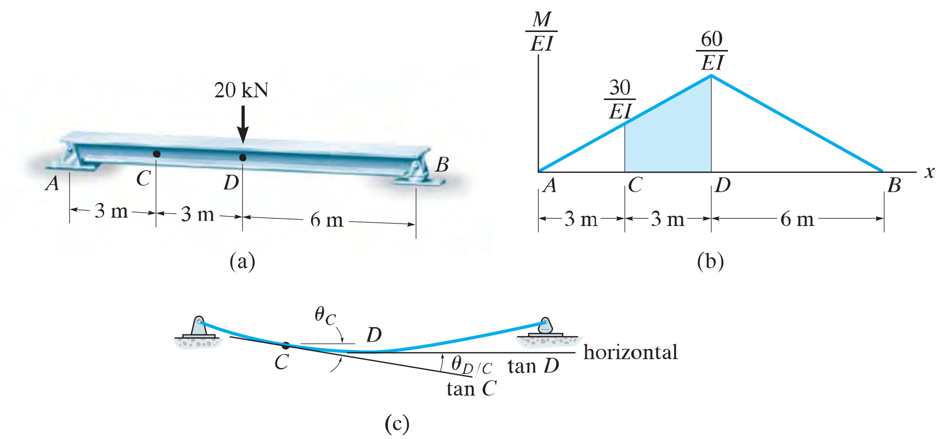


Fig. 8–18

SOLUTION

***M/EI* Diagram.** Fig. 8–18*b*.

Elastic Curve. Since the loading is applied symmetrically to the beam, the elastic curve is symmetric, as shown in Fig. 8–18*c*. We are required to find θ_C. This can easily be done, realizing that the tangent at D is *horizontal*, and therefore, by the construction, the angle $\theta_{D/C}$ between tan C and tan D is equal to θ_C; that is,

$$\theta_C = \theta_{D/C}$$

Moment-Area Theorem. Using Theorem 1, $\theta_{D/C}$ is equal to the shaded area under the M/EI diagram between points C and D. We have

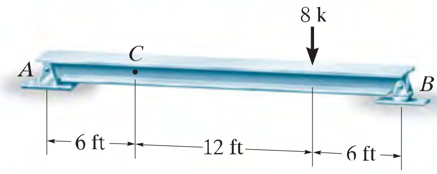
$$\theta_C = \theta_{D/C} = 3\text{ m}\left(\frac{30\text{ kN}\cdot\text{m}}{EI}\right) + \frac{1}{2}(3\text{ m})\left(\frac{60\text{ kN}\cdot\text{m}}{EI} - \frac{30\text{ kN}\cdot\text{m}}{EI}\right)$$

$$= \frac{135\text{ kN}\cdot\text{m}^2}{EI}$$

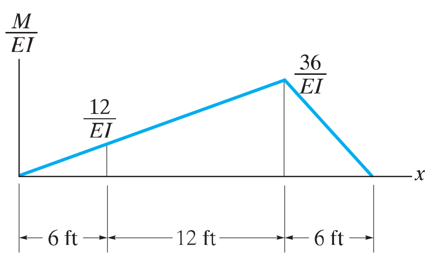
Thus,

$$\theta_C = \frac{135\text{ kN}\cdot\text{m}^2}{[200(10^6)\text{ kN/m}^2][6(10^6)(10^{-12})\text{ m}^4]} = 0.112\text{ rad} \qquad \textit{Ans.}$$

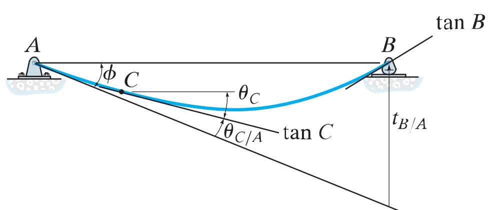
EXAMPLE 8.9



(a)



(b)



(c)

Fig. 8–19

Determine the slope at point C of the beam in Fig. 8–19*a*. $E = 29(10^3)$ ksi, $I = 600$ in^4.

SOLUTION

***M/EI* Diagram.** Fig. 8–19*b*.

Elastic Curve. The elastic curve is shown in Fig. 8–19*c*. We are required to find θ_C. To do this, establish tangents at A, B (the supports), and C and note that $\theta_{C/A}$ is the angle between the tangents at A and C. Also, the angle ϕ in Fig. 8–19*c* can be found using $\phi = t_{B/A}/L_{AB}$. This equation is valid since $t_{B/A}$ is actually very small, so that $t_{B/A}$ can be approximated by the length of a circular arc defined by a radius of $L_{AB} = 24$ ft and sweep of ϕ. (Recall that $s = \theta r$.) From the geometry of Fig. 8–19*c*, we have

$$\theta_C = \phi - \theta_{C/A} = \frac{t_{B/A}}{24} - \theta_{C/A} \qquad (1)$$

Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(6\text{ ft})\left(\frac{12\text{ k}\cdot\text{ft}}{EI}\right) = \frac{36\text{ k}\cdot\text{ft}^2}{EI}$$

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B, since this is the point where the tangential deviation is to be determined. We have

$$\begin{aligned} t_{B/A} &= \left[6\text{ ft} + \frac{1}{3}(18\text{ ft})\right]\left[\frac{1}{2}(18\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right] \\ &\quad + \frac{2}{3}(6\text{ ft})\left[\frac{1}{2}(6\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right] \\ &= \frac{4320\text{ k}\cdot\text{ft}^3}{EI} \end{aligned}$$

Substituting these results into Eq. 1, we have

$$\theta_C = \frac{4320\text{ k}\cdot\text{ft}^3}{(24\text{ ft})\,EI} - \frac{36\text{ k}\cdot\text{ft}^2}{EI} = \frac{144\text{ k}\cdot\text{ft}^2}{EI}$$

so that

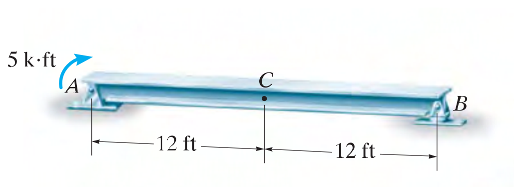
$$\begin{aligned} \theta_C &= \frac{144\text{ k}\cdot\text{ft}^2}{29(10^3)\text{ k/in}^2(144\text{ in}^2/\text{ft}^2)\ 600\text{ in}^4(1\text{ ft}^4/(12)^4\text{ in}^4)} \\ &= 0.00119\text{ rad} \end{aligned}$$

Ans.

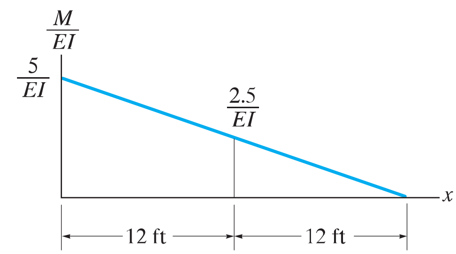
8

EXAMPLE 8.10

Determine the deflection at C of the beam shown in Fig. 8–20*a*. Take $E = 29(10^3)$ ksi, $I = 21$ in^4.



(a)



(b)

SOLUTION

M/EI Diagram. Fig. 8–20*b*.

Elastic Curve. Here we are required to find Δ_C, Fig. 8–20*c*. This is not necessarily the maximum deflection of the beam, since the loading and hence the elastic curve are *not symmetric*. Also indicated in Fig. 8–20*c* are the tangents at A, B (the supports), and C. If $t_{A/B}$ is determined, then Δ' can be found from proportional triangles, that is, $\Delta'/12 = t_{A/B}/24$ or $\Delta' = t_{A/B}/2$. From the construction in Fig. 8–20*c*, we have

$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B} \qquad (1)$$

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{A/B}$ and $t_{C/B}$. Here $t_{A/B}$ is the moment of the M/EI diagram between A and B about point A,

$$t_{A/B} = \left[\frac{1}{3}(24 \text{ ft})\right]\left[\frac{1}{2}(24 \text{ ft})\left(\frac{5 \text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{480 \text{ k}\cdot\text{ft}^3}{EI}$$

and $t_{C/B}$ is the moment of the M/EI diagram between C and B about C.

$$t_{C/B} = \left[\frac{1}{3}(12 \text{ ft})\right]\left[\frac{1}{2}(12 \text{ ft})\left(\frac{2.5 \text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{60 \text{ k}\cdot\text{ft}^3}{EI}$$

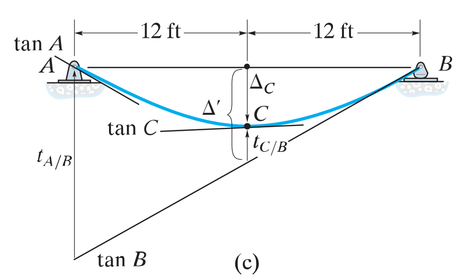
Substituting these results into Eq. (1) yields

$$\Delta_C = \frac{1}{2}\left(\frac{480 \text{ k}\cdot\text{ft}^3}{EI}\right) - \frac{60 \text{ k}\cdot\text{ft}^3}{EI} = \frac{180 \text{ k}\cdot\text{ft}^3}{EI}$$

Working in units of kips and inches, we have

$$\Delta_C = \frac{180 \text{ k}\cdot\text{ft}^3(1728 \text{ in}^3/\text{ft}^3)}{29(10^3) \text{ k/in}^2(21 \text{ in}^4)}$$

$$= 0.511 \text{ in.} \qquad \textit{Ans.}$$



(c)

Fig. 8–20

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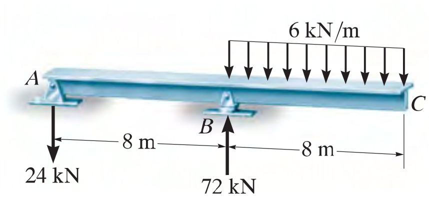
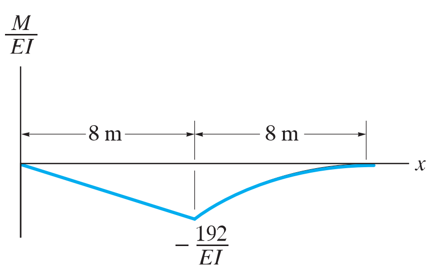
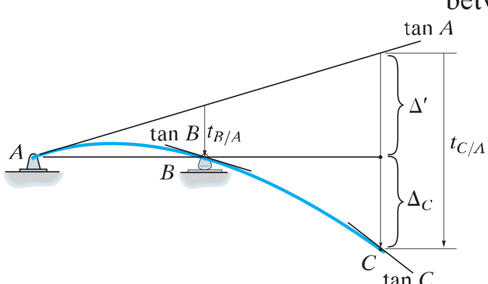
EXAMPLE 8.11



(a)



(b)



(c)

Fig. 8–21

Determine the deflection at point C of the beam shown in Fig. 8–21a. $E = 200$ GPa, $I = 250(10^6)$ mm^4.

SOLUTION

***M/EI* Diagram.** As shown in Fig. 8–21b, this diagram consists of a triangular and a parabolic segment.

Elastic Curve. The loading causes the beam to deform as shown in Fig. 8–21c. We are required to find Δ_C. By constructing tangents at A, B (the supports), and C, it is seen that $\Delta_C = t_{C/A} - \Delta'$. However, Δ' can be related to $t_{B/A}$ by proportional triangles, that is, $\Delta'/16 = t_{B/A}/8$ or $\Delta' = 2t_{B/A}$. Hence

$$\Delta_C = t_{C/A} - 2t_{B/A} \qquad (1)$$

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{C/A}$ and $t_{B/A}$. Using the table on the inside back cover for the parabolic segment and considering the moment of the M/EI diagram between A and C about point C, we have

$$t_{C/A} = \left[\frac{3}{4}(8\text{ m})\right]\left[\frac{1}{3}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right]$$
$$+\left[\frac{1}{3}(8\text{ m}) + 8\text{ m}\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right]$$
$$= -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI}$$

The moment of the M/EI diagram between A and B about point B gives

$$t_{B/A} = \left[\frac{1}{3}(8\text{ m})\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] = -\frac{2048\text{ kN}\cdot\text{m}^3}{EI}$$

Why are these terms negative? Substituting the results into Eq. (1) yields

$$\Delta_C = -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI} - 2\left(-\frac{2048\text{ kN}\cdot\text{m}^3}{EI}\right)$$
$$= -\frac{7168\text{ kN}\cdot\text{m}^3}{EI}$$

Thus,

$$\Delta_C = \frac{-7168\text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2][250(10^6)(10^{-12})\text{ m}^4]}$$
$$= -0.143\text{ m} \qquad \textit{Ans.}$$

8