University of Engineering & Technology Peshawar, Pakistan

CE301: Structure Analysis II

Module 09:

Analysis of S.I Pin Jointed Frames (Trusses) Using Stiffness method

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Topics to be Covered

- Introduction
- Prerequisites for using stiffness method for Trusses
- Kinematic Indeterminacy of trusses
- Stiffness method procedure for truss analysis
- Procedure of computing stiffness matrix in local and global coordinates
- Analysis of trusses Problem 1
- Problem 2
- Problem 3
- Problem 4

Stiffness Method for Trusses Analysis

❑ Introduction:

Trusses are analyzed with stiffness method due to

- To solve the problem in matrix notation, which is more systematic
- To compute reactions at all the supports.
- To compute internal resisting axial force in any member of the indeterminate Truss.

Stiffness Method for Trusses Analysis

The basic method for the analysis of indeterminate truss by stiffness method is similar to the indeterminate beam and rigid frame analysis discussed in the previous lessons, However fixed end actions are not taken in order to have simplicity in the structural model. Determine the degree of kinematic indeterminacy of the structure and then analyze the structure.

Stiffness Method for Trusses Analysis

- **Prerequisites for Analysis with Stiffness method:**

  It is necessary that students must have strong background of the following concepts before starting analysis with stiffness or any other matrix method.

- Enough concept of Matrix Algebra
- Must be able to find the kinematical Indeterminacy of trusses
- Enough concept of trigonometry

Stiffness Method for Trusses Analysis

❑ Kinematic Indeterminacy of Trusses:

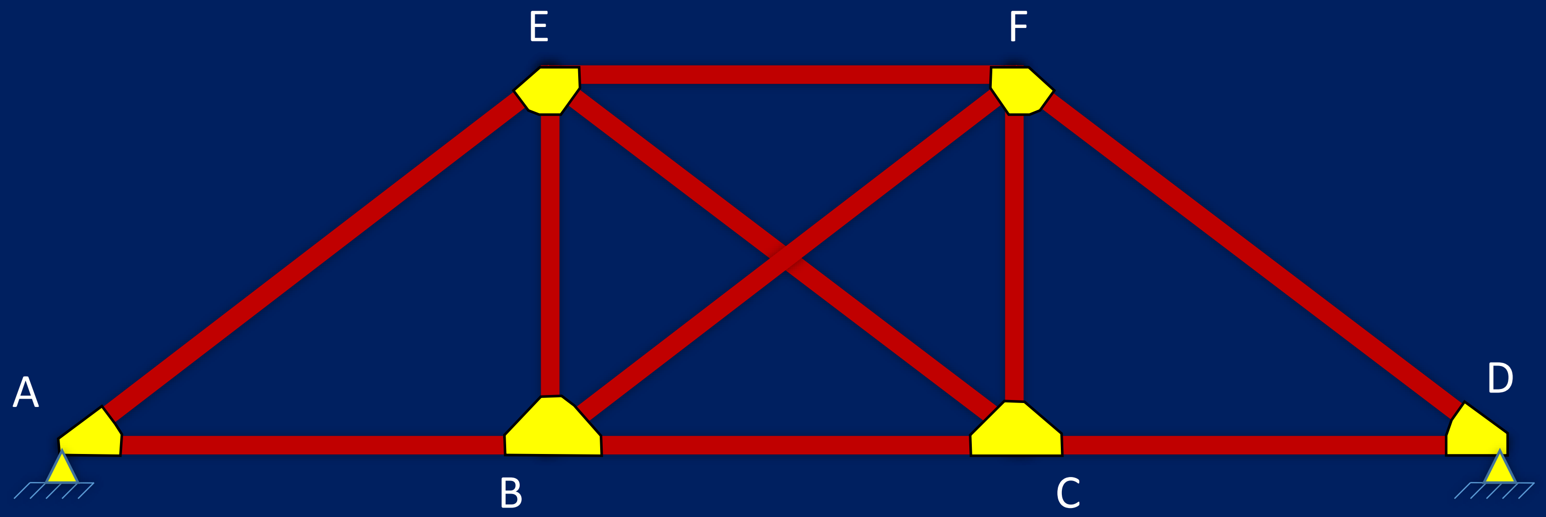
- The truss is said to be kinematiclly indeterminate when the total number of degrees of freedom at joints exceeds the no of reactions forces.
- At each joint there are three degrees of freedom but joints where supports are present , reduce the degree of freedom according to their nature at that joint , Roller support can restrained only perpendicular settlement to it & hinge support can restrained both.

$$\mathrm{K.I} = 2j - r$$

where j and r are number of joints and unknown reaction components respectively.

Stiffness Method for Trusses Analysis

- **Example:** Find the KI of the truss shown in fig.



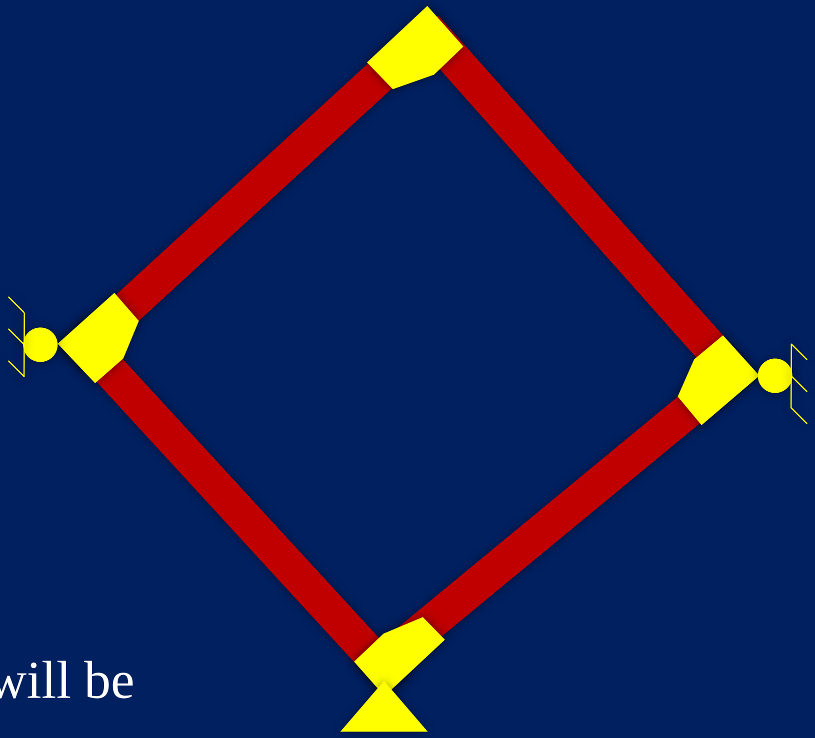
For the given truss , KI will be

$$r = 4 , \quad j = 6$$

$$\text{K.I} = 2j - r = 2 * 6 - 4 = 8 \circ$$

Stiffness Method for Trusses Analysis

- **Example:** Find the KI of the truss shown in fig.



For the given truss , KI will be

r = 4 ,   j = 4

$$K.I = 2j - r = 2 * 4 - 4 = 4 \circ$$

Stiffness Method for Trusses Analysis

❑ Analysis procedure:

The basic method for the analysis of indeterminate truss by stiffness method is similar to the indeterminate beam and rigid frame analysis discussed in the previous lessons, However fixed end actions are not taken in order to have simplicity in the structural model.

$$[AD] = [ADL] + [S] \bullet [D]$$

But in trusses fixed end actions are equal to zero so ADL = 0

$$[AD] = [S] \bullet [D]$$

Similarly for member end actions

$$[AM] = [AML] + [AMD] \bullet [D]$$

so

$$[AM] = [AMD] \bullet [D]$$

Stiffness Method for Trusses Analysis

❑ **Computation of stiffness matrix [S]:**

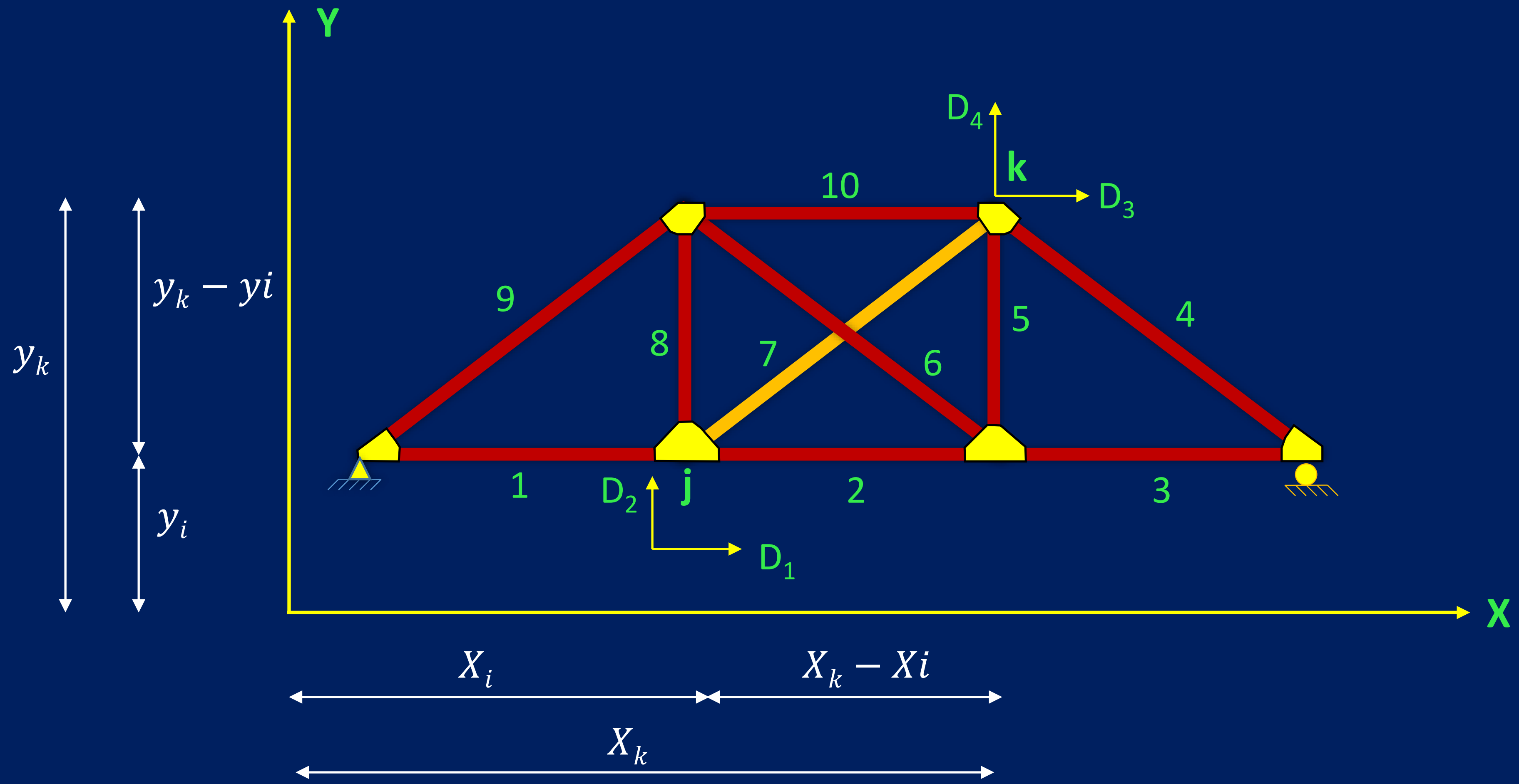
The stiffness matrix computation in truss analysis is the key to analysis . The stiffness matrix may be

- Stiffness matrix in local coordinate system
- Structural stiffness matrix in global coordinate system

Stiffness Method for Trusses Analysis

❑ Computation of stiffness matrix [S]:

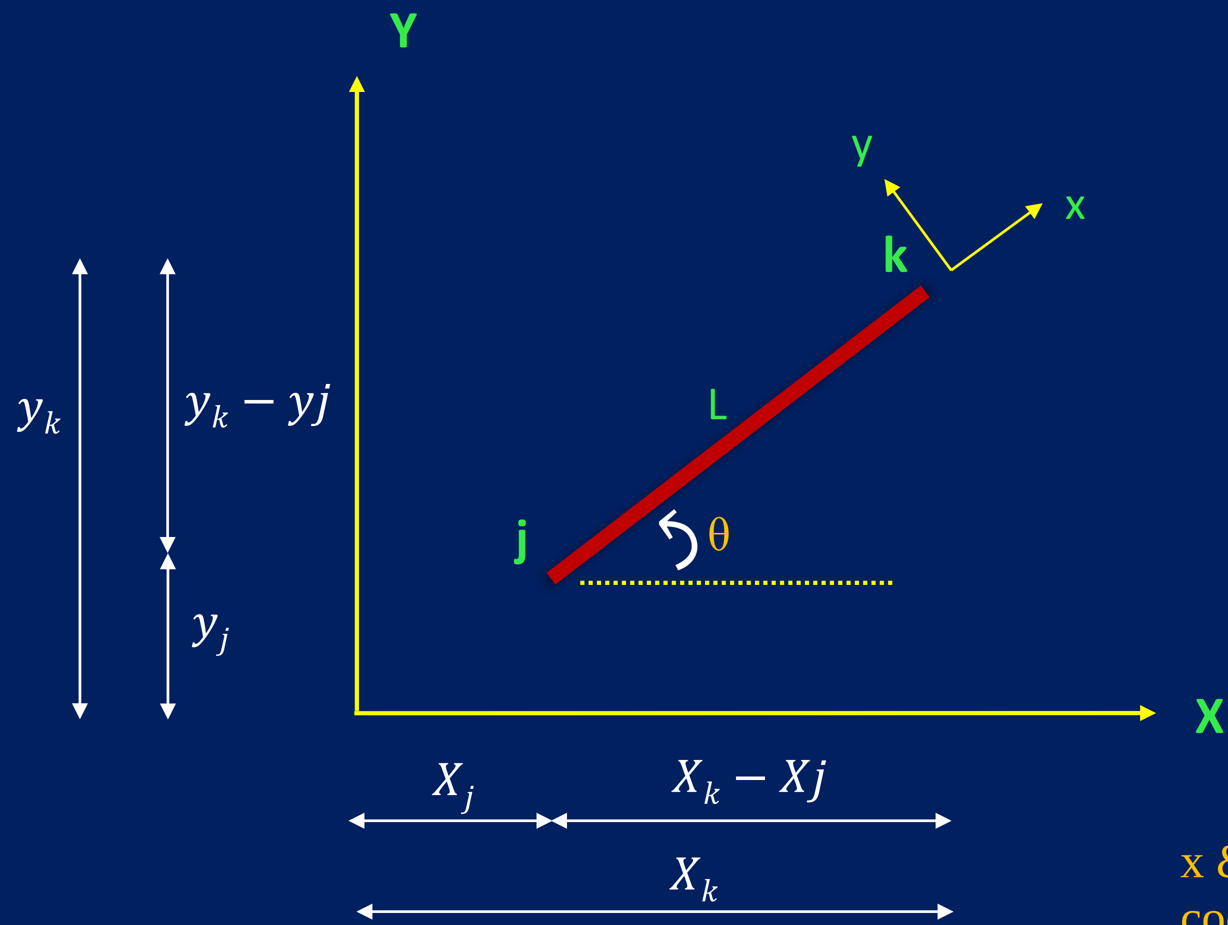
Lets we have a truss in global coordinate system as shown



Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S]:**

Isolate the member 7



$$\text{Cos}\,\theta = \frac{X_k - Xj}{L}$$

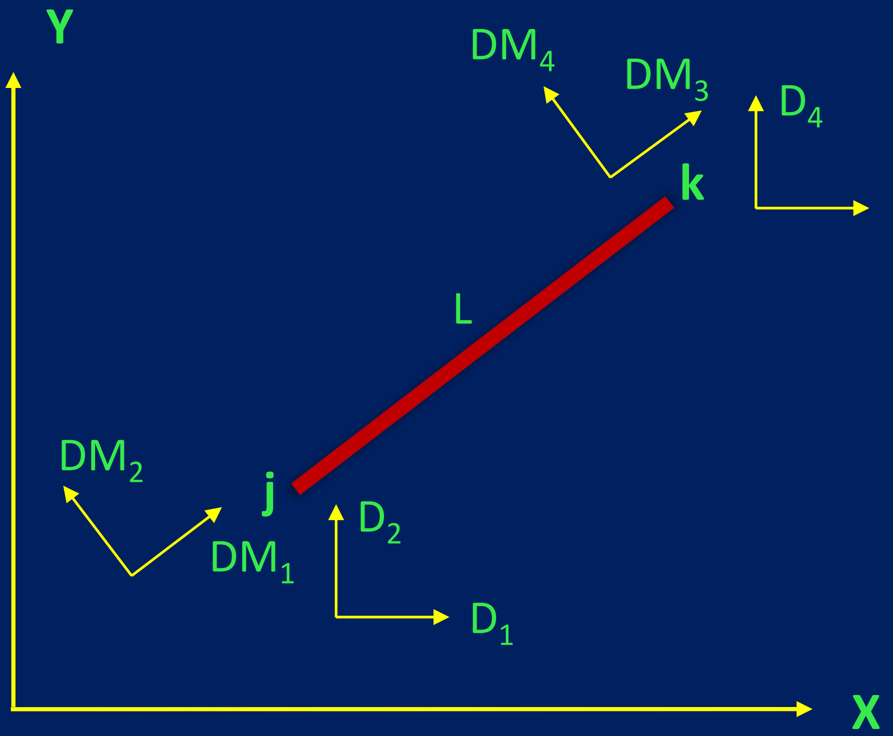
$$\text{Sin}\,\theta = \frac{y_k - yj}{L}$$

x & y are member/local coordinate system.

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S]:**

Isolate the member 7

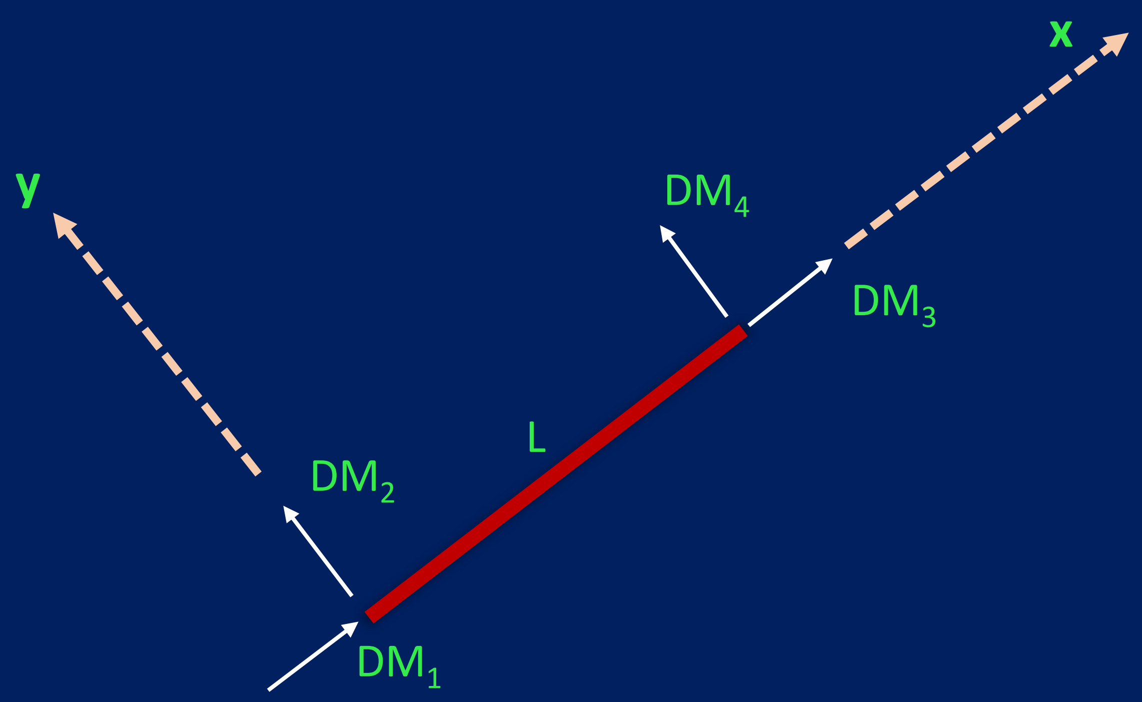


$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

D_1, D_2, D_3 & D_4 are the degrees of freedom in global axes

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in the member or in local coordinate system:**

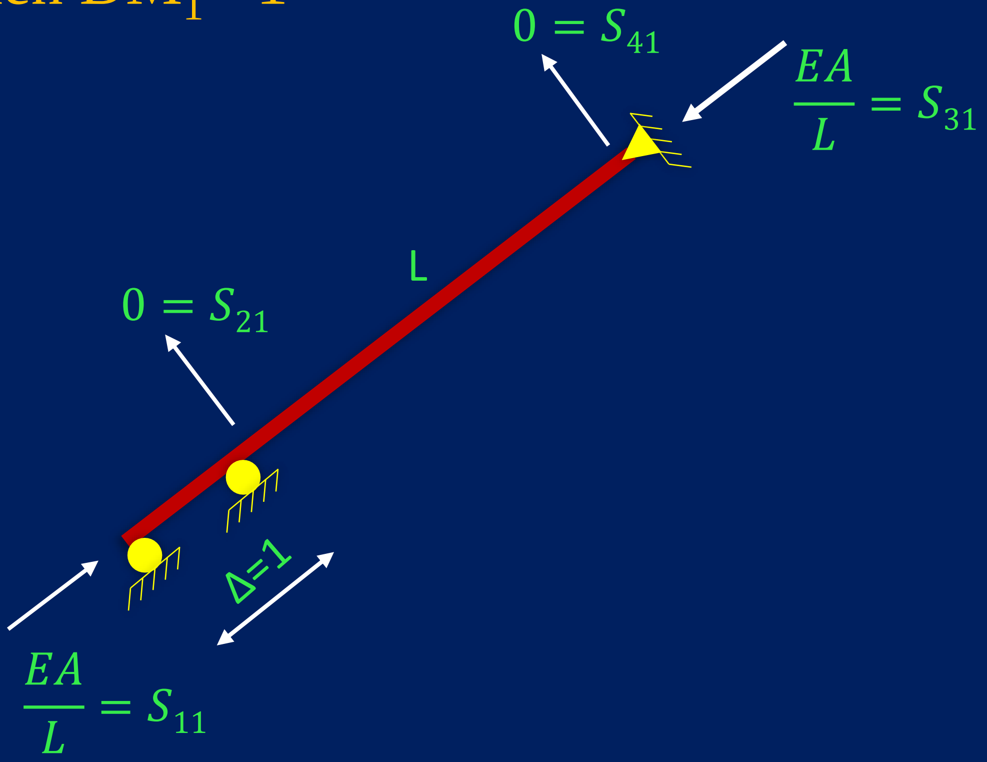


DM_1, DM_2, DM_3 & DM_4 are the degrees of freedom in local axes

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in the member or in local coordinate system:**

i. When $DM_1 = 1$



As we know that

$$\Delta = PL/AE$$

So when Δ=1 then

$$P = L/AE$$

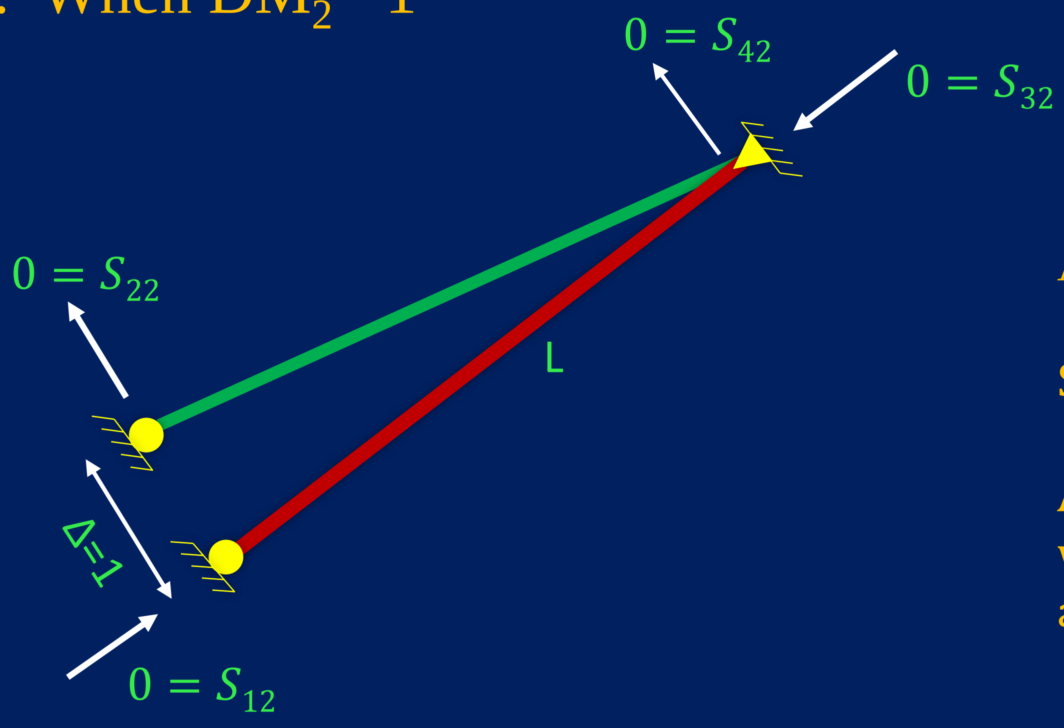
And this P will be the stiffness coefficient S.

$S_{11} = EA/L$ $\quad S_{21} = 0$ $\quad S_{31} = -EA/L$ $\quad S_{41} = 0$

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in the member or in local coordinate system:**

ii. When $DM_2 = 1$



As we know that

$$S = P/\Delta$$

So when P=0 then

$$S = 0$$

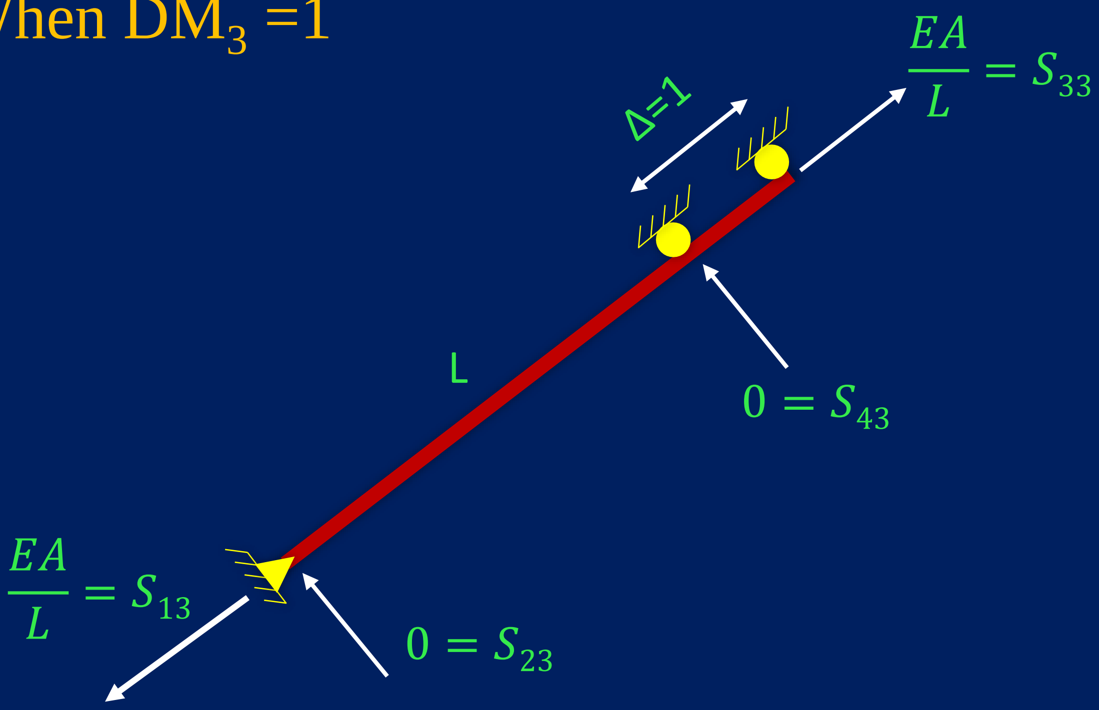
And P=0 because only axial action will be present and no transverse action

$S_{12} = 0$ $S_{22} = 0$ $S_{32} = 0$ $S_{42} = 0$

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in the member or in local coordinate system:**

iii. When $DM_3 = 1$

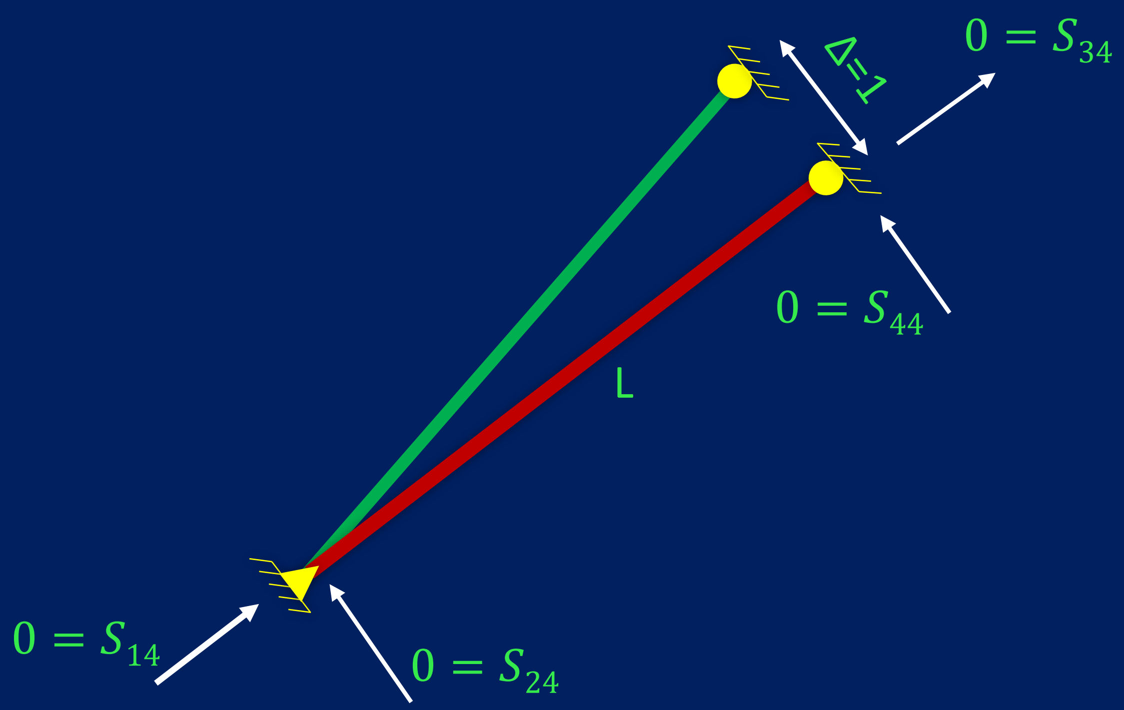


$S_{13} = -EA/L$ $S_{23} = 0$ $S_{33} = EA/L$ $S_{43} = 0$

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in the member or in local coordinate system:**

iv. When $DM_4 = 1$



$S_{14} = 0$ $S_{24} = 0$ $S_{34} = 0$ $S_{44} = 0$

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in the member or in local coordinate system:**

$S_{11} = EA/L$ $S_{21} = 0$ $S_{31} = -EA/L$ $S_{41} = 0$

$S_{12} = 0$ $S_{22} = 0$ $S_{32} = 0$ $S_{42} = 0$

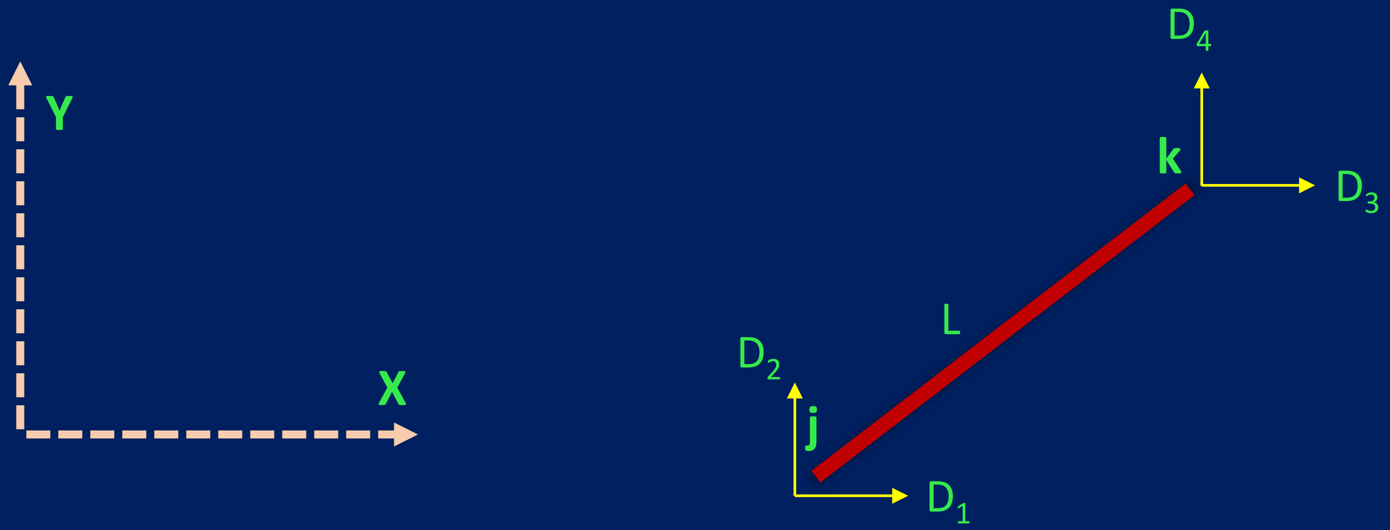
$S_{13} = -EA/L$ $S_{23} = 0$ $S_{33} = -EA/L$ $S_{43} = 0$

$S_{14} = 0$ $S_{24} = 0$ $S_{34} = 0$ $S_{44} = 0$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} = \begin{bmatrix} EA/L & 0 & -EA/L & 0 \\ 0 & 0 & S_{23} & 0 \\ -EA/L & 0 & EA/L & 0 \\ 0 & 0 & S_{43} & 0 \end{bmatrix}$$

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in global coordinate system:**

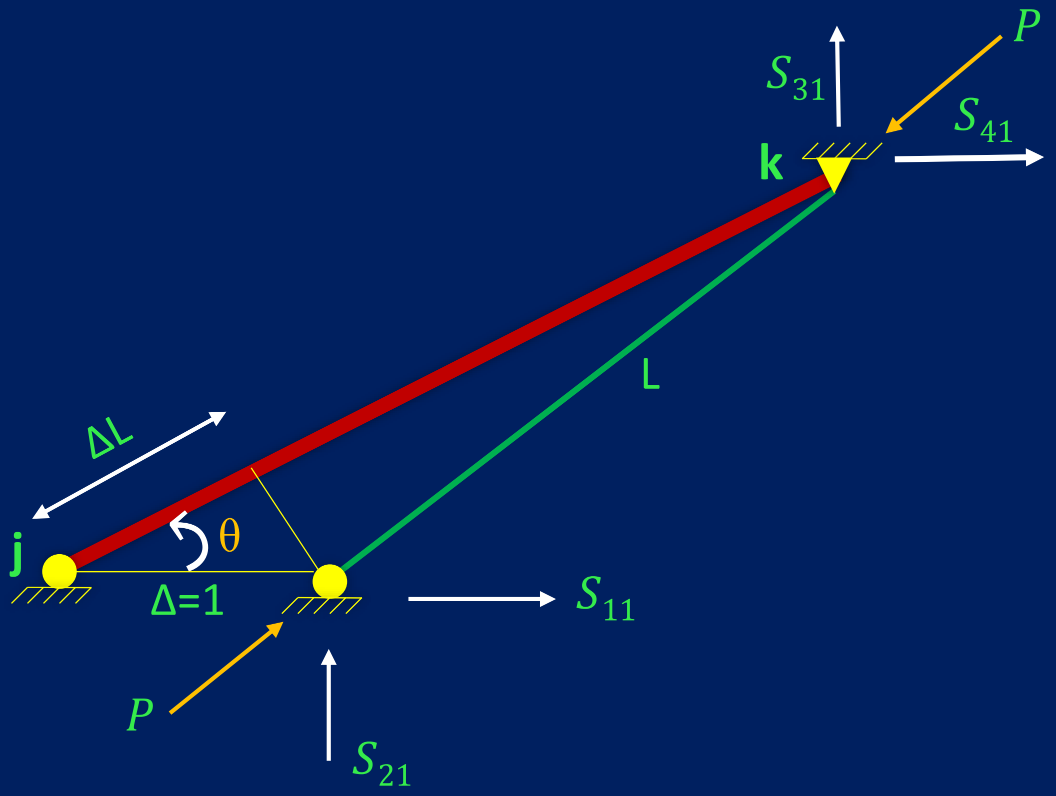


D_1, D_2, D_3 & D_4 are the degrees of freedom global axes

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in global coordinate system:**

i. When $D_1 = 1$



Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in global coordinate system:**

i. When $D_1 = 1$

From fig it is clear that $\cos\theta = \Delta L/1$

$$\Rightarrow \cos\theta = \Delta L$$

As $\Delta L = PL/AE$

From this $P = \frac{EA}{L}\Delta L = \frac{EA}{L}cos\theta$

As we know that $\cos\theta = \frac{x_k - xj}{L}$

$$P = \frac{EA}{L}\left(\frac{xk - xj}{L}\right) = \frac{EA}{L^2}(x_k - xj) = AMD_{71}$$

So S_{11} , S_{21} , S_{31}, S_{41} will be

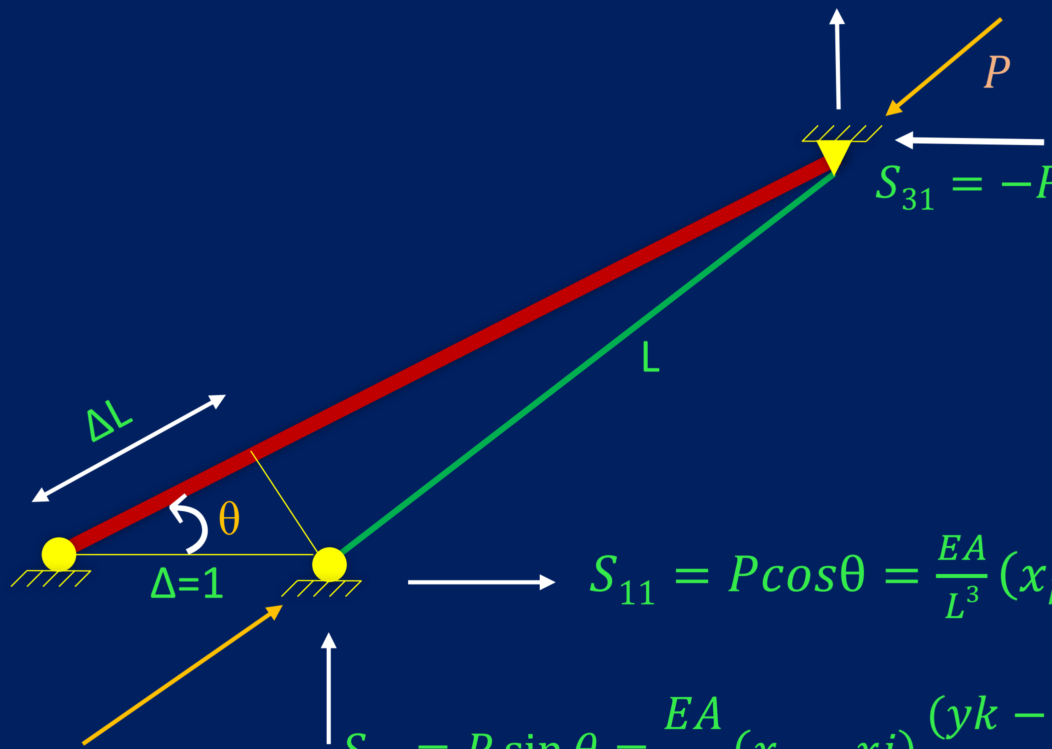
Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in global coordinate system:**

i. When $D_1 = 1$

$$S_{41} = -P\sin\theta = -\frac{EA}{L^3}(x_k - xj)(yk - yj)$$

$$S_{31} = -Pcos\theta = -\frac{EA}{L^3}(x_k - xj)^2$$



$$S_{11} = Pcos\theta = \frac{EA}{L^3}(x_k - xj)^2$$

$$P = \frac{EA}{L^2}(x_k - xj) = AMD_{71}$$

$$S_{21} = P\sin\theta = \frac{EA}{L^2}(x_k - xj)\frac{(yk - yj)}{L}$$

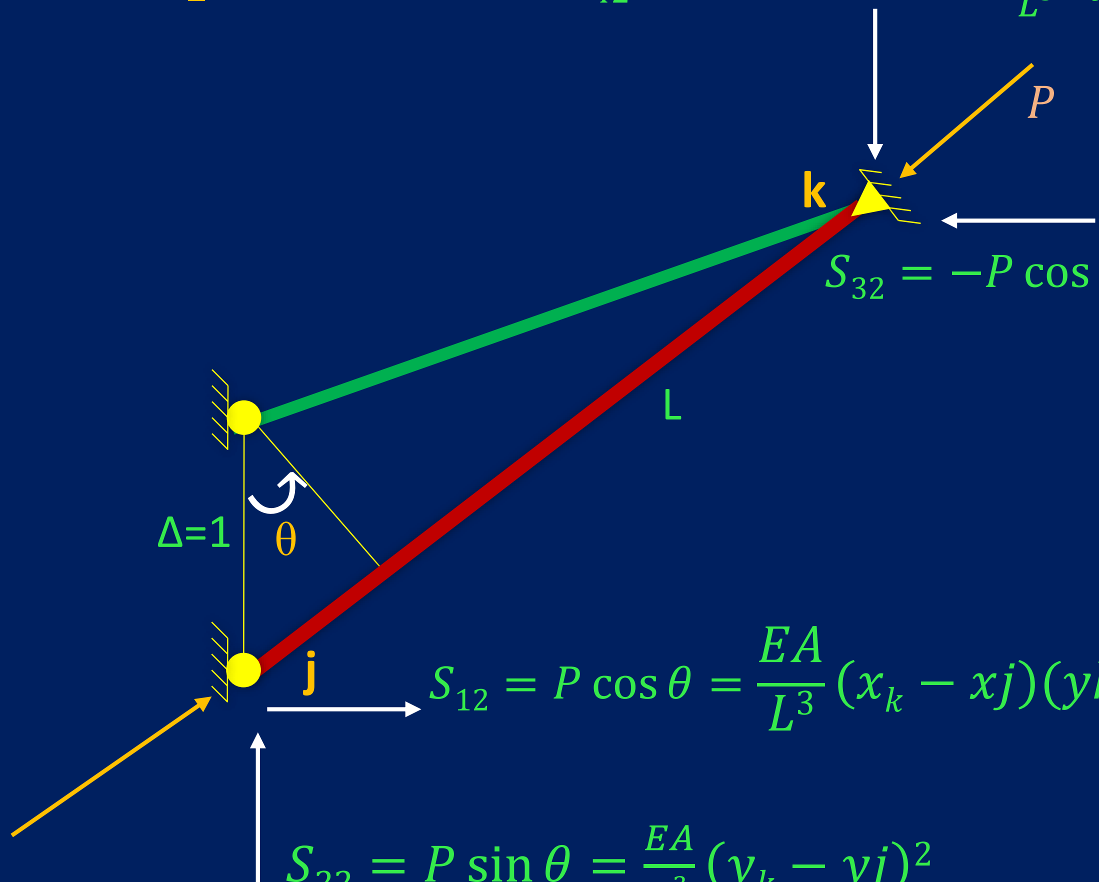
$$= \frac{EA}{L^3}(x_k - xj)(yk - yj)$$

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in global coordinate system:**

ii. When $D_2 = 1$

$$S_{42} = -P\sin\theta = -\frac{EA}{L^3}(y_k - yj)^2$$



$$S_{32} = -P\cos\theta = -\frac{EA}{L^3}(x_k - xj)(yk - yj)$$

$$S_{12} = P\cos\theta = \frac{EA}{L^3}(x_k - xj)(yk - yj)$$

$$S_{22} = P\sin\theta = \frac{EA}{L^3}(y_k - yj)^2$$

$$P = \frac{EA}{L^2}(y_k - yj) = AMD_{72}$$

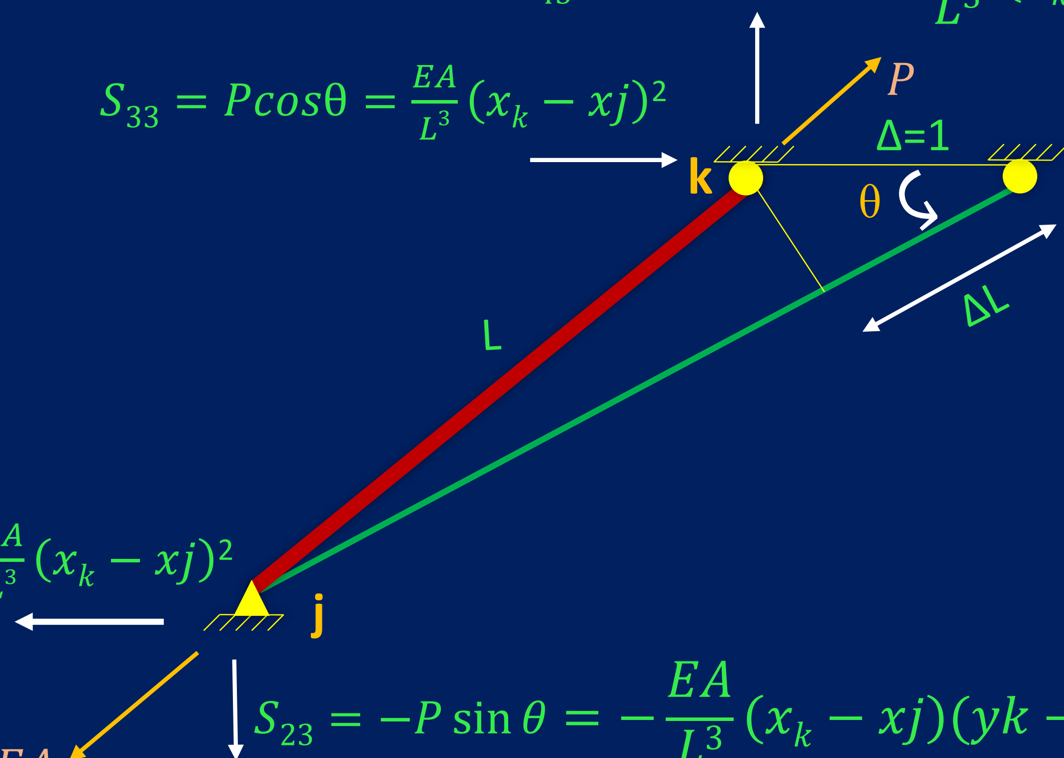
Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in global coordinate system:**

iii. When $D_3 = 1$

$$S_{43} = -P\sin\theta = -\frac{EA}{L^3}(x_k - xj)(yk - yj)$$

$$S_{33} = P\cos\theta = \frac{EA}{L^3}(x_k - xj)^2$$



$$S_{13} = -P\cos\theta = -\frac{EA}{L^3}(x_k - xj)^2$$

$$S_{23} = -P\sin\theta = -\frac{EA}{L^3}(x_k - xj)(yk - yj)$$

$$P = AMD_{73} = -\frac{EA}{L^2}(x_k - xj)$$

Stiffness Method for Trusses Analysis

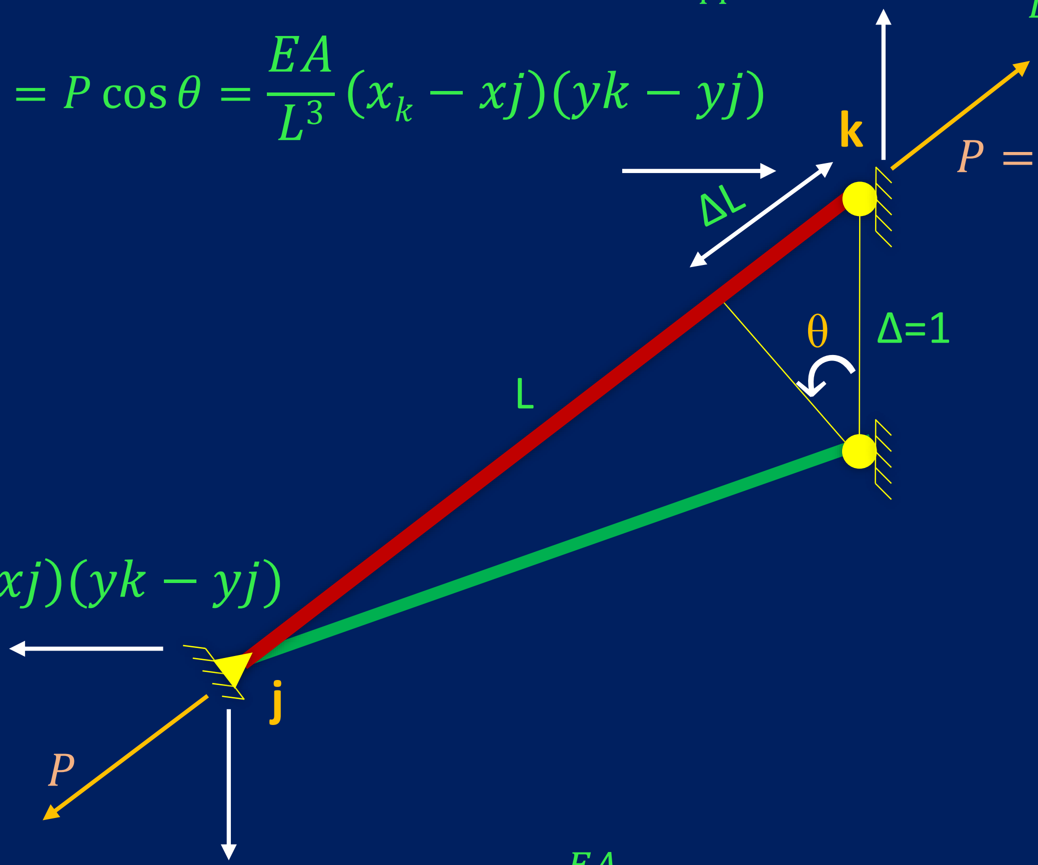
- **Computation of stiffness matrix [S] in global coordinate system:**

iv. When $D_4 = 1$

$$S_{44} = P\sin\theta = \frac{EA}{L^3}(y_k - yj)^2$$

$$S_{34} = P\cos\theta = \frac{EA}{L^3}(x_k - xj)(yk - yj)$$

$$P = AMD_{74} = \frac{EA}{L^2}(y_k - yj)$$



$$S_{14} = -P\cos\theta = -\frac{EA}{L^3}(x_k - xj)(yk - yj)$$

$$S_{24} = -P\sin\theta = -\frac{EA}{L^3}(y_k - yj)^2$$

Stiffness Method for Trusses Analysis

- **Computation of stiffness matrix [S] in global coordinate system:**

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

$$= \frac{EA}{L^3}\begin{bmatrix} (x_k - xj)^2 & (x_k - xj)(yk - yj) & -(x_k - xj)^2 & -(x_k - xj)(yk - yj) \\ (x_k - xj)(yk - yj) & (y_k - yj)^2 & -(x_k - xj)(yk - yj) & -(y_k - yj)^2 \\ -(x_k - xj)^2 & -(x_k - xj)(yk - yj) & (x_k - xj)(yk - yj) & (x_k - xj)(yk - yj) \\ -(x_k - xj)(yk - yj) & -(y_k - yj)^2 & (x_k - xj)^2 & (y_k - yj)^2 \end{bmatrix}$$

$$[S] = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

$$\text{Cos}\,\theta = \frac{X_k - Xj}{L}$$

$$\text{Sin}\,\theta = \frac{y_k - yj}{L}$$

Stiffness Method for Trusses Analysis

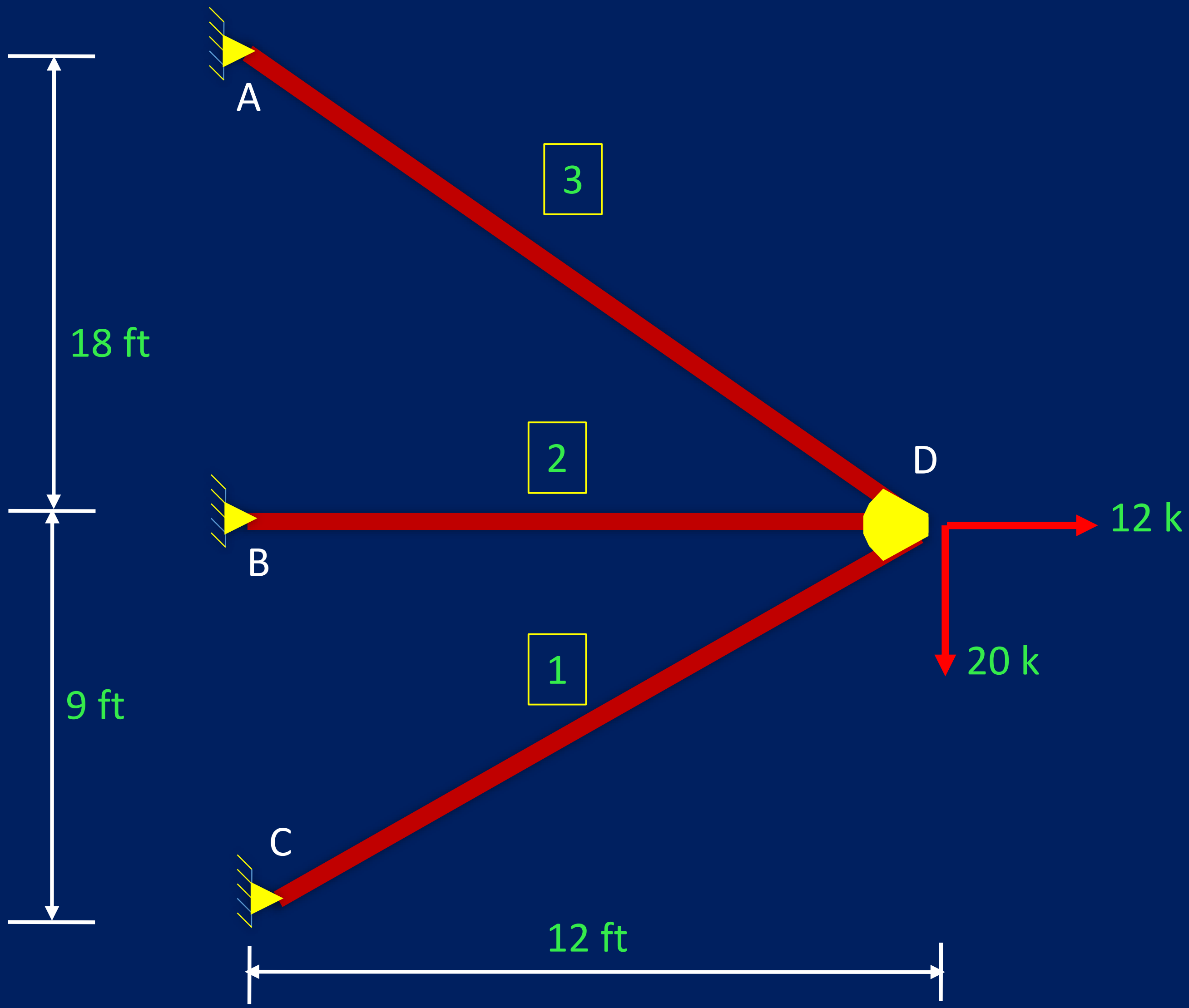
Problem 01: Analyze the given pin jointed frame using stiffness method.

Take EA = constant

$L_1 = \sqrt{9^2 + 12^2} = 15ft$

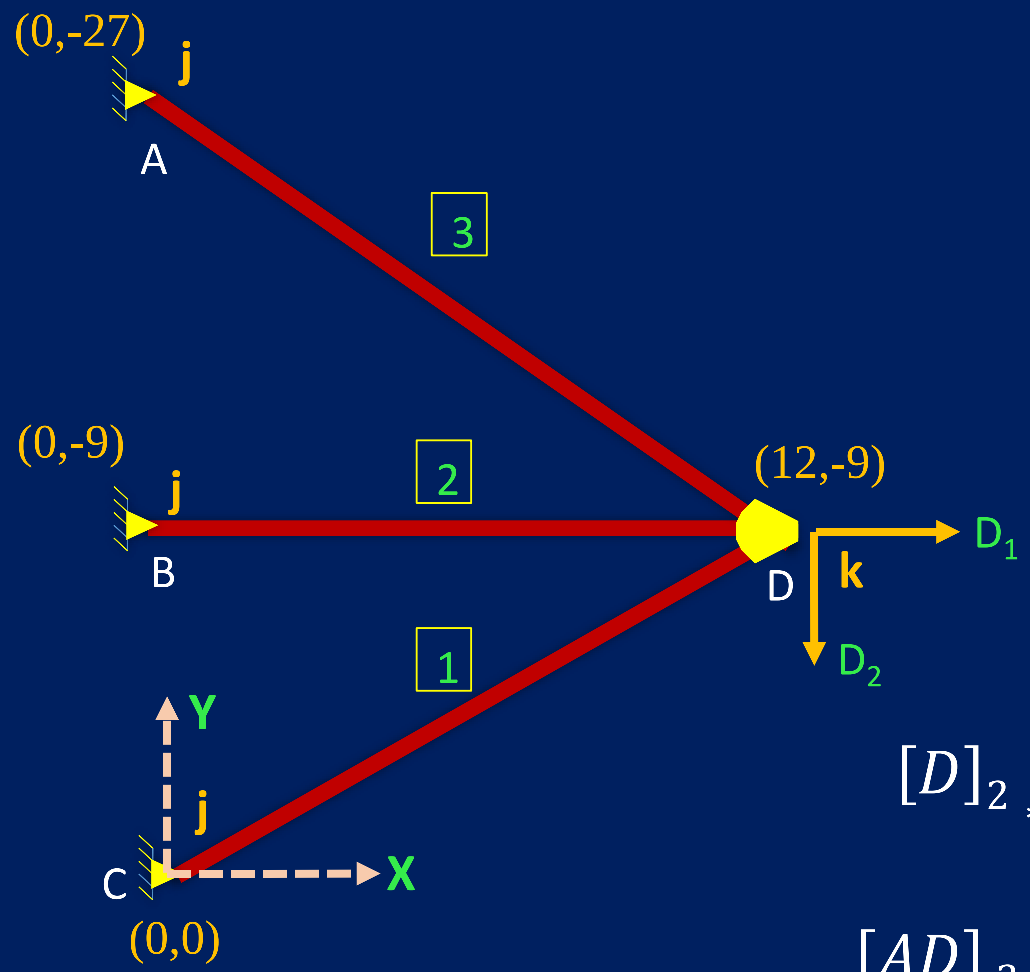
$L_2 = 12ft$

$L_3 = 21.63ft$



Stiffness Method for Trusses Analysis

- **Step # 01:** Identify the unknown joint displacements and compute the values of [AD] matrix.



Chose one point as an origin and assign coordinates to each joint w.r.t the chosen origin. Here point c is taken as origin.

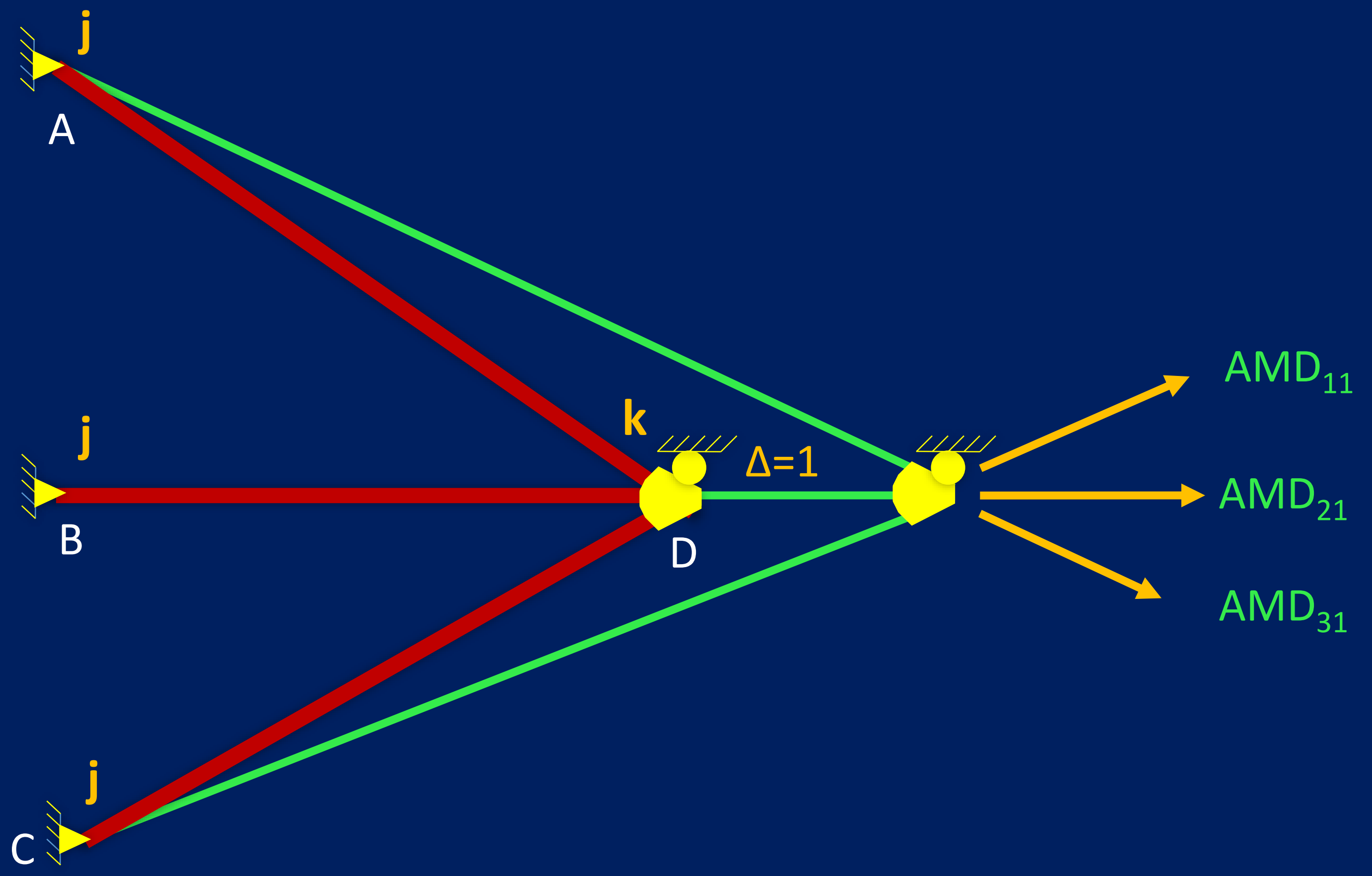
$$[D]_{2*1} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$[AD]_{2*1} = \begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = 0$



Compute the values of AMD & then stiffness coefficients.
S_{11} will be the sum of all horizontal components & S_{21} will be the sum of all vertical components of AMD .

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = 0$

$$AMD_{11} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{15^2}(12-0) = 0.0533EA$$

$$AMD_{21} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{12^2}(12-0) = 0.0833EA$$

$$AMD_{31} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{21.63^2}(12-0) = 0.0256EA$$

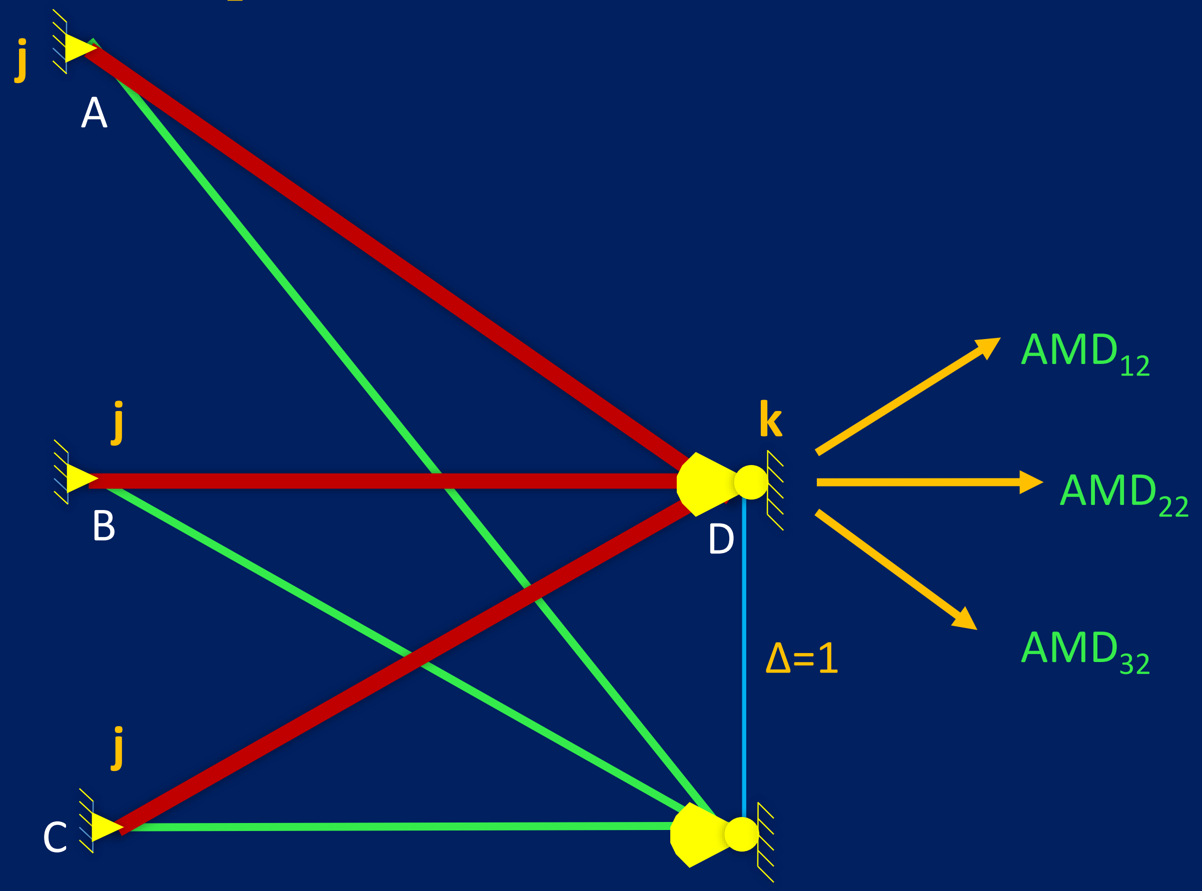
$$S_{11} = \frac{EA}{L^3}(x_k - xj)^2 = \frac{EA}{15^3}(12-0)^2 + \frac{EA}{12^3}(12-0)^2 + \frac{EA}{21.63^3}(12-0)^2$$
$$= 0.1402EA$$

$$S_{21} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{15^3}(12-0)(-9-0) + \frac{EA}{12^3}(12-0)\big(-9-(-9)\big)$$
$$+ \frac{EA}{21.63^3}(12-0)\big(-9-(-27)\big) = -0.0107EA$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = 0$



Compute the values of AMD & then stiffness coefficients.
S_{12} will be the sum of all horizontal components & S_{22} will be the sum of all vertical components of AMD .

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = 0$

$$AMD_{12} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{15^2}(-9+0) = -0.04EA$$

$$AMD_{22} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{12^2}(-9-(-9)) = 0$$

$$AMD_{32} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{21.63^2}(-9-(-12) = 0.0385EA$$

$$S_{12} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{15^3}(12-0)(-9-0) + \frac{EA}{12^3}(12-0)\left(-9-(-9)\right) + \frac{EA}{21.63^3}(12-0)\left(-9-(-27)\right) = -0.0107EA$$

$$S_{22} = \frac{EA}{L^3}(y_k - yj)^2 = \frac{EA}{15^3}(-9-0)^2 + \frac{EA}{12^3}(-9+9)^2 + \frac{EA}{21.63^3}(-9-27)^2 = 0.056EA$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

AMD matrix will be

$AMD_{11} = 0.0533EA$ $\quad AMD_{21} = 0.0833EA$ $\quad AMD_{31} = 0.0256EA$

$AMD_{12} = -0.04EA$ $\quad AMD_{22} = 0$ $\quad AMD_{32} = 0.0385EA$

$$[\,AMD\,] = EA\begin{bmatrix} 0.0533 & -0.04 \\ 0.0833 & 0 \\ 0.0256 & 0.0385 \end{bmatrix}$$

Stiffness matrix will be

$S_{11} = 0.1402EA$ $\quad S_{21} = -0.0107EA$ $\quad S_{12} = -0.0107EA$ $\quad S_{22} = 0.056EA$

$$[\,S\,] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad [\,S\,] = EA\begin{bmatrix} 0.1402 & -0.0107 \\ -0.0107 & 0.056 \end{bmatrix}$$

Stiffness Method for Truss Analysis

Step # 03: Apply equilibrium condition at the location of the redundant joint displacement to write equilibrium equations and solve for unknown joint displacement.

$$AD_1 = S_{11}D_1 + S_{12}D_2$$

$$AD_2 = S_{21}D_1 + S_{22}D_2$$

$$\begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$[AD]_{2\,*\,1} = [S]_{2\,*\,2} \bullet [D]_{2\,*\,1}$$

$$[\boldsymbol{D}] = [\boldsymbol{S}]^{-\mathbf{1}} \bullet [\boldsymbol{AD}]$$

Stiffness Method for Truss Analysis

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^{-1} \begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix}$$

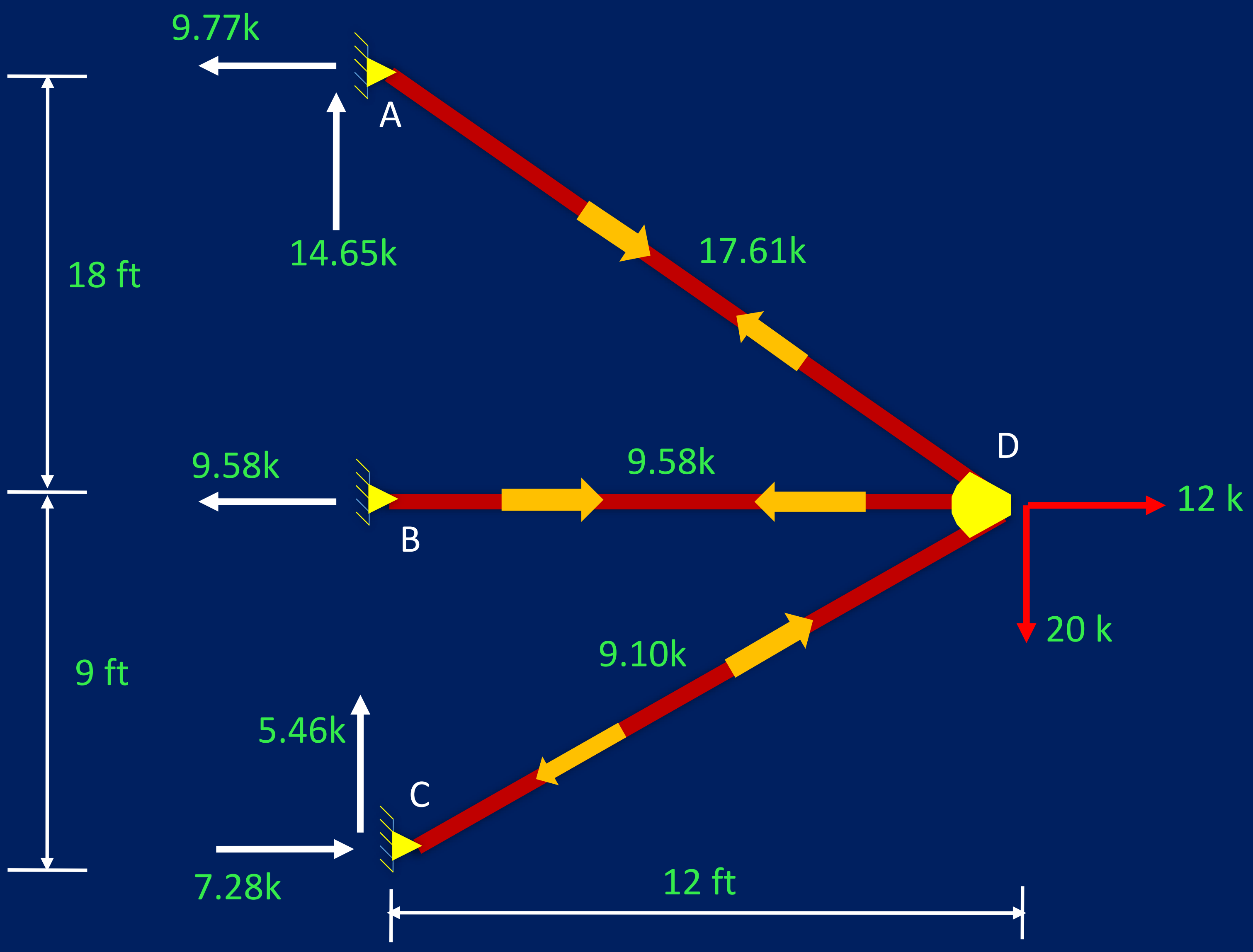
$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 0.1402 & -0.0107 \\ -0.0107 & 0.056 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 115.065 \\ 380.83 \end{bmatrix}$$

$$\begin{bmatrix} AM_1 \\ AM_2 \\ AM_3 \end{bmatrix} = \begin{bmatrix} AMD_{11} & AMD_{12} \\ AMD_{21} & AMD_{22} \\ AMD_{31} & AMD_{32} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$\begin{bmatrix} AM_1 \\ AM_2 \\ AM_3 \end{bmatrix} = \mathrm{EA} \begin{bmatrix} 0.0533 & -0.04 \\ 0.0833 & 0 \\ 0.0256 & 0.0385 \end{bmatrix} \begin{bmatrix} 115.065 \\ 380.83 \end{bmatrix} \frac{1}{EA} = \begin{bmatrix} -9.10k \\ 9.58k \\ 17.61k \end{bmatrix}$$

Stiffness Method for Truss Analysis



Final Analyzed structure

Stiffness Method for Truss Analysis

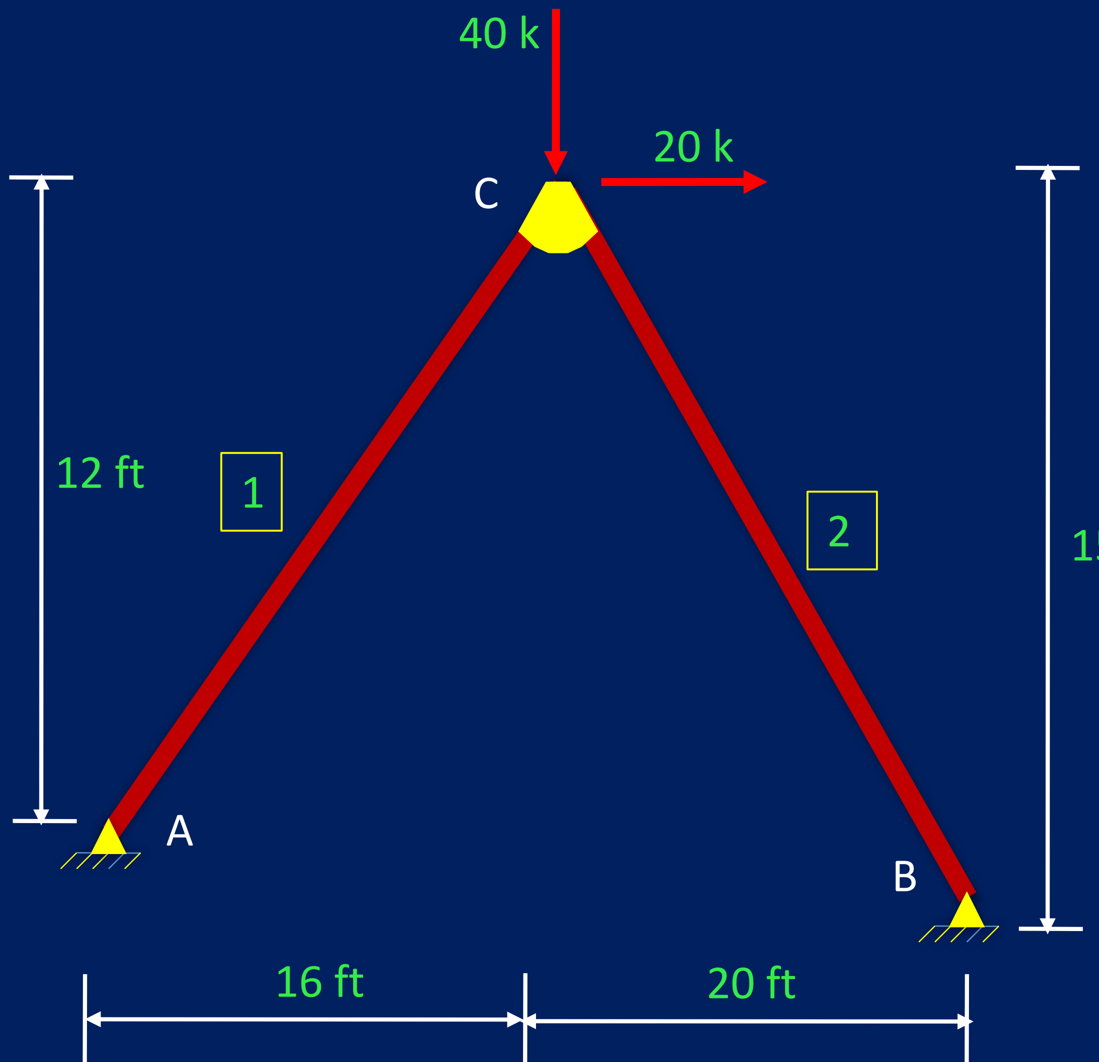
Problem 02: Analyze the given pin jointed frame using stiffness method.

K.I = 2 degrees

Take EA = constant

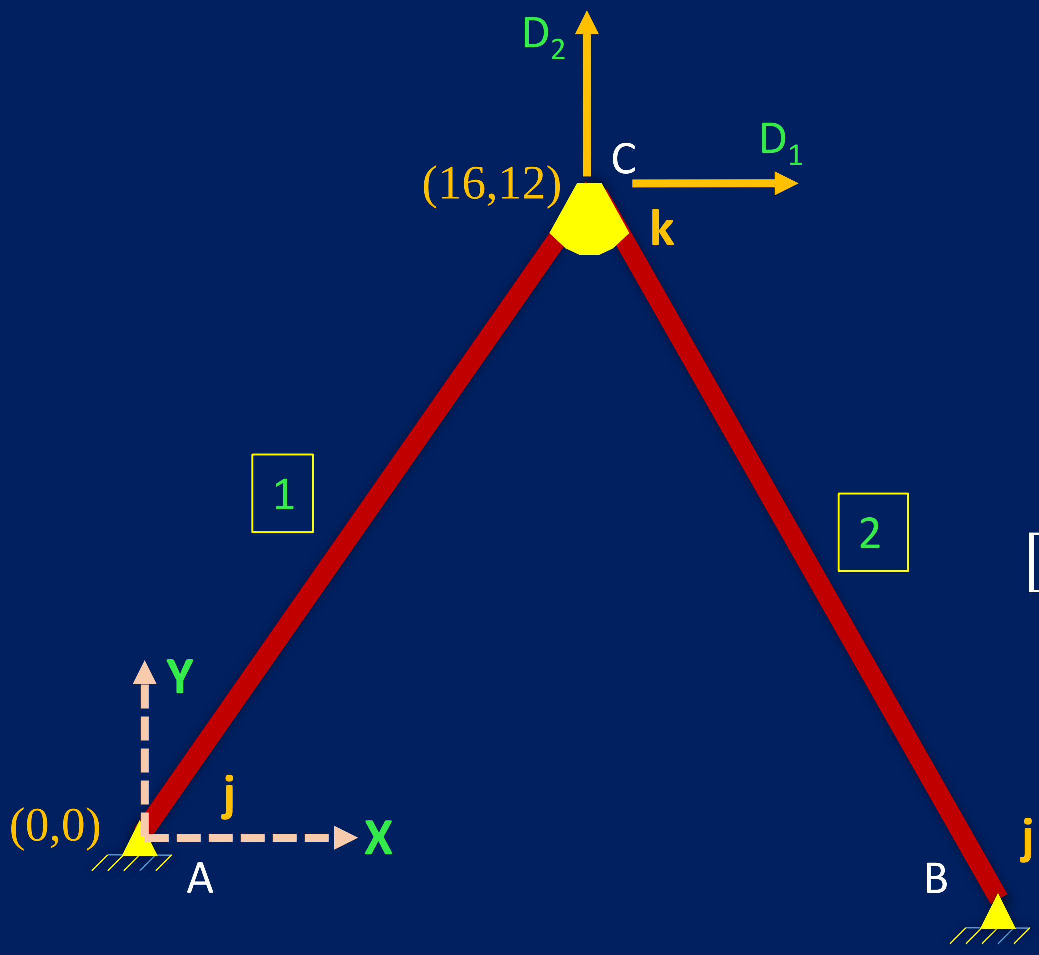
$L_1 = \sqrt{16^2 + 12^2} = 20ft$

$L_2 = 25ft$



Stiffness Method for Trusses Analysis

- **Step # 01:** Identify the unknown joint displacements and compute the values of [AD] matrix.



Chose one point as an origin and assign coordinates to each joint w.r.t the chosen origin. Here point c is taken as origin.

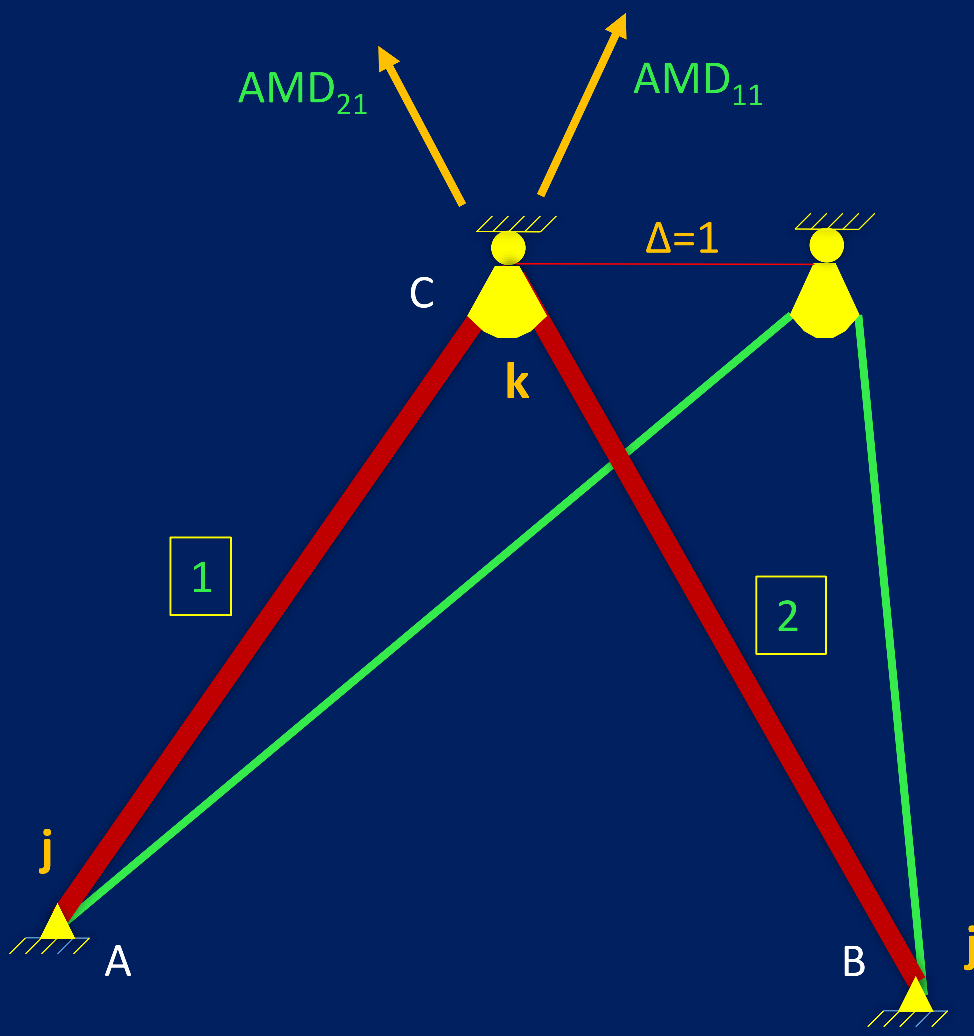
$$[AD]_{2*1} = \begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} 20 \\ -40 \end{bmatrix}$$

$$[D]_{2*1} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = 0$



Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = 0$

$$AMD_{11} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{20^2}(16 - 0) = 0.04EA$$

$$AMD_{21} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{25^2}(16 - 36) = -0.032EA$$

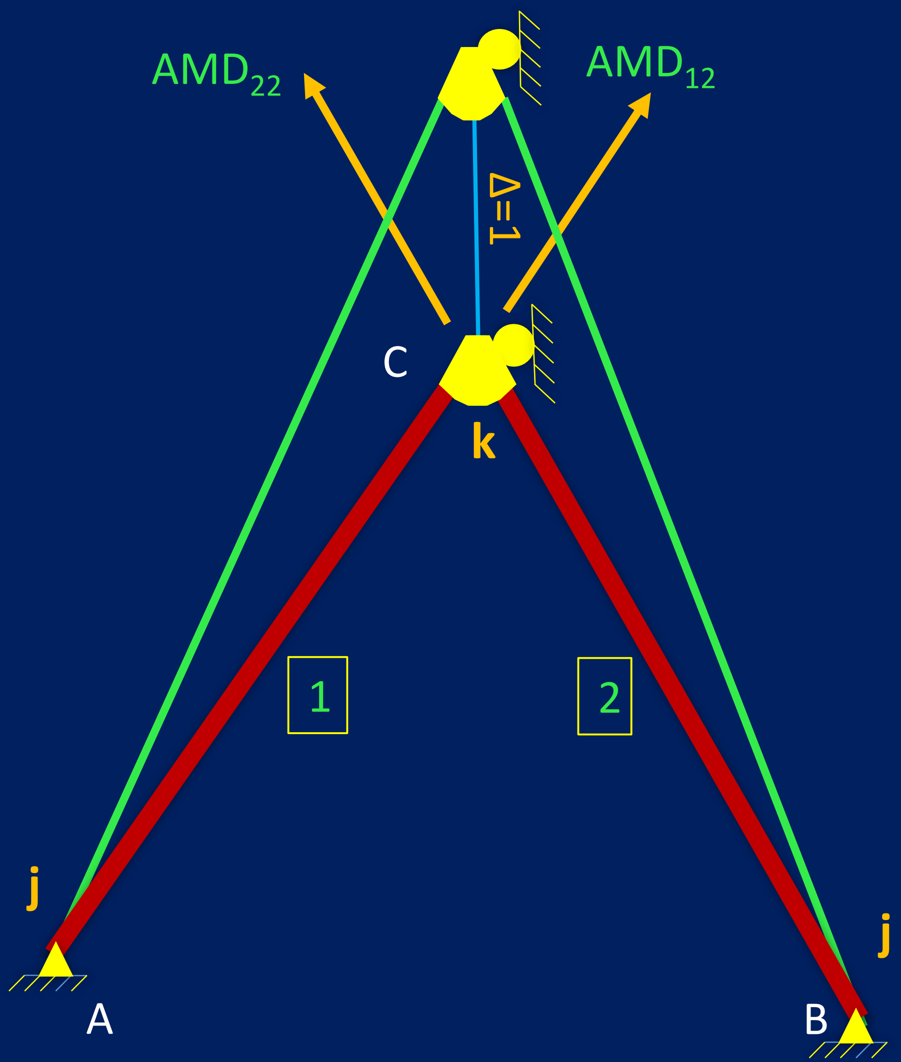
$$S_{11} = \frac{EA}{L^3}(x_k - xj)^2 = \frac{EA}{20^3}(16 - 0)^2 + \frac{EA}{25^3}(16 - 36)^2 = 0.0576EA$$

$$S_{21} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{20^3}(16 - 0)(12 - 0) + \frac{EA}{25^3}(16 - 36)\big(12 - (-3)\big)$$
$$= 0.0048EA$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = 0$



Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = 0$

$$AMD_{12} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{20^2}(12 + 0) = 0.03EA$$

$$AMD_{22} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{25^2}(12 - (-3)) = 0.024EA$$

$$S_{12} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{20^3}(16 - 0)(12 - 0) + \frac{EA}{25^3}(16 - 36)\left(12 - (-3)\right)$$
$$= 0.0048EA$$

$$S_{22} = \frac{EA}{L^3}(y_k - yj)^2 = \frac{EA}{20^3}(12 - 0)^2 + \frac{EA}{25^3}(12 + 3)^2 = 0.0324EA$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

AMD matrix will be

$$AMD_{11} = 0.04EA \qquad AMD_{21} = -0.032EA$$

$$AMD_{12} = 0.03EA \qquad AMD_{22} = 0.024EA$$

$$[\,\text{AMD}\,] = \text{EA}\begin{bmatrix} 0.04 & 0.03 \\ -0.032 & 0.024 \end{bmatrix}$$

Stiffness matrix will be

$$S_{11} = 0.0576EA \qquad S_{21} = 0.0048EA \qquad S_{12} = 0.0048EA \qquad S_{22} = 0.0324EA$$

$$[\,S\,] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad [\,\text{S}\,] = EA\begin{bmatrix} 0.0576 & 0.0048 \\ 0.0048 & 0.0324 \end{bmatrix}$$

Stiffness Method for Truss Analysis

Step # 03: Apply equilibrium condition at the location of the redundant joint displacement to write equilibrium equations and solve for unknown joint displacement.

$$AD_1 = S_{11}D_1 + S_{12}D_2$$

$$AD_2 = S_{21}D_1 + S_{22}D_2$$

$$\begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$[AD]_{2 * 1} = [S]_{2 * 2} \bullet [D]_{2 * 1}$$

$$[\boldsymbol{D}] = [\boldsymbol{S}]^{-\mathbf{1}} \bullet [\boldsymbol{AD}]$$

Stiffness Method for Truss Analysis

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^{-1} \begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix}$$

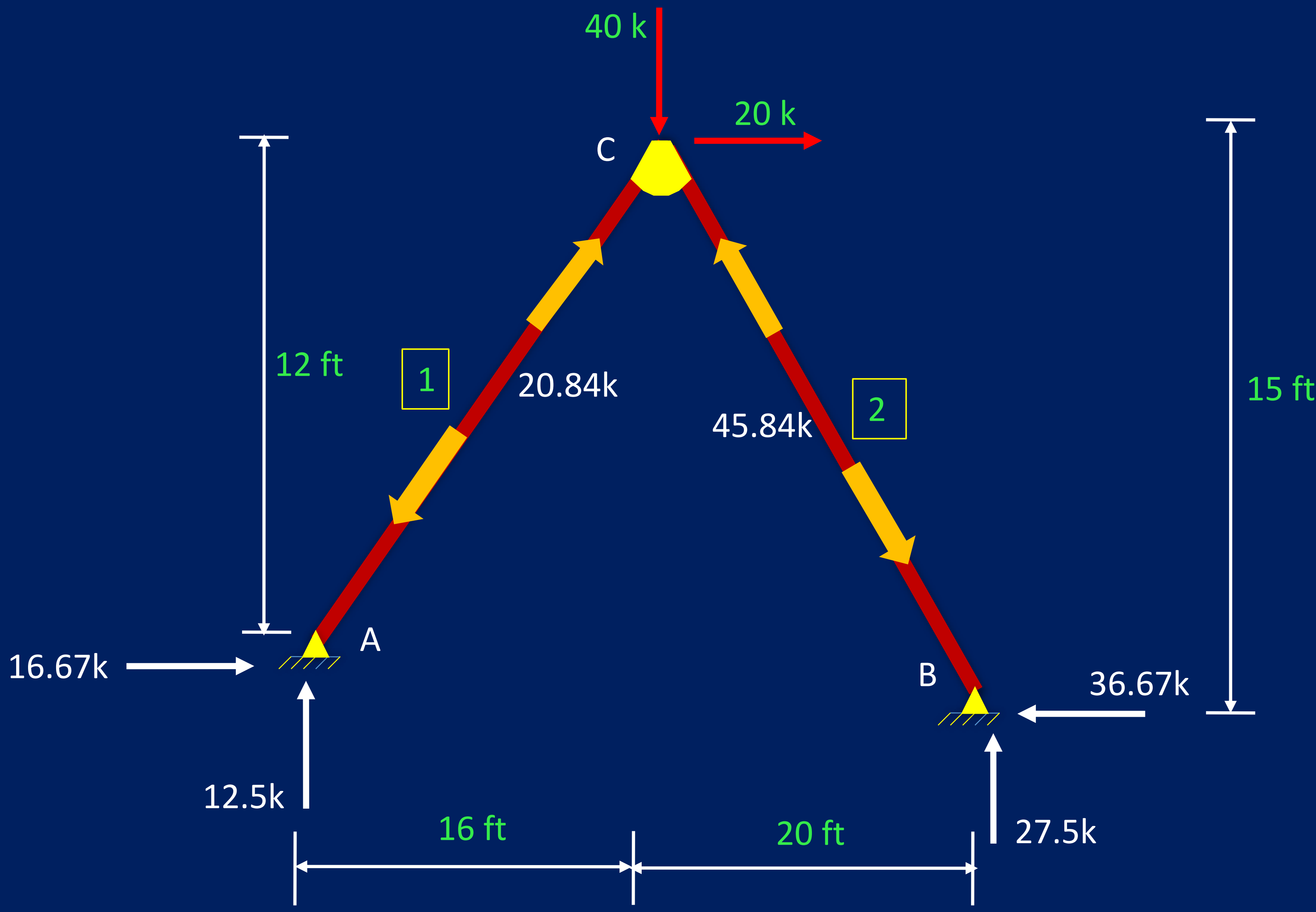
$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 0.0576 & 0.0048 \\ 0.0048 & 0.0324 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 455.73 \\ -1302.10 \end{bmatrix}$$

$$\begin{bmatrix} AM_1 \\ AM_2 \end{bmatrix} = \begin{bmatrix} AMD_{11} & AMD_{12} \\ AMD_{21} & AMD_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$\begin{bmatrix} AM_1 \\ AM_2 \end{bmatrix} = \mathrm{EA} \begin{bmatrix} 0.04 & 0.03 \\ -0.032 & 0.024 \end{bmatrix} \begin{bmatrix} 455.73 \\ -1302.10 \end{bmatrix} \frac{1}{EA} = \begin{bmatrix} -20.84k \\ -45.84k \end{bmatrix}$$

Flexibility Method for Trusses Analysis



Final Analyzed structure

Stiffness Method for Trusses Analysis

Problem 03: Analyze the given pin jointed frame using stiffness method.

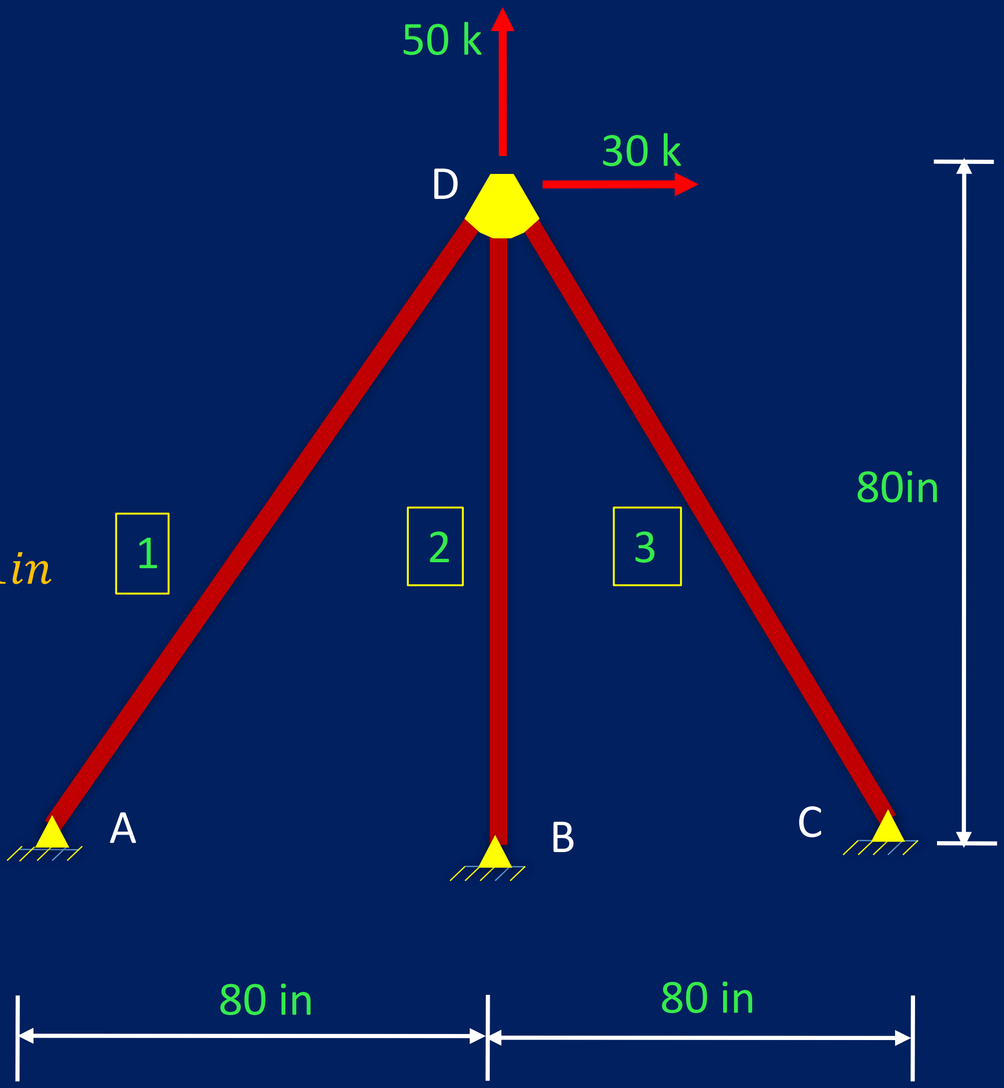
K.I= 2 degrees

Take EA = constant

$L_1 = \sqrt{80^2 + 80^2} = 113.1in$

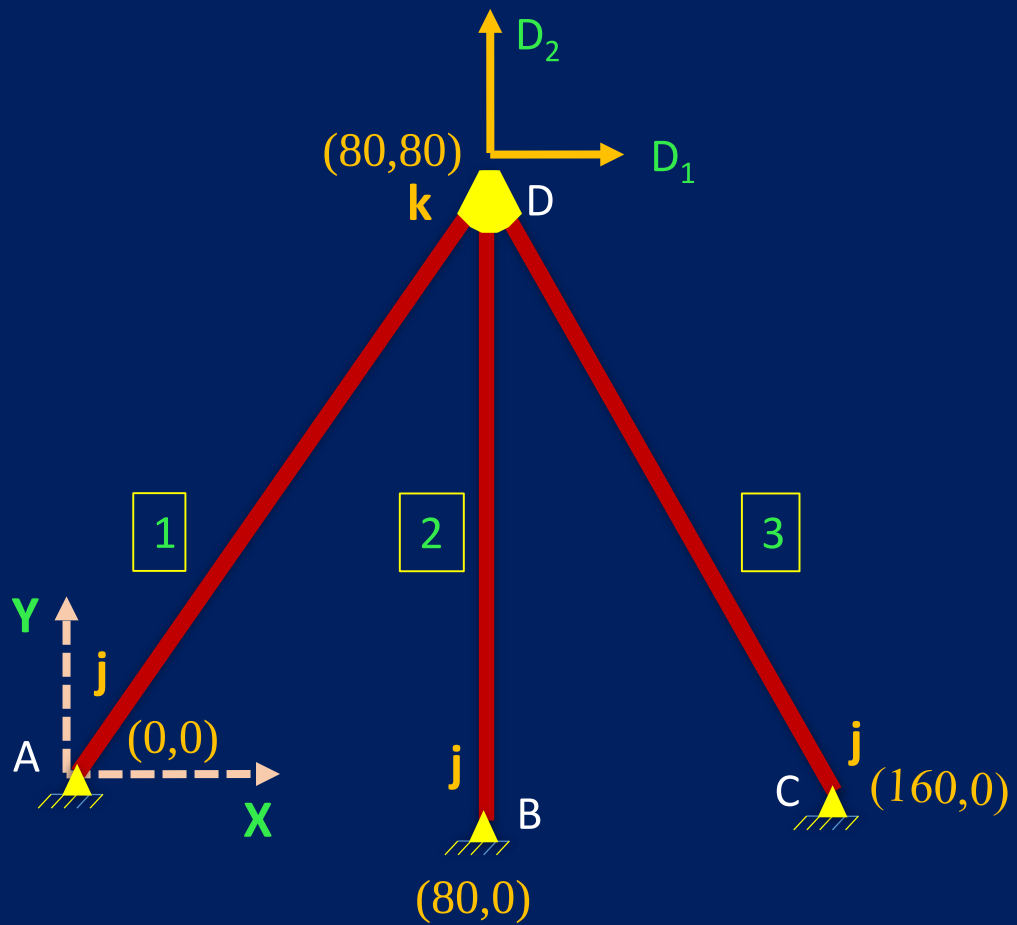
$L_2 = 80in$

$L_3 = 113.1in$



Stiffness Method for Trusses Analysis

- **Step # 01:** Identify the unknown joint displacements and compute the values of [AD] matrix.



Chose one point as an origin and assign coordinates to each joint w.r.t the chosen origin. Here point c is taken as origin.

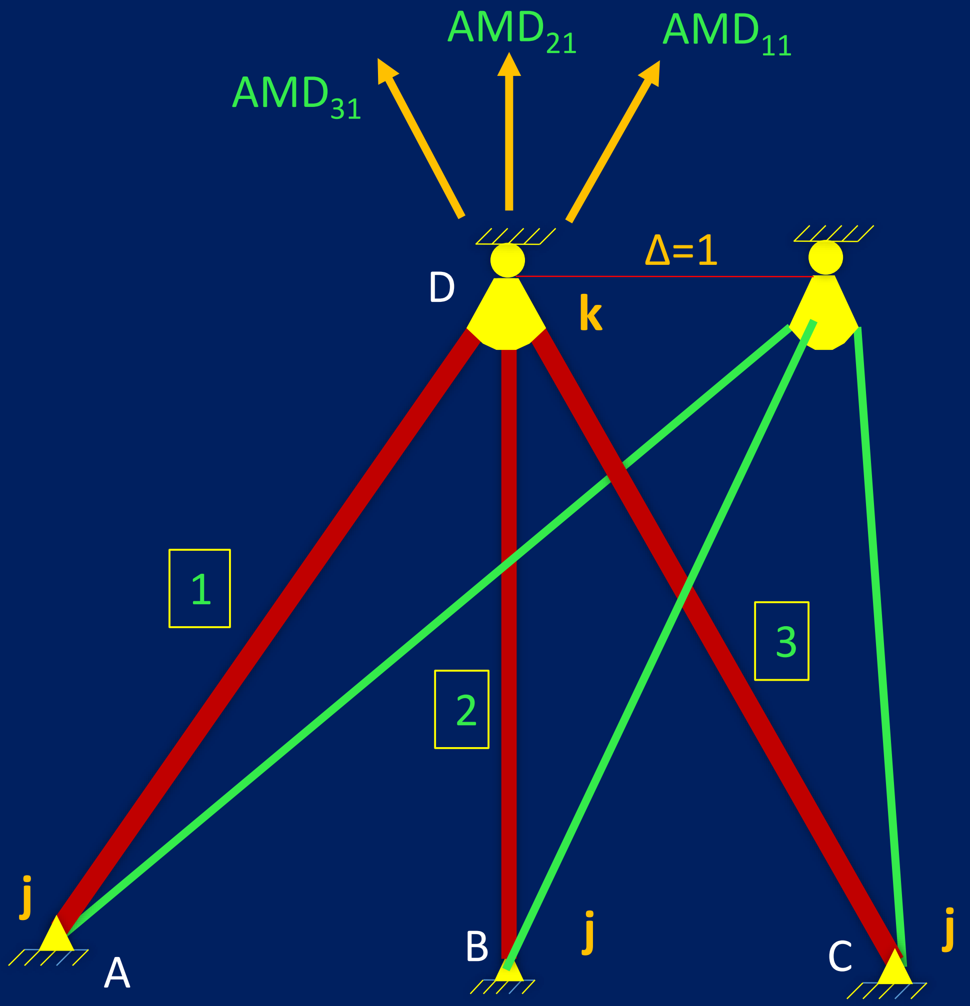
$$[D]_{2*1} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$[AD]_{2*1} = \begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 50 \end{bmatrix}$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = 0$



Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = 0$

$$AMD_{11} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{113.13^2}(80 - 0) = 0.00625EA$$

$$AMD_{21} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{80^2}(80 - 80) = 0$$

$$AMD_{31} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{113.13^2}(80 - 160) = -0.00625EA$$

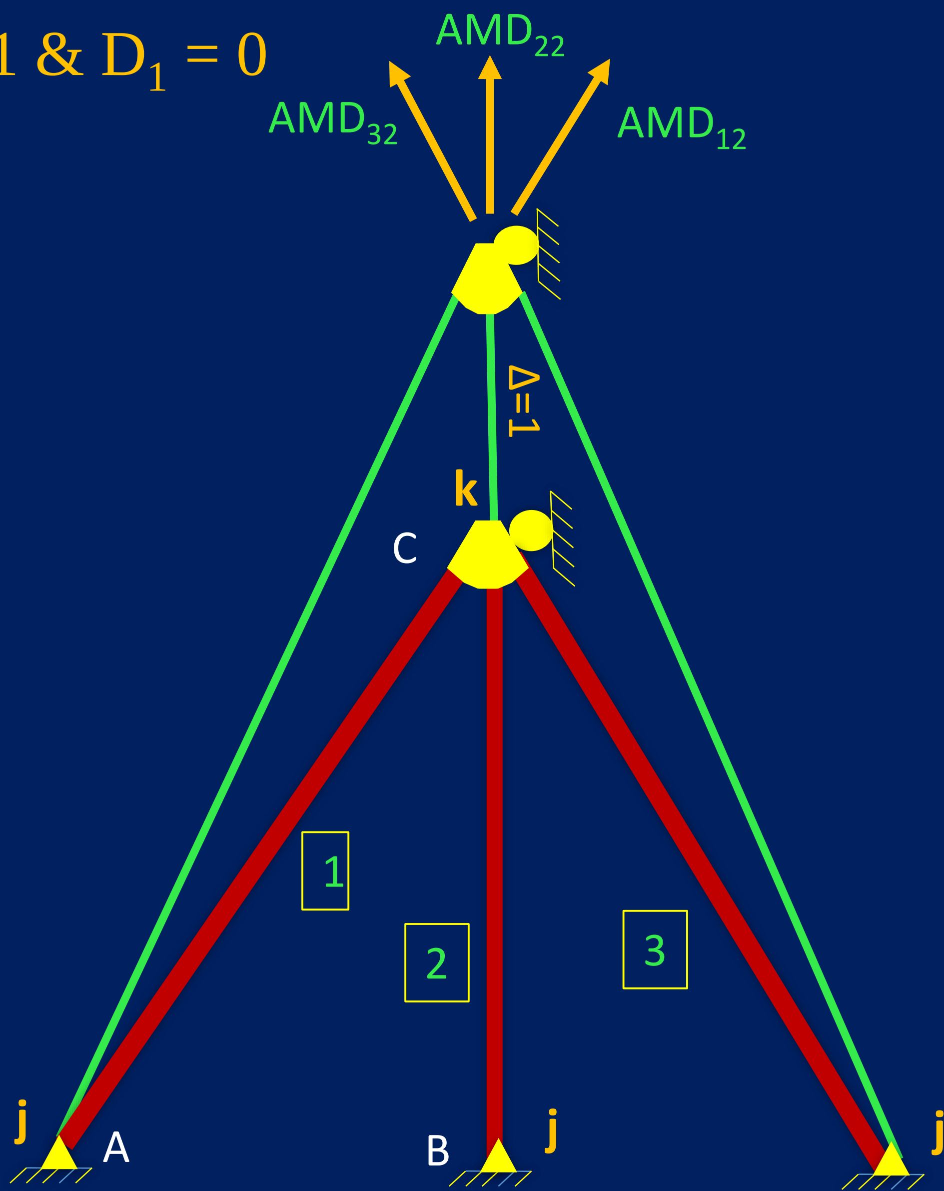
$$S_{11} = \frac{EA}{L^3}(x_k - xj)^2 = \frac{EA}{113.13^3}(80 - 0)^2 + \frac{EA}{80^3}(80 - 80)^2 + \frac{EA}{113.13^3}(80 - 160)^2 = 0.00884EA$$

$$S_{21} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{113.13^3}(80 - 0)(80 - 0) + \frac{EA}{80^3}(80 - 80)(80 - 0) + \frac{EA}{113.13^3}(80 - 160)(80 - 0) = 0$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = 0$



Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = 0$

$$AMD_{12} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{113.13^2}(80 - 0) = 0.00625EA$$

$$AMD_{22} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{80^2}(80 - 0) = 0.125EA$$

$$AMD_{32} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{113.13^2}(80 - 0) = 0.00625EA$$

$$S_{21} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{113.13^3}(80 - 0)(80 - 0) + \frac{EA}{80^3}(80 - 80)(80 - 0) + \frac{EA}{113.13^3}(80 - 160)(80 - 0) = 0$$

$$S_{22} = \frac{EA}{L^3}(y_k - yj)^2 = \frac{EA}{113.13^3}(80 - 0)^2 + \frac{EA}{80^3}(80 - 0)^2 + \frac{EA}{113.13^3}(80 - 0)^2 = 0.0213EA$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

AMD matrix will be

$AMD_{11} = 0.00625EA$ $AMD_{21} = 0$ $AMD_{31} = -0.00625EA$

$AMD_{12} = 0.00625EA$ $AMD_{22} = 0.0125EA$ $AMD_{32} = 0.00625EA$

$$[\text{ AMD }] = \text{EA}\begin{bmatrix} 0.00625 & 0.00625 \\ 0 & 0.0125 \\ -0.00625 & -0.00625 \end{bmatrix}$$

Stiffness matrix will be

$S_{11} = 0.0084EA$ $S_{21} = 0$ $S_{12} = 0$ $S_{22} = 0.0213EA$

$$[\, S \,] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad [\, \text{S} \,] = EA\begin{bmatrix} 0.0084 & 0 \\ 0 & 0.0213 \end{bmatrix}$$

Stiffness Method for Truss Analysis

Step # 03: Apply equilibrium condition at the location of the redundant joint displacement to write equilibrium equations and solve for unknown joint displacement.

$$AD_1 = S_{11}D_1 + S_{12}D_2$$

$$AD_2 = S_{21}D_1 + S_{22}D_2$$

$$\begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$[AD]_{2 * 1} = [S]_{2 * 2} \bullet [D]_{2 * 1}$$

$$[\boldsymbol{D}] = [\boldsymbol{S}]^{-1} \bullet [\boldsymbol{AD}]$$

Stiffness Method for Truss Analysis

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^{-1} \begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix}$$

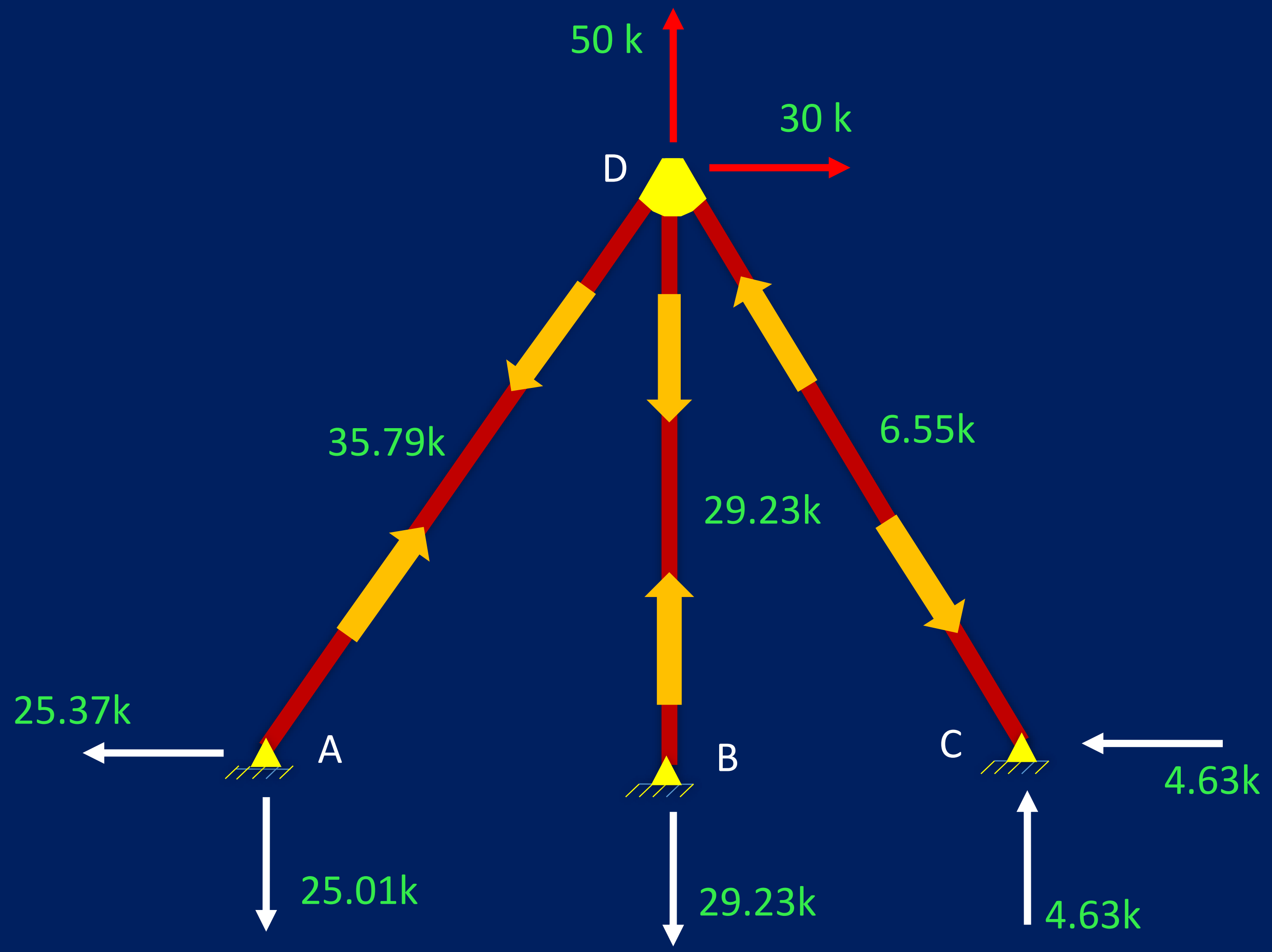
$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 0.0084 & 0 \\ 0 & 0.0213 \end{bmatrix}^{-1} \begin{bmatrix} 30 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 3387.30 \\ 2338.62 \end{bmatrix}$$

$$\begin{bmatrix} AM_1 \\ AM_2 \\ AM_3 \end{bmatrix} = \begin{bmatrix} AMD_{11} & AMD_{12} \\ AMD_{21} & AMD_{22} \\ AMD_{31} & AMD_{32} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$\begin{bmatrix} AM_1 \\ AM_2 \\ AM_3 \end{bmatrix} = \mathrm{EA} \begin{bmatrix} 0.00625 & 0.00625 \\ 0 & 0.0125 \\ -0.00625 & -0.00625 \end{bmatrix} \begin{bmatrix} 3387.30 \\ 2338.62 \end{bmatrix} \frac{1}{EA} = \begin{bmatrix} 35.79k \\ 29.23k \\ -6.55k \end{bmatrix}$$

Stiffness Method for Trusses Analysis



Final Analyzed structure

Stiffness Method for Trusses Analysis

Problem 04: Analyze the given pin jointed frame using stiffness method.

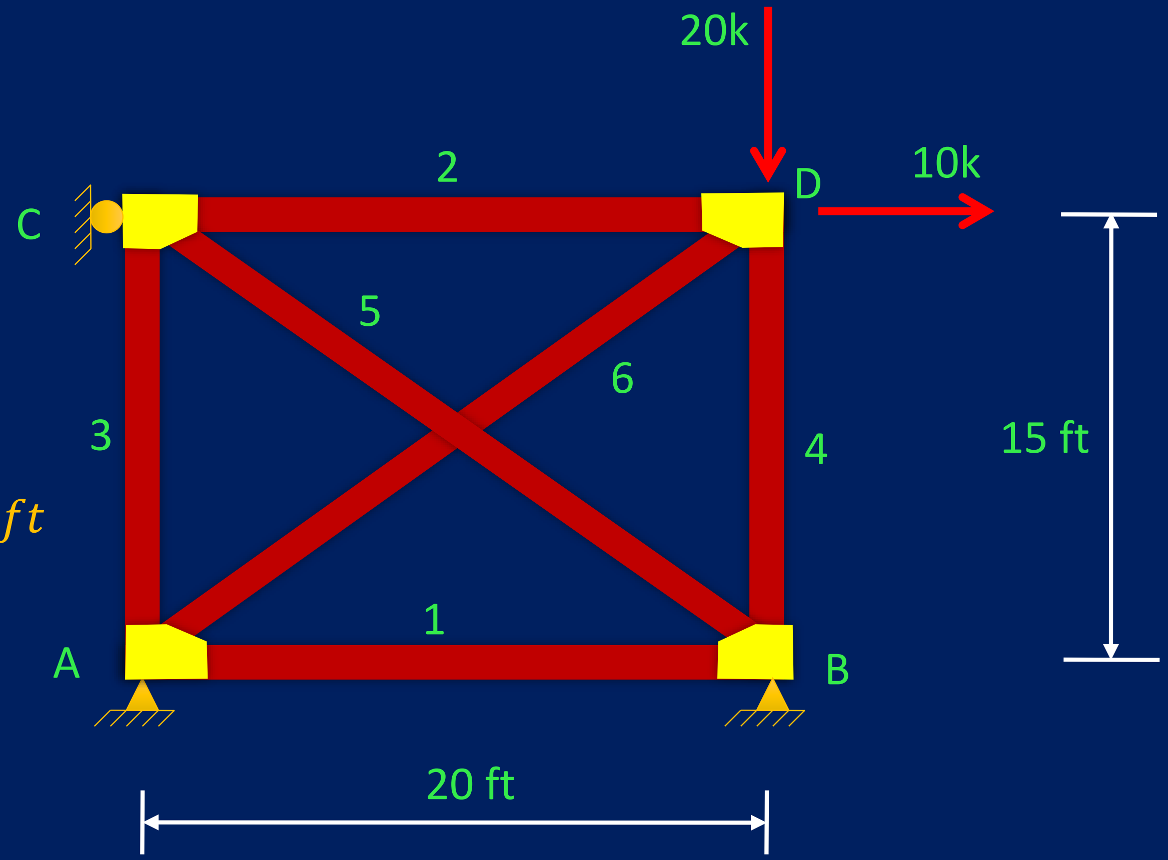
K.I= 3 degrees

Take EA = constant

$L_5 = L_6 = \sqrt{20^2 + 15^2} = 25ft$

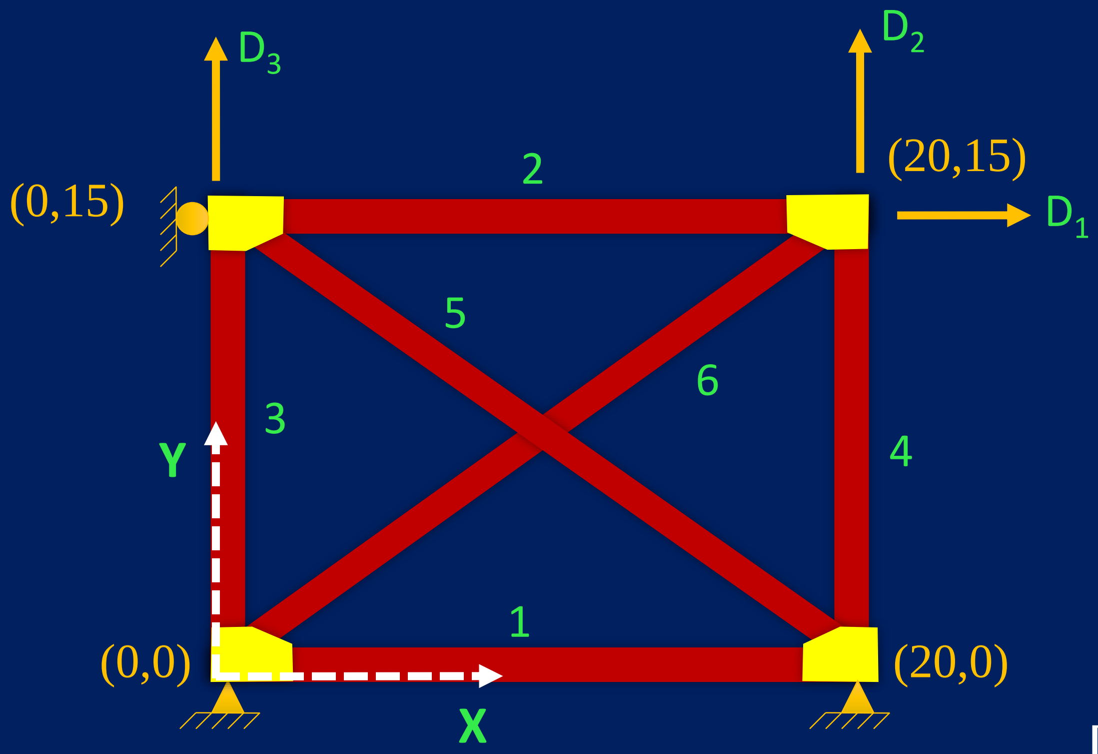
$L_1 = L_2 = 20ft$

$L_3 = L_4 = 15ft$



Stiffness Method for Trusses Analysis

- **Step # 01:** Identify the unknown joint displacements and compute the values of [AD] matrix.



Chose one point as an origin and assign coordinates to each joint w.r.t the chosen origin. Here point c is taken as origin.

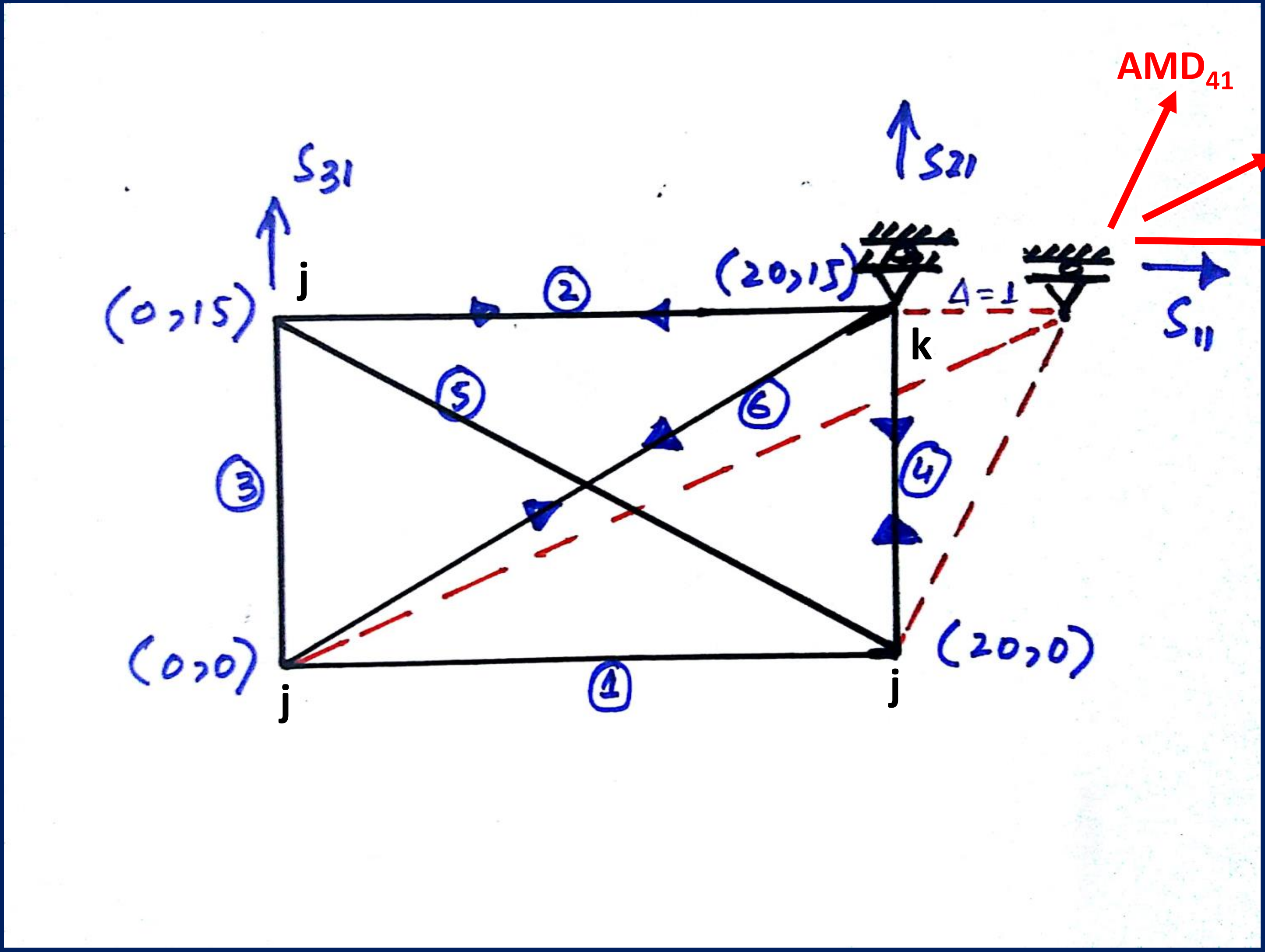
$$[D]_{3*1} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

$$[AD]_{3*1} = \begin{bmatrix} AD_1 \\ AD_2 \\ AD_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -20 \\ 0 \end{bmatrix}$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = D_3 = 0$



Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

i. When $D_1 = 1$ & $D_2 = D_3 = 0$

$$AMD_{21} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{20^2}(20 - 0) = 0.05EA$$

$$AMD_{41} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{15^2}(20 - 20) = 0$$

$$AMD_{61} = \frac{EA}{L^2}(x_k - xj) = \frac{EA}{25^2}(20 - 0) = 0.032EA$$

$$AMD_{11} = AMD_{31} = AMD_{51} = 0 \text{ (By observation as there is no change in length)}$$

$$S_{11} = \frac{EA}{L^3}(x_k - xj)^2 = \frac{EA}{20^3}(20 - 0)^2 + \frac{EA}{25^3}(20 - 0)^2 + \frac{EA}{15^3}(20 - 20)^2 = 0.00884EA$$

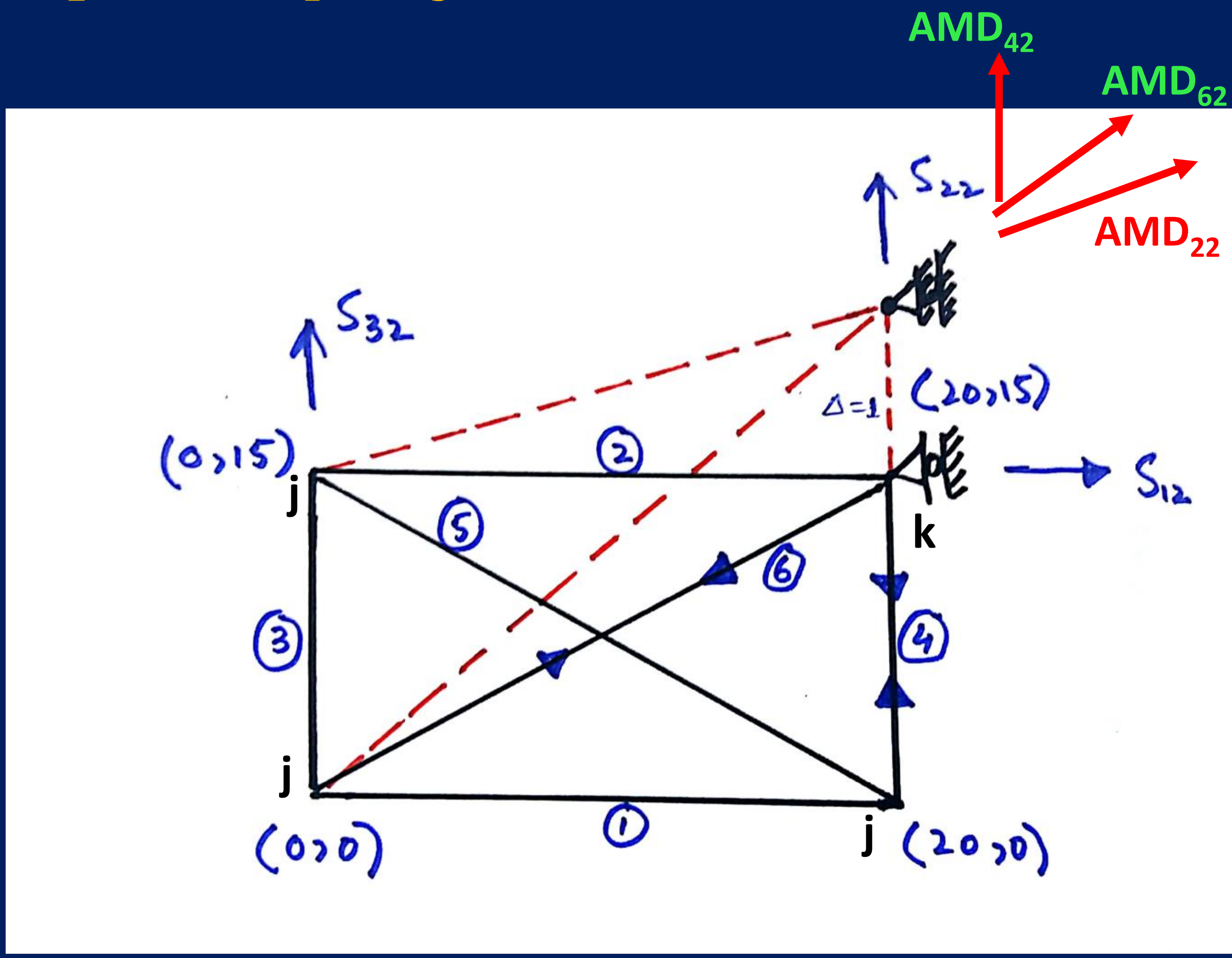
$$S_{21} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{20^3}(20 - 0)(15 - 15) + \frac{EA}{25^3}(20 - 0)(15 - 0) + \frac{EA}{15^3}(20 - 20)(15 - 0) = 0.0192EA$$

$$S_{31} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{20^3}(20 - 0)(15 - 15) = 0$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = D_3 = 0$



Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

ii. When $D_2 = 1$ & $D_1 = D_3 = 0$

$$AMD_{22} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{20^2}(15 - 15) = 0$$

$$AMD_{42} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{15^2}(15 - 0) = 0.066EA$$

$$AMD_{62} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{25^2}(15 - 0) = 0.024EA$$

$$AMD_{12} = AMD_{32} = AMD_{52} = 0(\textit{ By observation as there is no change in length})$$

$$S_{12} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{20^3}(20 - 0)(15 - 15) + \frac{EA}{25^3}(20 - 0)(15 - 0) + \frac{EA}{15^3}(20 - 20)(15 - 0) = 0.0192EA$$

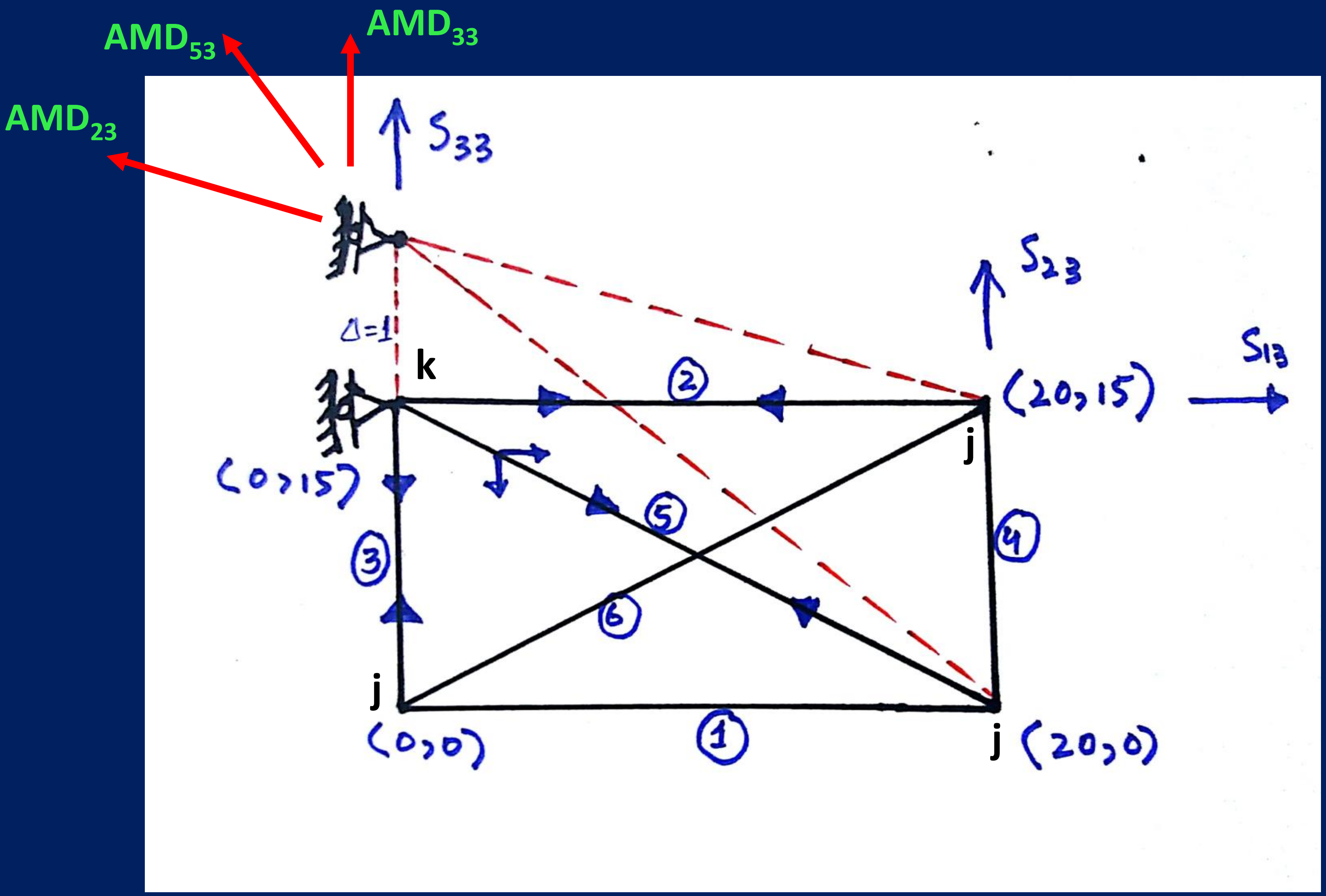
$$S_{22} = \frac{EA}{L^3}(y_k - yj)^2 = \frac{EA}{20^3}(15 - 15)^2 + \frac{EA}{15^3}(15 - 0)^2 + \frac{EA}{25^3}(15 - 0)^2 = 0.0810EA$$

$$S_{32} = \frac{EA}{L^3}(y_k - yj)^2 = \frac{EA}{20^3}(15 - 15)^2 = 0$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

iii. When $D_3 = 1$ & $D_1 = D_2 = 0$



Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

iii. When $D_3 = 1$ & $D_1 = D_2 = 0$

$$AMD_{23} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{15^2}(15 - 15) = 0$$

$$AMD_{33} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{15^2}(15 - 0) = 0.066EA$$

$$AMD_{53} = \frac{EA}{L^2}(y_k - yj) = \frac{EA}{25^2}(15 - 0) = 0.024EA$$

$AMD_{13} = AMD_{43} = AMD_{63} = 0$ (*By observation as there is no change in length*)

$$S_{13} = \frac{EA}{L^3}(x_k - xj)(y_k - yj) = \frac{EA}{20^3}(20 - 0)(15 - 15) = 0$$

$$S_{23} = \frac{EA}{L^3}(y_k - yj)^2 = \frac{EA}{20^3}(15 - 15)^2 = 0$$

$$S_{33} = \frac{EA}{L^3}(y_k - yj)^2 = \frac{EA}{15^3}(15 - 0)^2 + \frac{EA}{25^3}(15 - 0)^2 + + \frac{EA}{20^3}(15 - 15)^2 = 0.08106EA$$

Stiffness Method for Trusses Analysis

- **Step # 02:** Computation of AMD and stiffness matrices.

AMD matrix will be

$$[\,AMD\,] = EA\begin{bmatrix} 0 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.066 \\ 0 & 0.066 & 0 \\ 0 & 0 & 0.024 \\ 0.032 & 0.024 & 0 \end{bmatrix}$$

Stiffness matrix will be

$$[\,S\,] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$[\,S\,] = EA\begin{bmatrix} 0.0756 & 0.0192 & 0 \\ 0.0192 & 0.08106 & 0 \\ 0 & 0 & 0.08106 \end{bmatrix}$$

Stiffness Method for Truss Analysis

Step # 03: Apply equilibrium condition at the location of the redundant joint displacement to write equilibrium equations and solve for unknown joint displacement.

$$AD_1 = S_{11}D_1 + S_{12}D_2 + S_{13}D_3$$

$$AD_2 = S_{21}D_1 + S_{22}D_2 + S_{23}D_3$$

$$AD_3 = S_{31}D_1 + S_{32}D_2 + S_{33}D_3$$

$$\begin{bmatrix} AD_1 \\ AD_2 \\ AD_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

$$[\boldsymbol{D}] = [\boldsymbol{S}]^{-1} \bullet [\boldsymbol{AD}]$$

Stiffness Method for Truss Analysis

So the unknown joint displacement will be

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}^{-1} \begin{bmatrix} AD_1 \\ AD_2 \\ AD_3 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 0.0756 & 0.0192 & 0 \\ 0.0192 & 0.08106 & 0 \\ 0 & 0 & 0.08106 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ -20 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 207.41 \\ -295.86 \\ 0 \end{bmatrix}$$

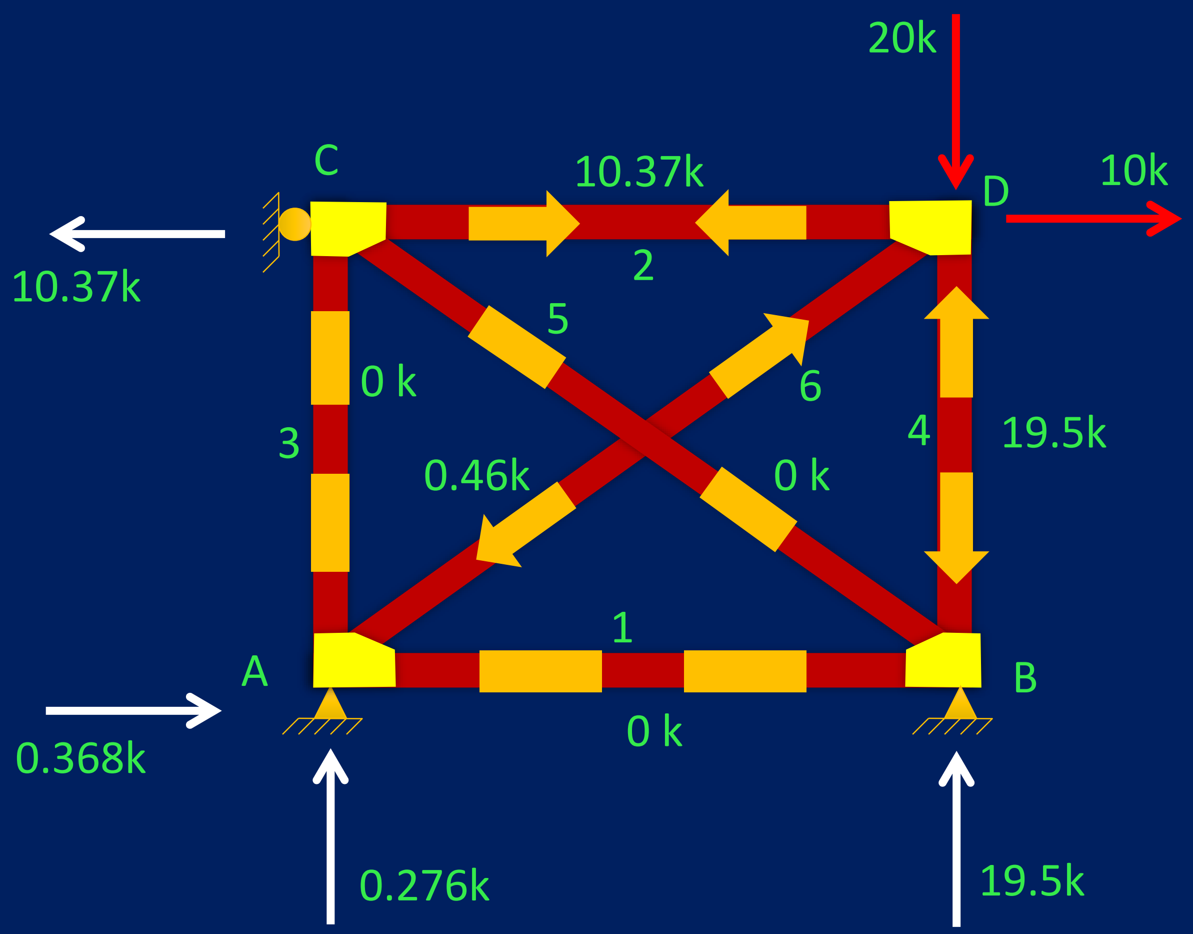
Stiffness Method for Truss Analysis

The Member forces will be

$$\begin{bmatrix} AM_1 \\ AM_2 \\ AM_3 \\ AM_4 \\ AM_5 \\ AM_6 \end{bmatrix} = \begin{bmatrix} AMD_{11} & AMD_{12} & AMD_{13} \\ AMD_{21} & AMD_{22} & AMD_{23} \\ AMD_{31} & AMD_{32} & AMD_{33} \\ AMD_{41} & AMD_{42} & AMD_{43} \\ AMD_{51} & AMD_{52} & AMD_{53} \\ AMD_{61} & AMD_{62} & AMD_{63} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

$$\begin{bmatrix} AM_1 \\ AM_2 \\ AM_3 \\ AM_4 \\ AM_5 \\ AM_6 \end{bmatrix} = \text{EA} \begin{bmatrix} 0 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.066 \\ 0 & 0.066 & 0 \\ 0 & 0 & 0.024 \\ 0.032 & 0.024 & 0 \end{bmatrix} \begin{bmatrix} 207.41 \\ -295.86 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10.37\ k \\ 0 \\ -19.5\ k \\ 0 \\ -0.46\ k \end{bmatrix}$$

Stiffness Method for Trusses Analysis



Final Analyzed structure

References

- Structural Analysis by R. C. Hibbeler
- Matrix structural analysis by William Mc Guire
- Matrix analysis of frame structures by William Weaver
- Online Civil Engineering blogs