Module-2
Plane stress Transformation

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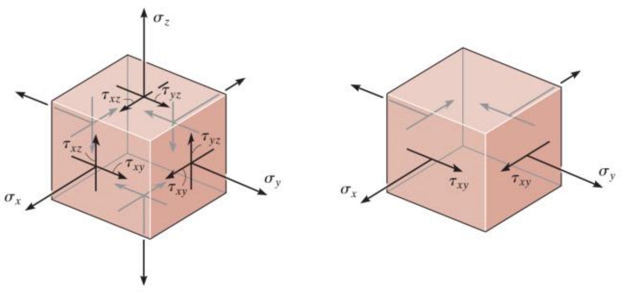
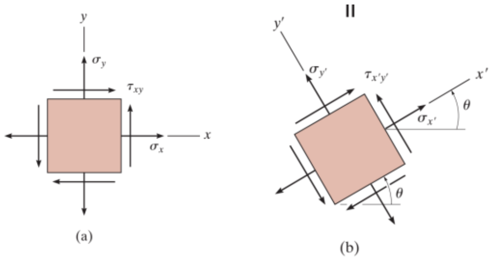
- The general state of stress at a point is characterized by *six* independent normal and shear stress components.
- This state of stress, however, is not often encountered in engineering practice. Instead, engineers frequently make approximations or simplifications of the loadings on a body in order that the stress produced in a structural member or mechanical element can be analyzed in a *single plane* .When this is the case, the material is said to be subjected to *plane stress*.



General state of stress
(a)

Plane stress
(b)

- The general state of ***plane stress*** at a point is therefore represented by a combination of two normal-stress component Ϭx and Ϭy, and one shear stress component Ґxy, which act on four faces of the element. For convenience, in this text we will view this state of stress in the *x–y* plane as shown in the figure below. If this state of stress is defined on an element having a *different orientation* then it will be subjected to three *different* stress components defined as Ϭx΄, Ϭy΄ and Ґx΄y΄.
- ***The state of plane stress at the point is uniquely represented by two normal stress components and one shear stress component acting on an element that has a specific orientation at the point***.
- This is like knowing two force components, say, **Fx** and **Fy**, directed along the *x*, *y* axes, that produce a resultant force **Fr,** and then trying to find the force components **Fx΄** and **Fy΄** directed along the axes **x΄** and **y΄**, so they produce the *same* resultant. The transformation for force must only account for the force component's magnitude and direction.



Procedure for analysis:

If the state of stress at a point is known for a given orientation of an element of material, Fig. 1-*a*, then the state of stress in an element having some other orientation, θ Fig. 1-*b*, can be determined using the following procedure

1. To determine the normal and shear stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on the +x´ face of the element, Fig. 1-*b*, section the element in Fig. 1-*a* as shown in Fig. 1-*c*. If the sectioned area is ΔA then the adjacent areas of the segment will be ΔAsinθ and ΔAcosθ.
2. Draw the free-body diagram of the segment, which requires showing the *forces* that act on the segment, Fig. 1-*d*.This is done by multiplying the stress components on each face by the area upon which they act.
3. Apply the force equations of equilibrium in the x´ and y´ directions. The area ΔA will cancel from the equations and so the two unknown stress components $\sigma_{x'}$ and $\tau_{x'y'}$ can be determined.
4. If $\sigma_{y'}$ acting on the face y+ of the element in Fig. 1-*b*, is to be determined, then it is necessary to consider a segment of the element as shown in Fig. 1-*e* and follow the same procedure just described. Here, however, the shear stress $\tau_{x'y}$ does not have to be determined if it was previously calculated, since it is complementary, that is, it must have the same magnitude on each of the four faces of the element, Fig. 1-*b*.

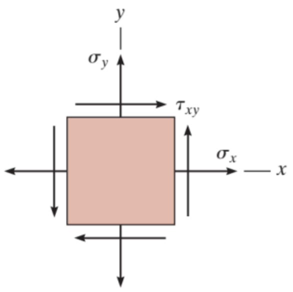


Figure 1-a

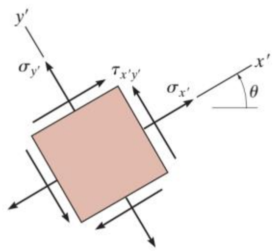
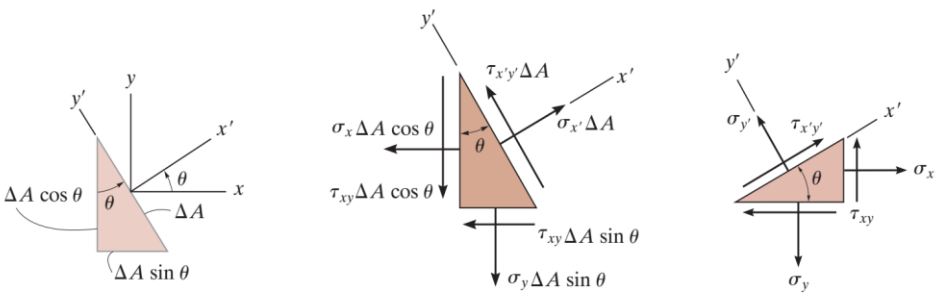


Figure 1-b



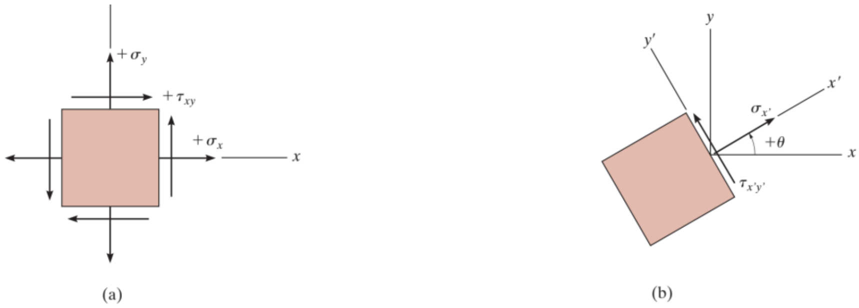
(c) (d) (e)

Figure 1-c, 1-d and 1-e

General equations for plane stress transformation:

The method of transforming the normal and shear stress components from the *x*, *y* to the coordinate axes x´ and y´, as discussed in the previous section, can be developed in a general manner and expressed as a set of stress-transformation equations

Sign convention: The orientation on which the normal and shear stresses are to be determined is defined by the angle θ, which is measured from x to x´ axis. The angle θ will be positive if it is in counter-clock wise direction and vice versa.



Normal and shear stress components:

Using the established sign convention, the element in Fig. 2-*a* is sectioned along the inclined plane and the segment shown in Fig. 2-*b* is isolated. Assuming the sectioned area ΔA, then the horizontal and vertical faces of the segment have an area of ΔAsinθ and ΔAcosθ respectively

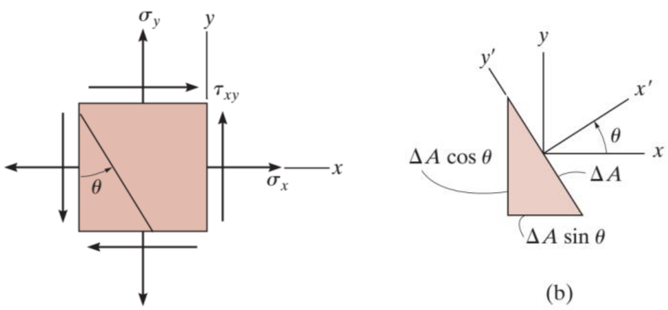


Figure 2-a and figure 2-b

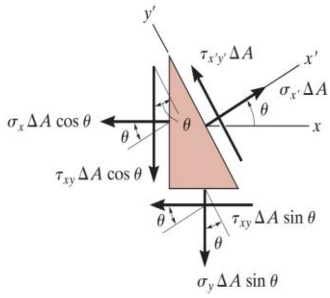
The resulting *free-body diagram* of the segment is shown in Fig. 2-*c*.Applying the equations of equilibrium to determine the unknown normal and shear stress components Ϭx´ and Гx´y´ we have

$$+\nearrow\Sigma F_{x'} = 0; \quad \sigma_{x'}\,\Delta A - (\tau_{xy}\,\Delta A \sin\theta)\cos\theta - (\sigma_y\,\Delta A\sin\theta)\sin\theta - (\tau_{xy}\,\Delta A\cos\theta)\sin\theta - (\sigma_x\,\Delta A\cos\theta)\cos\theta = 0$$

$$\sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

$$+\nwarrow\Sigma F_{y'} = 0; \quad \tau_{x'y'}\,\Delta A + (\tau_{xy}\,\Delta A\sin\theta)\sin\theta - (\sigma_y\,\Delta A\sin\theta)\cos\theta - (\tau_{xy}\,\Delta A\cos\theta)\cos\theta + (\sigma_x\,\Delta A\cos\theta)\sin\theta = 0$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$



These two equations may be simplified by using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin^2\theta = (1 - \cos 2\theta)/2$, and $\cos^2\theta = (1 + \cos 2\theta)/2$, in which case,

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

In order to apply the above stress transformation equations (Eq1, 2 and 3) , the known values of Ϭx, Ϭy and Ґxy and θ in accordance with the established sign conventions. If Ϭx´ and Ґx´y´ calculated are positive quantities, then these stresses act in the positive direction of x´ and y´ axes.

See Example 9.2.

Principle Stresses and maximum in plane shear stresses:

From stress transformation equations, it can be seen that the magnitudes of If Ϭx´ and Ґx´y depend on the angle of inclination of the planes θ on which these stresses act. In engineering practice it is often important to determine the orientation of the element that causes the normal stress to be a maximum and a minimum and the orientation that causes the shear stress to be a maximum.

Maximum in-plane shear stresses: The stresses transformation equation can be differentiated w.r.t θ to obtain the orientation of θ= θp for maximum and minimum normal stresses

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

The solution has two roots θp1 and θp2 and they are 90º apart .

To determine the maximum/minimum normal stresses the value of θp1 and θp2 must be substituted in the stress transformation equations to obtain the following equations.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The above equations gives the maximum or minimum in-plane normal stress acting at a point, where $\sigma_1 > \sigma_2$.This particular set of values are called the in-plane ***principal stresses***, and the corresponding planes on which they act are called the ***principal planes o***f stress, Fig. 1-3. Furthermore, if the trigonometric relations for θ_{p1} or θ_{p2} are substituted into Eq. 2, it can be seen that $\tau_{xy}=0$. In other words, ***no shear stress acts on the principal planes***.

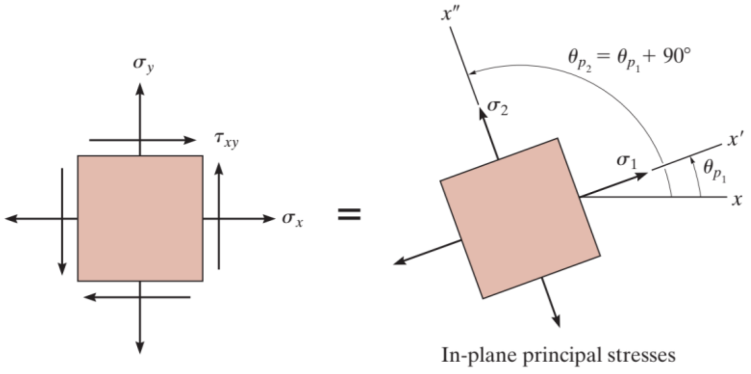


Figure 1-3

Maximum in-plane shear stresses: The orientation of an element that is subjected to maximum shear stress on its sides is given by

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

The roots of θ_s and θ_p are 45° apart. Therefore, an element subjected to ***maximum shear stress will be 45° from the position of an element that is subjected to the principal stress.***

Similarly, the maximum in plane shear stresses at an orientation θ_s is given by

$$\tau_{\substack{\text{max} \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

The value of $\tau_{xy\,max}$ as calculated from this equation is referred to as the ***maximum in-plane shear stress*** because it acts on the element in the *x*–*y* plane.

Important points:

- The *principal stresses* represent the maximum and minimum normal stress at the point.
- When the state of stress is represented by the principal stresses, *no shear stress* will act on the element.
- The state of stress at the point can also be represented in terms of the *maximum in-plane shear stress*. In this case an *average normal stress* will also act on the element.

• The element representing the maximum in-plane shear stress with the associated average normal stresses is oriented 45° from the element representing the principal stresses.

• A brittle member applied to pure axial loading (normal stress) will fail due to principal stresses.

• A ductile member subjected to pure shear loading will fail due to Absolute maximum shear stresses.

See Example 9.3, 9.4, 9.5 and 9.6.