University of Engineering & Technology Peshawar, Pakistan

CE301: Structure Analysis II

Module 05:

Introduction to Stiffness Method (displacement method)

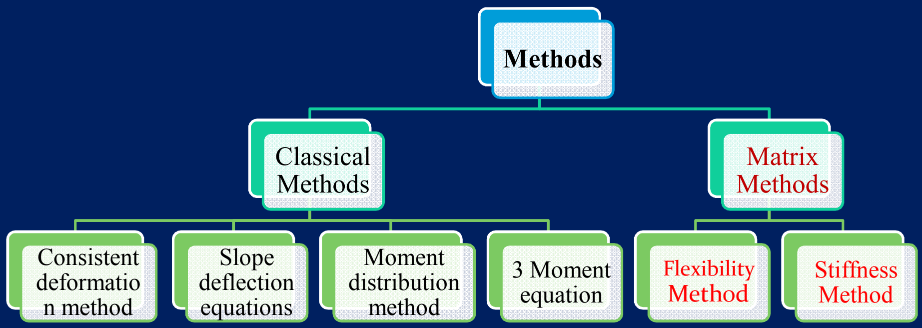
By:
Prof. Dr. Bashir Alam
Civil Engineering Department
UET , Peshawar

Topics to be Covered

- Methods of analysis for indeterminate structures
- Matrix analysis and methods
- Introduction to Stiffness method of analysis
- Stiffness of a member
- Kinematic indeterminacy
- General procedure of stiffness method
- Step wise procedure of stiffness method
- Example 1 & 2 (comparison of stiffness & flexibility method)
- Computation of member end actions
- Fixed end actions

Analysis of S.I Structures

Following are the methods used for the analysis of indeterminate structures as discussed earlier.



Analysis of S.I Structures

There are two ways to analyze indeterminate structures

1. Force Based Analysis
2. Deformation Based Analysis

- Force Based Analysis

  It involves the calculation of reaction of the supports and determination of internal action (Normal force, shear force, and bending moment) within the structure.

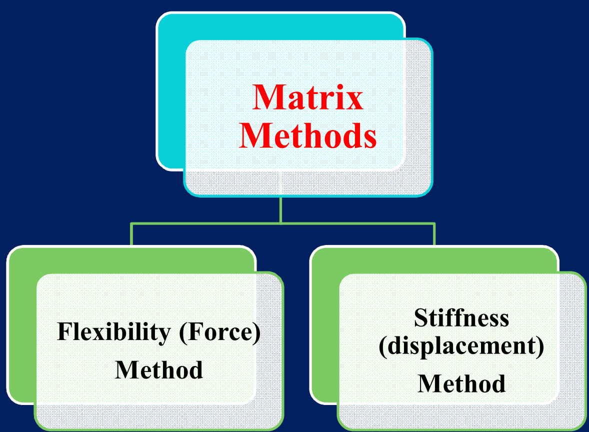
- Deformation Based Analysis

  Deformation analysis involves the evaluation of deformation (displacements and strains) of the elements of a structure as well as whole of the structure.

Matrix Methods of Analysis

- Matrix analysis of a structure is a branch of structural analysis in which matrix algebra is used as a tool for the analysis of structure.
- Objective of analysis of structure is to predict the behavior of a structure due to external loads, temperature changes, settlement of supports, vibrations etc.…

Matrix Methods of Analysis



Matrix Methods of Analysis

- **Flexibility Method**

In this method redundant constraints are removed and corresponding redundant actions (forces or moment) are placed. An equation of compatibility of deformation is written in terms of these redundants and corresponding displacements. The redundants are determined from these simultaneous equation. Equations of statics are then used for calculation of designed internal actions. In this method, actions are treated as basic unknowns.

Matrix Methods of Analysis

- **Stiffness or Displacement Method**

In the displacement method of analysis the equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load displacement relations. The unknown joint displacements (the degrees of freedom of the structure) are calculated by solving equilibrium equations. displacement method of analysis is preferred to computer implementation.

Stiffness Method of Analysis

❑ **Stiffness:**

Stiffness of a member measure its resistance to deformation.

The stiffness of a member is defined as the force which is to be applied at some point to produce a unit displacement when all other displacement are restrained to be zero.

If a member which behaves elastically is subjected to varying axial tensile load (W) as shown in fig. 5.1 and a graph is drawn of load (W) versus displacement (Δ) the result will be a straight line as shown in fig. 5.2, the slope of this line is called stiffness.

Stiffness Method of Analysis

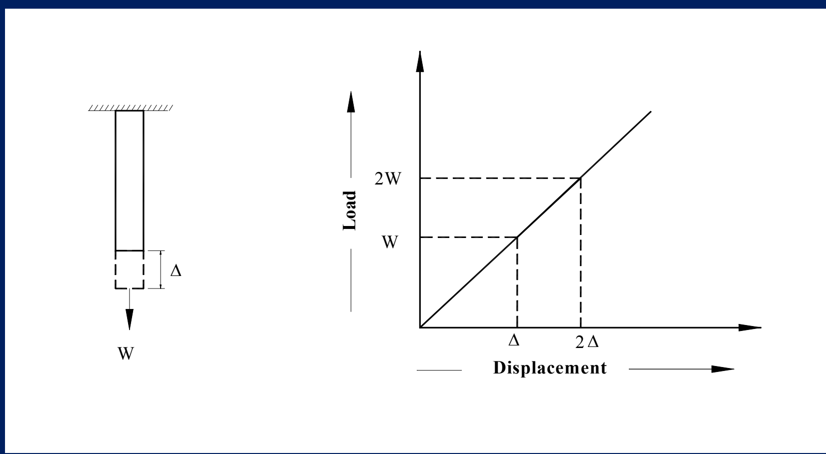


Fig.5.1
(Members subjected to varying axial load)

Fig.5.2
(load vs displacement graph)

Stiffness Method of Analysis

Mathematically it can be expressed as

$$S = W/\Delta \qquad \text{----------5.1}$$

In other words Stiffness 'S' is the force required at a certain point to cause a unit displacement at that point.

Equation 5.1 can be written in the following form

$$W = S\,\Delta \qquad \text{--------- 2.2}$$

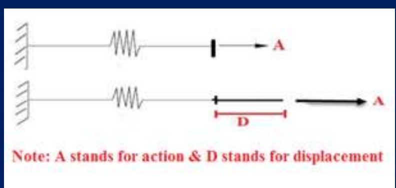
Where,

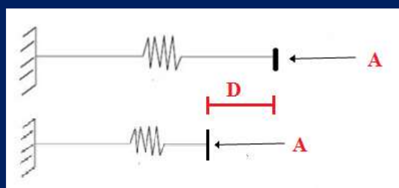
W = Force at a particular point
S = Stiffness (k)
Δ = Unit displacement of the particular point

The above equation relates the force and displacement at a single point. This can be extended for the development of a relationship between load and displacement for more than one point on a structure

Stiffness Method of Analysis

- Relationship between flexibility & stiffness:





Since the spring is linearly elastic i.e actions are directly proportional to displacement i.e

$$D \propto A \;\Rightarrow\; D = f\,A \text{ ---------(a)} \;\Rightarrow\; f = \frac{D}{A}$$

- Where f is the flexibility of spring

Also ;

$$A \propto D \;\Rightarrow\; A = s\,D \text{ ---------(b)} \;\Rightarrow\; s = \frac{A}{D}$$

- Where s is called stiffness of spring

Stiffness Method of Analysis

As from equation “b” in previous slide

$$A = sD$$

Also from equation “a”

$$D = fA$$

putting the value of “A” from equation “b” in equation “a” we will get

$$D = fsD \quad \Rightarrow \quad 1 = fs$$

from this

$$f = \frac{1}{s} \quad \& \quad s = \frac{1}{f}$$

From this it is clear that flexibility and stiffness are the reciprocal of each other.

Stiffness Method of Analysis

- **Kinematic indeterminacy**

Degree of kinematical indeterminacy is the number of unconstrained degrees of freedom (displacements).

In Kinematic Indeterminacy we measure the total number of degree of freedom possible at joints. A structure is said to be kinematically indeterminate if the displacement components of its joints cannot be determined by compatibility conditions alone. i.e. no. of unknown joint displacements over and above the compatibility conditions will give the degree of kinematic indeterminacy.

Stiffness Method of Analysis

K.I for 2 dimension,

- Fixed support gives 3 reactions , one rotation and 2 translation hence degree of freedom at fixed support is 0,
- Hinge gives 2 reaction both translations and no rotation reaction hence degree of freedom at hinge is 1 , rotation.
- Roller gives 1 reaction in translation direction hence degree of freedom at roller is 2 , one rotation and one translation.
- For a 2 dimension rigid joint or free end the number of degree of freedom at a joint is 3 (one rotation and 2 translation).
- For internal hinge, the degree of freedom is 4, two rotational and 2 translational.

Stiffness Method of Analysis

K.I for 3 dimension,

- Fixed support gives 6 reactions , 3 rotation and 3 translation hence degree of freedom at fixed support is 0,
- Hinge gives 3 reactions all in translations and no rotation reaction hence degree of freedom at hinge is 3 , rotation in all directions.
- Roller gives 1 reaction in translation in one direction hence degree of freedom at roller is 5 , 3 rotations and 2 translation.
- For a 3 dimension rigid joint or free end the number of degree of freedom at a joint is 6 (3 rotation and 3 translation).
- For internal hinge, the degree of freedom is 9, 6 rotations and 3 translations.

Stiffness Method of Analysis

❑ Kinematic indeterminacy

Generally for a planer structure

$$K.I = 3j - r + i$$

where

j = no. of joints
r = no. of reactions
i = no. of internal hinges.

Ignoring axial deformation ⇒ if we ignore axial deformation than the kinematic indeterminacy will decrease.(why we are ignoring axial deformations? Any idea)

Kinematic indeterminacy after ignoring the axial deformation is

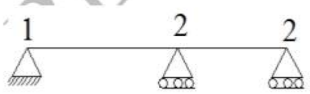
$$K.I = 3j - r + i - No\ of\ members$$

Stiffness Method of Analysis

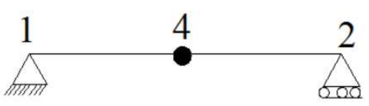
❑ Kinematic indeterminacy

K.I = 0+3=3 Ignoring axial deformation K.I = 3-1=2
OR j=2 r=3 K.I=3*2-3=3 Ignoring axial deformation K.I = 3-1=2

K.I = 1+2=3 Ignoring axial deformation K.I = 3-1=2
OR j=2 r=3 K.I=3*2-3=3 Ignoring axial deformation K.I = 3-1=2



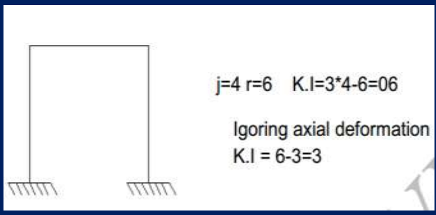
K.I = 1+2+2=5 Ignoring axial deformation K.I = 5-2=3
OR j=3 r=4 K.I=3*3-4=5 Ignoring axial deformation K.I = 5-2=3

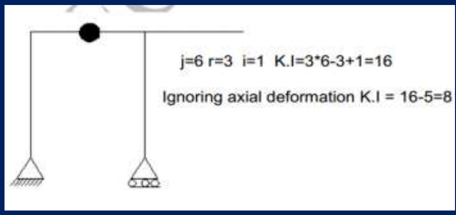


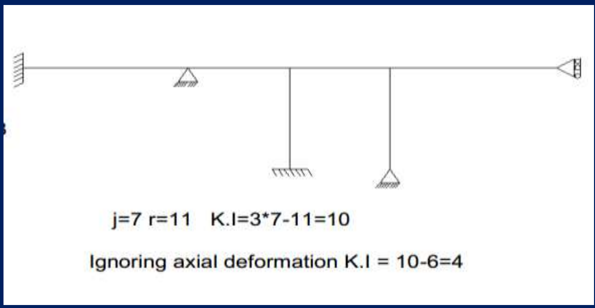
K.I = 1+4+2=7 Ignoring axial deformation K.I = 7-2=5
OR j=3 r=3 i=1 K.I=3*3-3+1=7
Ignoring axial deformation K.I = 7-2=5

Stiffness Method of Analysis

- **Kinematic indeterminacy**







Stiffness Method of Analysis

- **General procedure of stiffness method**

This method is also called displacement method. In this method the degree of K.I is determined and the displacement components are identified. A coordinate is assigned to each independent displacement component. Thus D_1, D_2, D_3.........D_n are the redundant displacement at 1,2,3..........n locations.

If all the redundant displacements restrained the resultant structure known as Restrained structure or Basic kinematically determinate structure.

Stiffness Method of Analysis

It first requires satisfying equilibrium equations for the structure. To do this the unknown displacements are written in terms of the loads by using the load-displacement relations, then these equations are solved for the displacements. Once the displacements are obtained, the unknown loads are determined from the compatibility equations using the load-displacement relations. For n independent displacements

$$AD_1 = ADL_1 + S_{11}D_1 + S_{12}D_2 + \ldots \ldots \ldots \ldots + S_{1nDn}$$

$$AD_2 = ADL_2 + S_{21}D_1 + S_{22}D_2 + \ldots \ldots \ldots \ldots + S_{2n}D_n$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$AD_n = ADL_n + S_{n1}D_1 + S_{n2}D_2 + \ldots \ldots \ldots \ldots + S_{nn}D_n$$

Stiffness Method of Analysis

Writing the equations on the previous slide in matrix form called matrix formulation of Stiffness method

$$\begin{bmatrix} AD_1 \\ AD_2 \\ \vdots \\ AD_n \end{bmatrix} = \begin{bmatrix} ADL_1 \\ ADL_2 \\ \vdots \\ ADL_n \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} & \ldots & S_{1n} \\ S_{21} & S_{22} & \ldots & S_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ S_{n1} & S_{n2} & \ldots & S_{nn} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix}$$

$$[\boldsymbol{AD}]_{\mathbf{n}\,*\,\mathbf{1}} = [\boldsymbol{ADL}]_{\mathbf{n}\,*\,\mathbf{1}} + [\boldsymbol{S}]_{\mathbf{n}\,*\,\boldsymbol{n}} \bullet [\boldsymbol{D}]_{\mathbf{n}\,*\,\mathbf{1}}$$

$$[\boldsymbol{D}]_{\mathbf{n}\,*\,\mathbf{1}} = [\boldsymbol{S}]^{-\mathbf{1}}{}_{\mathbf{n}\,*\,\boldsymbol{n}} \bullet [\boldsymbol{AD} - \boldsymbol{ADL}]_{\mathbf{n}\,*\,\mathbf{1}}$$

Where **AD =** External actions corresponding to the displacement joints

Stiffness Method of Analysis

ADL = **Actions** corresponding to the displacement joints in the restrained structure due to applied loads.(Fixed end actions)

D = The unknown joint displacements

S = Stiffness coefficient i.e displacement per unit force/action

Stiffness Method of Analysis

- ❑ **Step wise Solution Procedure using Stiffness method method:**

The following steps must be followed while solving a structure using Stiffness method.

- **Step # 01:** Make the structure kinametically determinate, by restraining the joints i.e select the redundant joint displacement.

Stiffness Method of Analysis

Step # 02: Apply the actual external loads on the BKDS (Basic kinametically determinate structure) and find the actions at the locations of redundant joints (compute fixed end actions) this will generate ADL matrix.

Step # 03: Apply the redundant joint displacement on the BKDS (To standardize the procedure, only a unit displacement is applied in the +ve direction) this will generate stiffness coefficient matrix.

Stiffness Method of Analysis

Step # 04: Apply equilibrium condition at the location of the redundant joint displacement to write equilibrium equations and solve for unknown joint displacement.

$$[\boldsymbol{AD}] = [\boldsymbol{ADL}] + [\boldsymbol{S}] \bullet [\boldsymbol{D}]$$

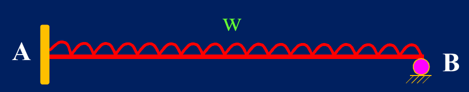
$$[\boldsymbol{D}] = [\boldsymbol{S}]^{-1} \bullet [\boldsymbol{AD} - \boldsymbol{ADL}]$$

Step # 05: Compute the member end actions .

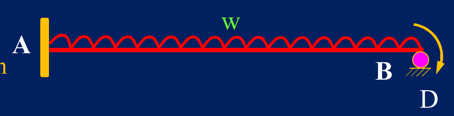
Stiffness Method of Analysis

- **Example: 01** let illustrate the above mentioned procedure with the help of general example.

K.I = 1 degree
So one redundant joint displacement should be chosen



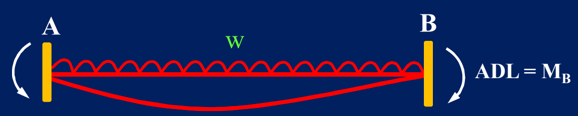
Step#01: selection of redundant joint displacement. Rotation at B is taken as independent joint displacement.



Basic kinametically determinate structure (BKDS) or Restrained structure

Stiffness Method of Analysis

Step#02: BKDS acted upon by actual load. Computation of ADL values.



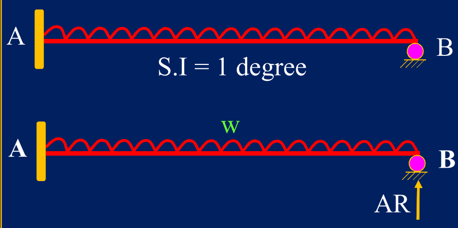
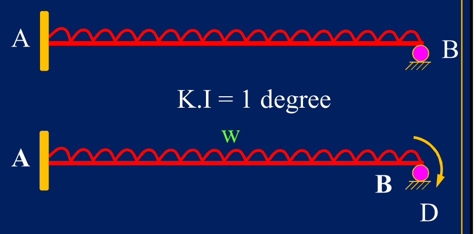
Step#03: BKDS structure acted upon by unit value of redundant joint displacement. Computation of stiffness coefficients values.

Step#04: Apply equilibrium at the location of the redundant joint displacement and solve the matrices to compute the values of unknown joint displacements.

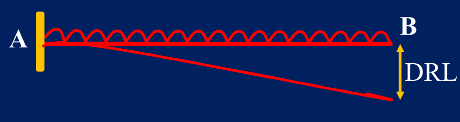
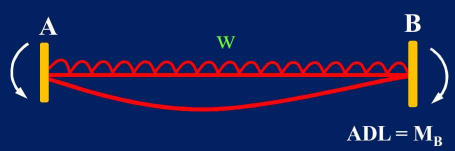
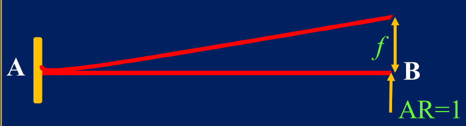
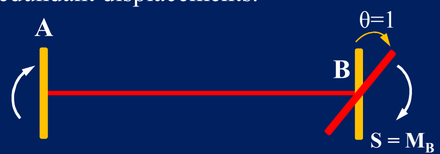
$$[AD] = [ADL] + [S] \bullet [D]$$

$$[D] = [S]^{-1} \bullet [AD - ADL]$$

Stiffness Method of Analysis

Comparison b/w Flexibility & stiffness method	
Flexibility method	**Stiffness method**
1. Determine the degree of S.I ,Compute the DRS values & choose the redundant action to obtain the BDS. 	1. Determine the degree of K.I , Compute the AD values & choose the redundant joint displacement to obtained restrained structure 
2. Assign coordinates to the chosen redundants.	2. Assign coordinates to the independent displacement components.

Stiffness Method of Analysis

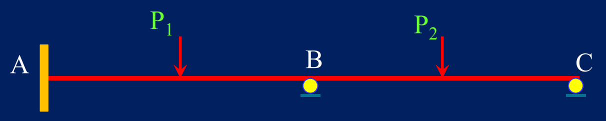
3. In the released structure compute the displacements (DRL) at coordinates due to actual loads. 	3. In the restrained structure compute the actions (ADL) at the coordinates due to the loads other than those acting at the coordinates. 
4. In the released structure compute the displacements (*f*) at coordinates due to the unit redundant actions. 	2. Compute the actions at the coordinates in the restrained structure due to a unit redundant displacements. 

Stiffness Method of Analysis

5. Use compatibility conditions at the coordinates to determine the redundants. $$[DRS] = [DRL] + [f] \bullet [AR]$$ $$[AR] = [f]^{-1} \bullet [DRS - DRL]$$	5. Use equilibrium conditions at the coordinates to determine the independent displacement. $$[AD] = [ADL] + [S] \bullet [D]$$ $$[D] = [S]^{-1} \bullet [AD - ADL]$$
6. Compute member end actions	6. Compute member end actions
7. Draw SF & BM diagrams	7. Draw SF & BM Diagrams

Stiffness Method of Analysis

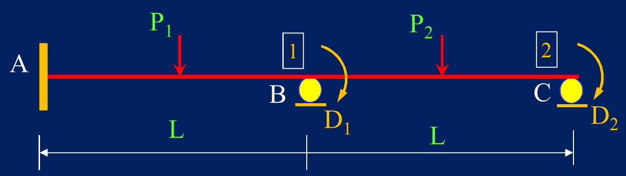
- **Example: 02** Formulate the beam shown in fig below using stiffness method.



K.I = 2 degree
So two independent joint displacement should be chosen.

Stiffness Method of Analysis

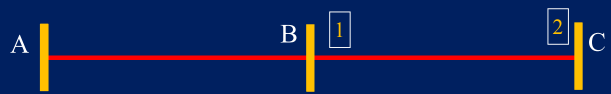
- **Step # 01:** Selection of redundant Joint displacements and assign coordinates at those locations. Also compute AD values.



$$[D] = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$[AD] = \begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

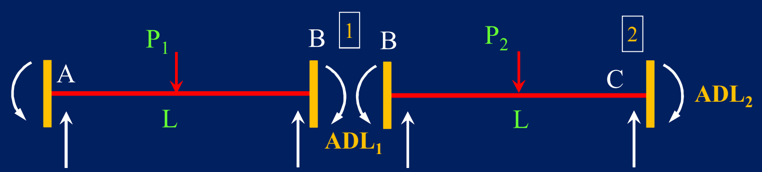
Rotation at B & C is taken as redundant joint displacement.



Basic Kinametically determinate structure (BKDS) or Restrained structure

Stiffness Method of Analysis

- **Step # 02 :** Restrained structure acted upon by the actual loads. compute the values of actions in the restrained structure corresponding to the redundant locations. This will generate ADL matrix.(Fixed end actions)

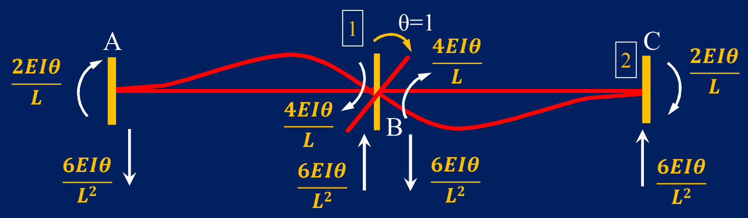


ADL_1 = moment in the restrained structure under the actual loads corresponding to redundant joint displacement 1.
ADL_2 = moment in the restrained structure under the actual loads corresponding to redundant joint displacement 2.

Stiffness Method of Analysis

Step # 03 : Primary structure acted upon by the by a unit value of D & computation of stiffness coefficients values in the BKDS corresponding to the redundant joint displacement locations.

- 1^{st} a unit rotation is applied at location 1 & prevent it at 2 as shown. Compute the values of S_{11} and S_{21} .



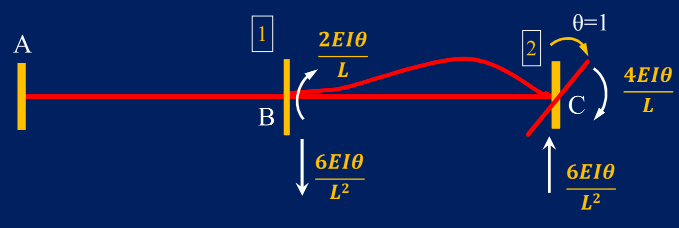
where

S_{11} = Action(sum of moments in this case) in the BKDS at redundant displacement location 1 due to unit rotation at that location.

S_{21} = Moment in the BDS at redundant displacement location 2 due to a unit rotation applied at location 1

Stiffness Method of Analysis

- Now a unit rotation is applied at the redundant displacement location 2 and prevented at 1 as shown. Compute the values of S_{12} & S_{22} .



where

S_{12} = Moment in the BKDS at redundant displacement location 1 due to unit rotation applied at redundant displacement location 2.

S_{22} = Moment in the BKDS at redundant displacement location 2 due to a unit rotation applied at that location.

Stiffness Method of Analysis

Step # 04: Apply equilibrium condition at the location of the redundant joint displacement to write equilibrium equations and solve for unknown joint displacement.

$$AD_1 = ADL_1 + S_{11}D_1 + S_{12}D_2$$

$$AD_2 = ADL_2 + S_{21}D_1 + S_{22}D_2$$

$$\begin{bmatrix} AD_1 \\ AD_2 \end{bmatrix} = \begin{bmatrix} ADL_1 \\ ADL_2 \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$[AD]_{2*1} = [ADL]_{2*1} + [S]_{2*2} \bullet [D]_{2*1}$$

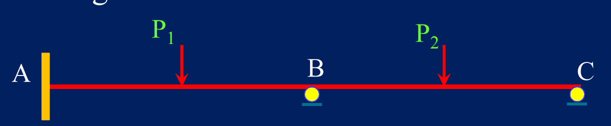
$$[\boldsymbol{D}] = [\boldsymbol{S}]^{-1} \bullet [\boldsymbol{AD - ADL}]$$

Step # 05: Compute the member end actions, same procedure as discussed in flexibility method.

Stiffness Method of Analysis

- **Member End Actions:** Member end actions are the moments and shear at each point where its going to be changed. We can find the member end actions using stiffness method when the redundant joint displacements are known.

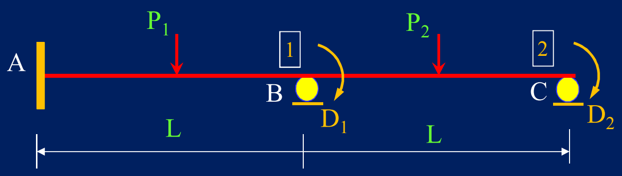
- Let us illustrate the procedure of finding member actions by stiffness method with the help of the above mentioned Example as shown in fig.



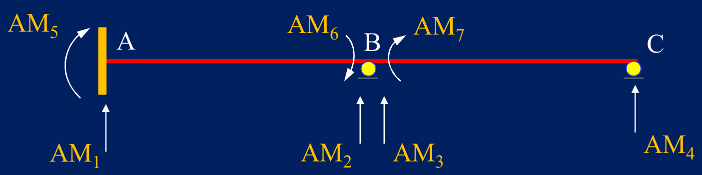
K.I = 2 degrees

Stiffness Method of Analysis

Joint rotation at B & C is taken as redundant joint displacements



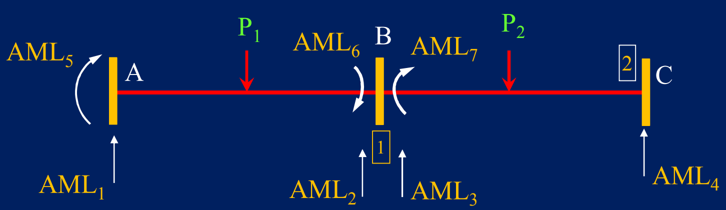
Step # 01: Specify the member end actions AM_n by assigning a No. to them. e.g. in the above problem



AM_n = is the member end action in the indeterminate structure at different location specified with a number "n". n shows the number of member end actions.

Stiffness Method of Analysis

Step # 02: Determine the member end actions (shear & moments) in the restrained structure under the actual applied loads (AML_n) at the specified locations.

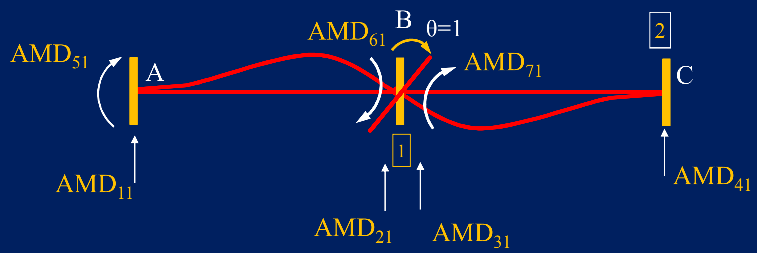


AML_n = is the member end actions in the restrained structure at location "n" when acted upon by the actual applied loads.

Stiffness Method of Analysis

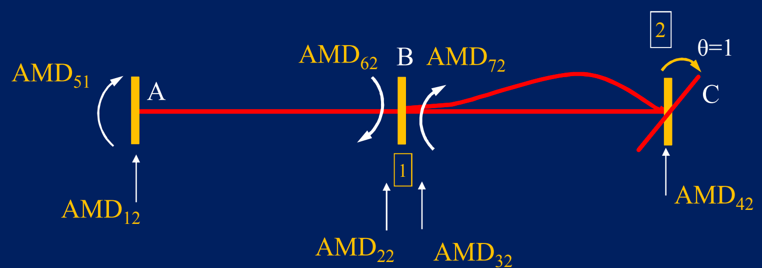
Step # 03: Determine the member end actions (shear & moments) in the restrained structure when acted upon by a unit at displacement 1st at redundant displacement location 1 and then at 2 (AMD_{n1} & AMD_{n2}).

When a unit action is applied at the redundant location 1 as shown.



Stiffness Method of Analysis

When a unit rotations is applied at the redundant location 2 and prevented at1 as shown.



AMD_{n1} & AMD_{n2} = is the member end actions in the restrained structure when acted upon by a unit displacement.
n shows the number member action and 1 & 2 shows the unit displacement position.

Stiffness Method of Analysis

$$AM_1 = AML_1 + AMD_{11}D_1 + AMD_{12}D_2$$

$$AM_2 = AML_2 + AMD_{21}D_1 + AMD_{22}D_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$AM_7 = AML_7 + AMD_{71}D_1 + AMD_{72}D_2$$

$$\begin{bmatrix} AM_1 \\ AM_2 \\ \vdots \\ AM_7 \end{bmatrix} = \begin{bmatrix} AML_1 \\ AML_2 \\ \vdots \\ AML_7 \end{bmatrix} + \begin{bmatrix} AMR_{11} & AMR_{12} \\ AMR_{21} & AMR_{22} \\ \vdots & \vdots \\ AMR_{71} & AMR_{72} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

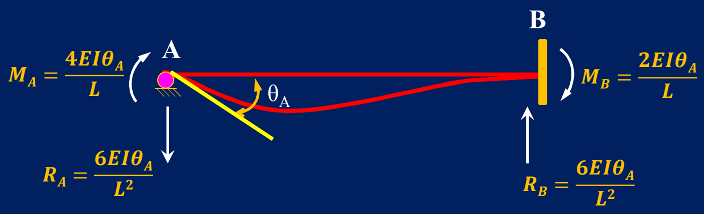
$$[AM] = [AML] + [AMD][D]$$

This method will be explained again in the coming problem

Stiffness Method of Analysis

- Actions produced in member due to a unit displacement:

Consider the beam in which is pinned at one end and fixed at the other. If a unit rotation is applied at joint A then the moment induced in the member is shown in the fig.



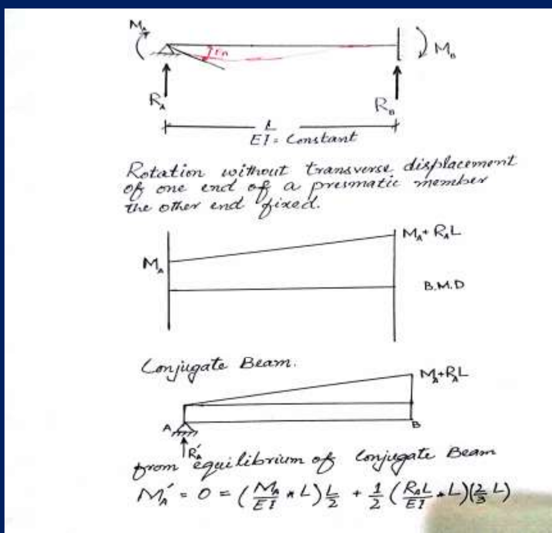
Class activity: Prove that $M_A = \frac{4EI\theta_A}{L}$ (Hint: use conjugate beam method) you have 10 mints

Stiffness Method of Analysis

If a unit translation is applied at joint A then the moment induced in the member is shown in the fig.

Stiffness Method of Analysis

- ❑ Prove that $M_A = \frac{4EI\theta_A}{L}$



Stiffness Method of Analysis

$$R_A = -\frac{1.5 M_A}{L} \quad \text{------- (i)}$$

$$R_A' = \left(\frac{\frac{M'}{EI} + \frac{M_A + R_A L}{EI}}{2} \right) L$$

$$R_A' = \frac{M_A L}{4EI}$$

θ_A in actual $= R_A'$ in conjugate

$$\theta_A = \frac{M_A L}{4EI}$$

OR

$$\boxed{M_A = \frac{4EI\theta_A}{L}}$$

Subtitute this value in eq. (i)

$$R_A = -1.5 \left(\frac{4EI\theta_A}{L}\right) / L$$

$$\boxed{R_A = -\frac{6EI\theta_A}{L^2}}$$

From equilibrium

$$\boxed{R_B = \frac{6EI\theta_A}{L^2}}$$

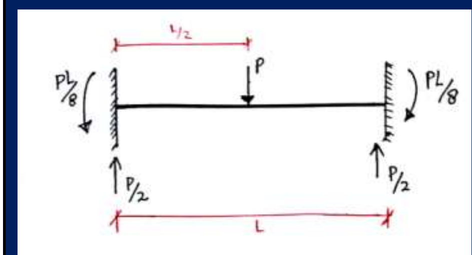
and

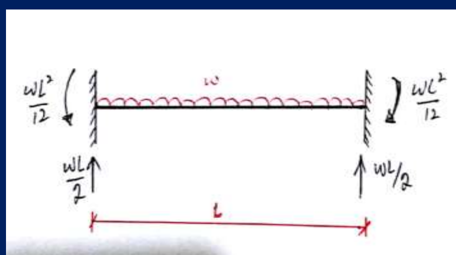
$$\boxed{M_B = \frac{2EI\theta_A}{L}}$$

Hence proved

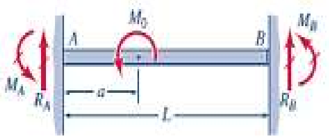
Stiffness Method of Analysis

- Fixed end actions:
  The actions(moments & shear) at the "walls" or fixed joints of a loaded member are called fixed-end actions.





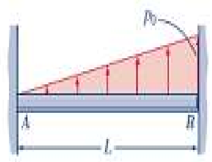
Stiffness Method of Analysis



$$R_A = \frac{6M_0 a}{L^2}\left(1 - \frac{a}{L}\right) \qquad R_B = -\frac{6M_0 a}{L^2}\left(1 - \frac{a}{L}\right)$$

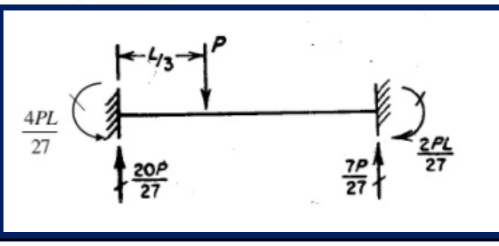
$$M_A = M_0\left(-1 + 4\frac{a}{L} - 3\frac{a^2}{L^2}\right) \qquad M_B = \frac{M_0 a}{L}\left(2 - 3\frac{a}{L}\right)$$

4

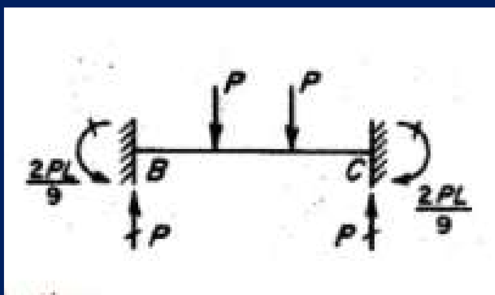


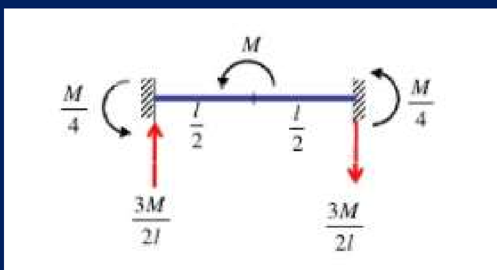
$$R_A = -\frac{3p_0L}{20} \qquad R_B = -\frac{7p_0L}{20}$$

$$M_A = -\frac{p_0L^2}{30} \qquad M_B = \frac{p_0L^2}{20}$$



Stiffness Method of Analysis





References

- Structural Analysis by R. C. Hibbeler
- Matrix structural analysis by William Mc Guire
- Matrix analysis of frame structures by William Weaver
- Online Civil Engineering blogs