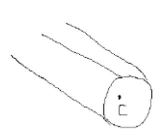
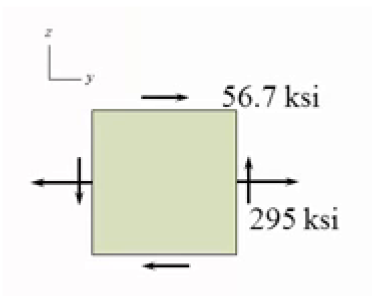
STRESS TRANSFORMATION.

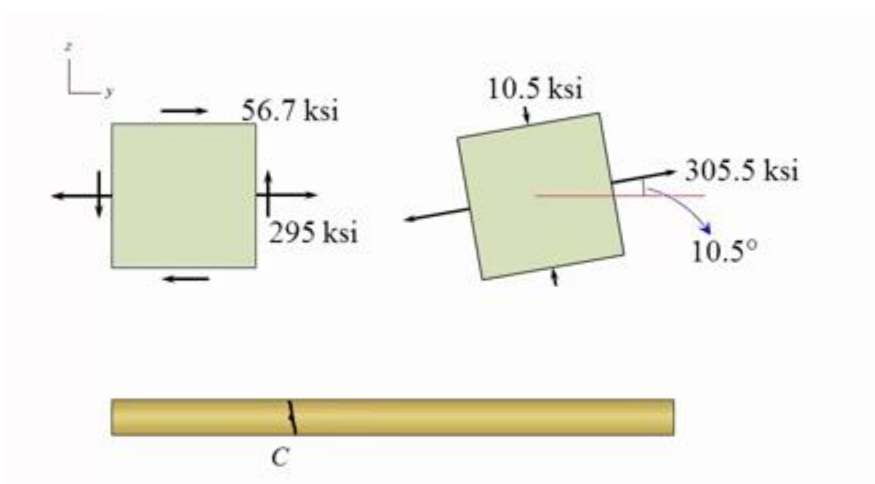


A member is subjected to different stresses. Consider a section cut through the member and a point C is analyzed. Point C represents a particle with no size, no shape and no orientation. We chose this volume element to represent this particle and orientation because it was the most convenient. But what if the orientation changes. Does the state of stress change as well?

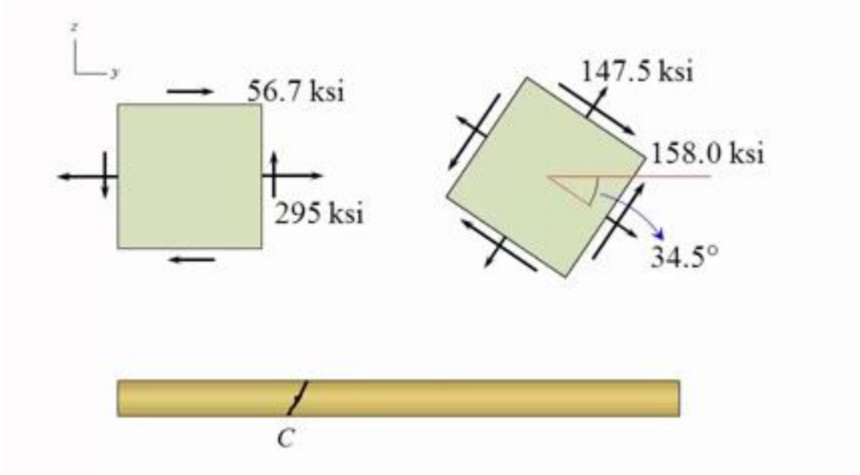
To answer this question we need to consider the topic plane stress transformation.



The same particle will now be discussed having a different orientation. We will also determine at what orientation the max normal stress or maximum in plane shear stress occurs. E.g. for the given values of normal sand shear stress combination in X and Y direction respectively, for this particular element it is determined that the maximum normal stress (305.5ksi) and 0 shear stress occurs at plane 10.5 degrees to the horizontal . And if point C happens to be the weakest point of the member and if the material happens to be brittle and has lower resistance to normal stress the member is likely to fail along this plane in accordance to this orientation.

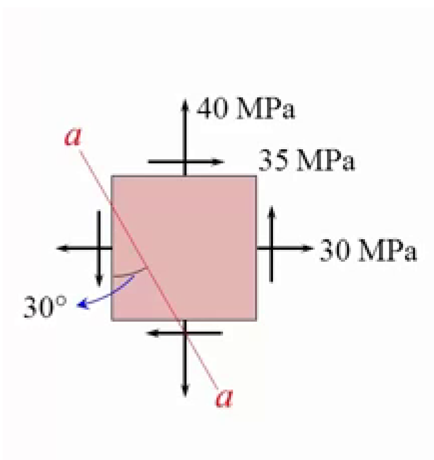


On the other hand for the same particle at a different orientation will have a maximum in-plane shear stresses of 158 ksi . If point C happens to be the weakest point in the member and if material is ductile having low resistance to shear stress will fail in the given plane.

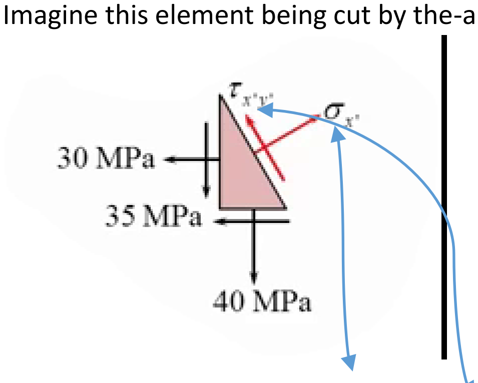


Therefore it is important to understand the stress transformation since this way we can better understand how the material behaves under loadings.

Problem: Determine the stress in a-a plane.



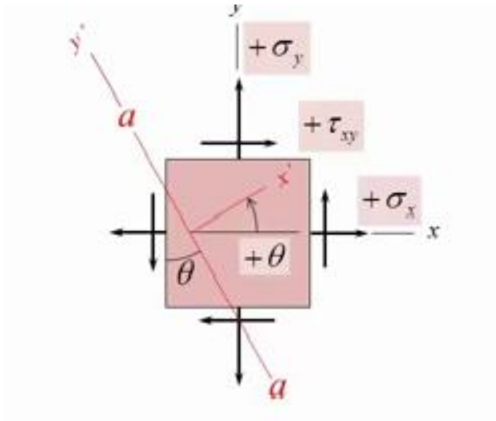
Imagine this element being cut by the-a plane and pick the triangular segment for our analysis.



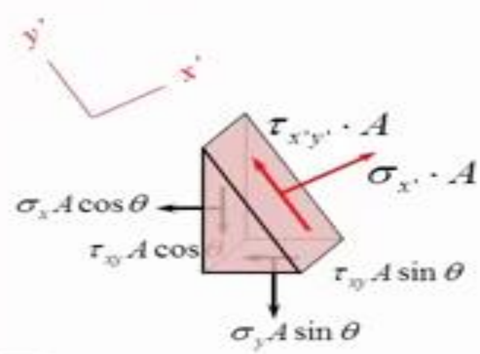
Normal stress in given plane

Shear stress in given plane

SIGN CONVENTION



So the general equation is given by



$$\sum F_{x'} = -\sigma_x A\cos\theta\cos\theta - \tau_{xy} A\cos\theta\sin\theta$$
$$-\tau_{xy} A\sin\theta\cos\theta - \sigma_y A\sin\theta\sin\theta + \sigma_{x'} \cdot A = 0$$
$$\sum F_{y'} = \sigma_x A\cos\theta\sin\theta - \tau_{xy} A\cos\theta\cos\theta$$
$$+\tau_{xy} A\sin\theta\sin\theta - \sigma_y A\sin\theta\cos\theta + \tau_{x'y'} \cdot A = 0$$

The encircled portion of the equation are the unknown normal and shear stress values in a given plane. Thus the final equation is given by.

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

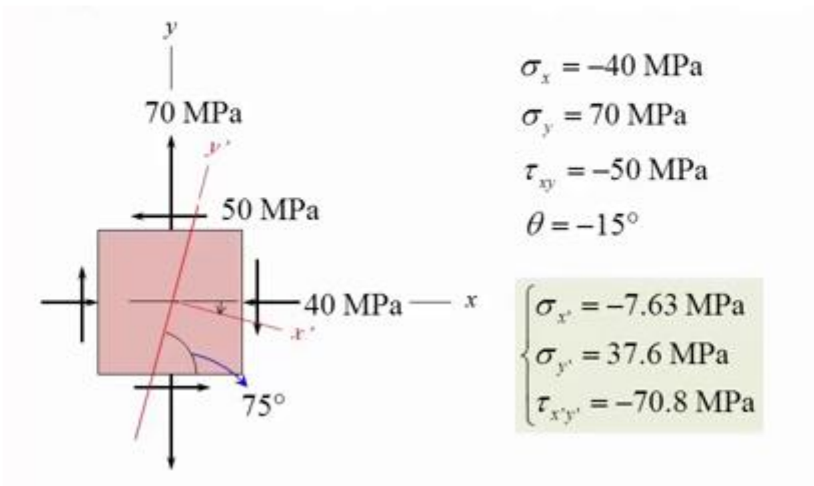
equation 1.1 , 1.2 and 1.3 respectively.

Also remember that.

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

These equations can be used to find the normal stress and shear stress in any plane.

Assignment: determine the stresses for the rotated orientation.



PRINCIPLE STRESSES AND MAXIMUM IN-PLANE SHEAR STRESSES:

If a plane is selected in such a way that only normal stresses exists (i-e no shear stresses) then the plane is called principal plane and the stresses are called principal stresses. For a 2-D analysis there are 2 principal planes mutually perpendicular to each other.

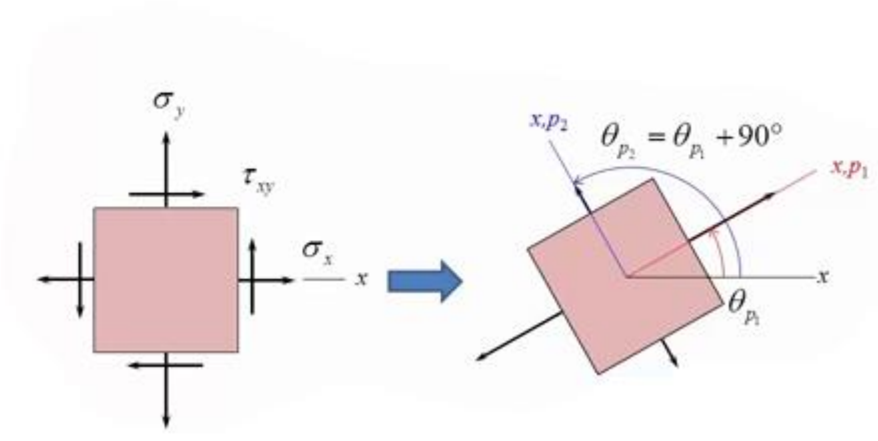
For the state of stress of an element within a specified plane we can determine its principle stress within this plane by rotating this element by an angle θ =θ p1 or θ p2. The difference between these angles by 90 degree and these two angle can both be evaluated from this equation.

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

For either orientation there will be only two pairs of normal stresses and no shear stress and these two pairs of normal stresses can be evaluated by the equation.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

One of these is maximum and other is minimum. These normal stresses are called normal stresses.



Maximum In-plane shear stresses: The orientation of an element that s subjected to maximum shear stresses on its sides can be determined by taking a derivative of equation 1.2 with respect to θ and setting the result equal to zero this gives

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

Therefore, an element subjected to
maximum shear stress will be 45° from the position of an element that
is subjected to the principal stress

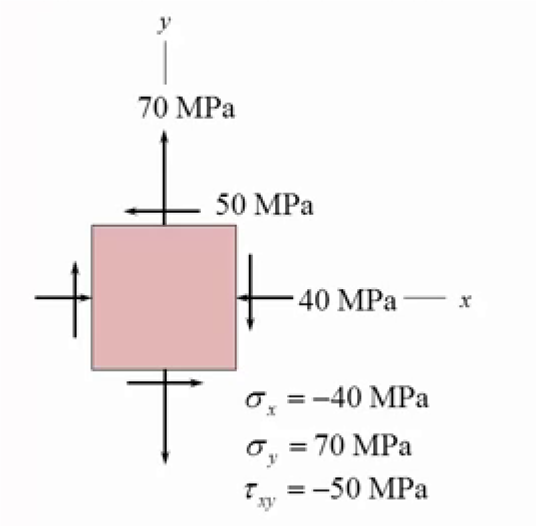
The equation for maximum in plane shear stress in given by.

$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The value of shear stress as calculated from this equation is referred to as the
maximum in-plane shear stress because it acts on the element in the x–y
plane.
There is also an average normal stress on the planes of maximum in-plane
shear stress. We get

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}$$

Problem: Determine the principle stresses and maximum in plane shear stresses of the element



Solution : The orientation of principle can be determined from the equation.

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$ = 0.909, θp =21.1 degrees

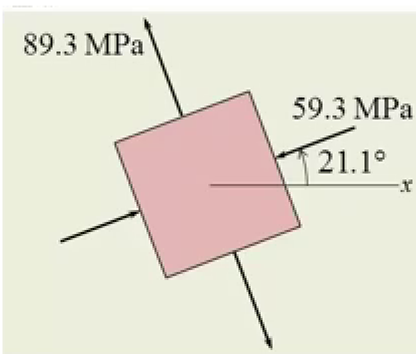
Using this angle we can calculate 2 critical stresses by using the equation.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$ = 83.9 Mpa or -59.3 Mpa

We have to know which of the 2 values of critical stress occurs at angle 21.1 degree so the fastest way to find out is by putting θp =21.1 in general equation.

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta_{p_1} + \tau_{xy}\sin 2\theta_{p_1}$$ = -59.3 Mpa

So now we know for sure that at this orientation the principle stresses are



Now let's look at the maximum   In-plane shear stresses
the orientation of maximum in-plane shear stress is given by

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

θ s1 = -23.9 degree.

The absolute value of maximum in-plane shear stress is given by

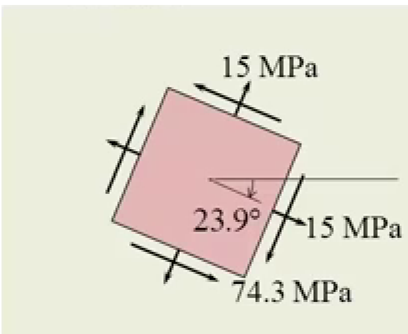
$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

= 74.3 Mpa

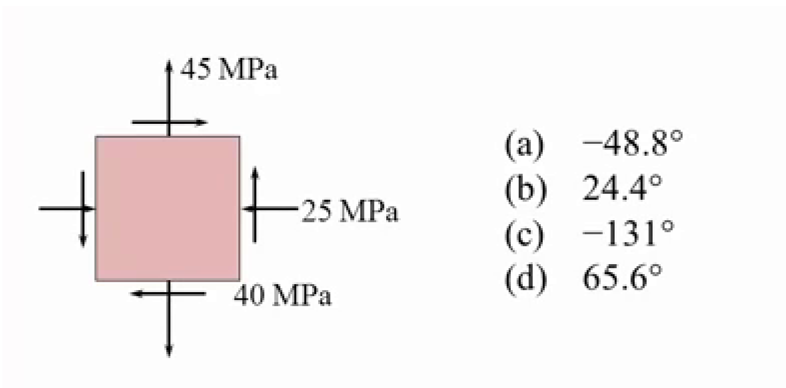
The corresponding normal stress which is same on all 4 sides of the element is given by

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

=15 MPa



The difference between the principal plane and maximum in plane shear stress in 45 dergees.



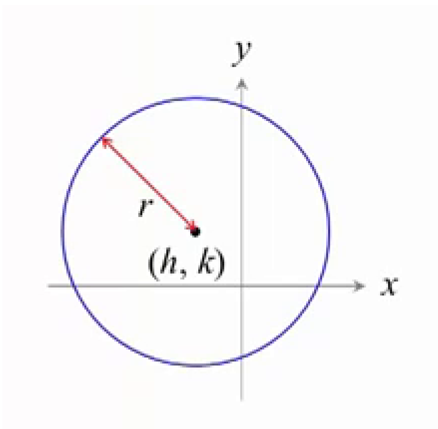
Assignment :

MOHR’S CIRCLE:

Mohr’s circle is a more convenient way to determine plane stress transformation.

Before that lest first review the standard equation of a circle. Ina a given X Y coordinate system if there is a circle with the center coordinates h and k and a radius r then the equation of circle is given in the standard form as:

(x-h)^2 + (y-k)^2 = r^2



In this equation only X and Y are the variables while h, k and r are constants.

Now coming back to the General equation, this equation can be rearranged and transformed into equation of a circle. The derivation is as

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

$$\begin{cases} \left(\sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2}\right)^2 = \left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 \cos^2 2\theta + (\sigma_x - \sigma_y)\tau_{xy} \sin 2\theta \cos 2\theta + \tau_{xy}^2 \sin^2 2\theta \\ (\tau_{x'y'})^2 = \left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 \sin^2 2\theta - (\sigma_x - \sigma_y)\tau_{xy} \sin 2\theta \cos 2\theta + \tau_{xy}^2 \cos^2 2\theta \end{cases}$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + (\tau_{x'y'})^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 \underbrace{(\sin^2 2\theta + \cos^2 2\theta)}_{=1} + \tau_{xy}^2 \underbrace{(\sin^2 2\theta + \cos^2 2\theta)}_{=1}$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + (\tau_{x'y'})^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

So the final equation is similar to the equation of a circle.

Standard equation of a circle:

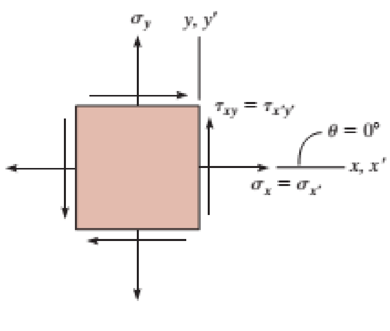
$$(x-h)^2+(y-k)^2=r^2$$

$$\left(\sigma_{x'}-\frac{\sigma_x+\sigma_y}{2}\right)^2+\left(\tau_{x'y'}\right)^2=\left(\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\right)^2$$

variables

This equation has center at ((Sx-Sy)/2 , 0) = (Savg , 0) and radius at

$$R=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

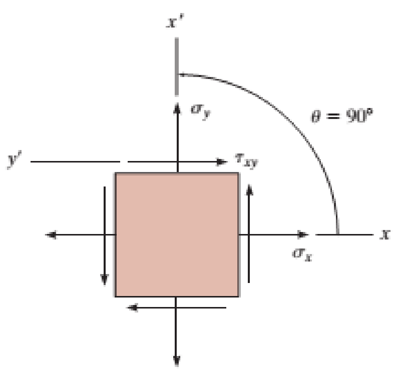


(a)

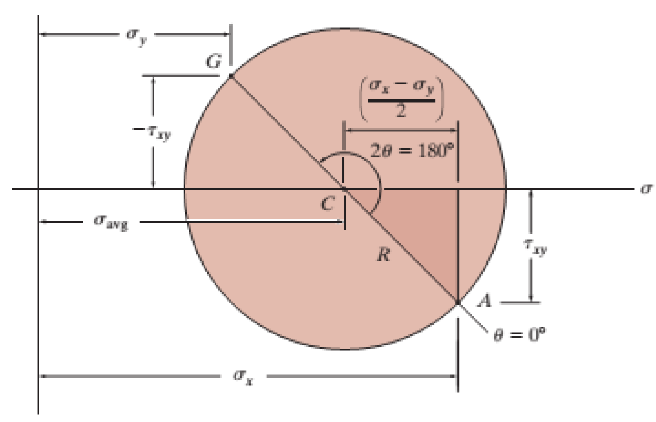
Each point on Mohr's circle represents the two stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on the side of the element defined by the x' axis, when the axis is in a specific direction θ. For example, when x' is coincident with the x axis as shown in Fig. 9–16*a*, then $\theta = 0°$ and $\sigma_{x'} = \sigma_x$, $\tau_{x'y'} = \tau_{xy}$. We will refer to this as the "reference point" A and plot its coordinates $A(\sigma_x, \tau_{xy})$, Fig. 9–16*c*.

Now consider rotating the x' axis 90° counterclockwise, Fig. 9–16*b*. Then $\sigma_{x'} = \sigma_y$, $\tau_{x'y'} = -\tau_{xy}$. These values are the coordinates of point $G(\sigma_y, -\tau_{xy})$ on the circle, Fig. 9–16*c*. Hence, the radial line CG is 180° counterclockwise from the "reference line" CA. In other words, a rotation θ of the x' axis on the element will correspond to a rotation 2θ on the circle in the *same direction*.*

Once constructed, Mohr's circle can be used to determine the principal stresses, the maximum in-plane shear stress and associated average normal stress, or the stress on any arbitrary plane.



(b)



(c)

Fig. 9–16

The following steps are required to draw and use Mohr's circle.

Construction of the Circle.

- Establish a coordinate system such that the horizontal axis represents the normal stress σ, with *positive to the right*, and the vertical axis represents the shear stress τ, with *positive downwards*, Fig. 9–17*a*.*
- Using the positive sign convention for σ_x, σ_y, τ_{xy}, as shown in Fig. 9–17*b*, plot the center of the circle *C*, which is located on the σ axis at a distance $\sigma_{avg} = (\sigma_x + \sigma_y)/2$ from the origin, Fig. 9–17*a*.
- Plot the "reference point" *A* having coordinates $A(\sigma_x, \tau_{xy})$. This point represents the normal and shear stress components on the element's right-hand vertical face, and since the x' axis coincides with the *x* axis, this represents $\theta = 0°$, Fig. 9–17*a*.
- Connect point *A* with the center *C* of the circle and determine *CA* by trigonometry. This distance represents the radius *R* of the circle, Fig. 9–17*a*.
- Once *R* has been determined, sketch the circle.

Principal Stress.

- The principal stresses σ_1 and σ_2 ($\sigma_1 \geq \sigma_2$) are the coordinates of points *B* and *D* where the circle intersects the σ axis, i.e., where $\tau = 0$, Fig. 9–17*a*.
- These stresses act on planes defined by angles θ_{p_1} and θ_{p_2}, Fig. 9–17*c*. They are represented on the circle by angles $2\theta_{p_1}$ (shown) and $2\theta_{p_2}$ (not shown) and are measured *from* the radial reference line *CA* *to* lines *CB* and *CD*, respectively.
- Using trigonometry, only one of these angles needs to be calculated from the circle, since θ_{p_1} and θ_{p_2} are 90° apart. Remember that the direction of rotation $2\theta_p$ on the circle (here it happens to be counterclockwise) represents the *same* direction of rotation θ_p from the reference axis ($+x$) to the principal plane ($+x'$), Fig. 9–17*c*.*

Maximum In-Plane Shear Stress.

- The average normal stress and maximum in-plane shear stress components are determined from the circle as the coordinates of either point *E* or *F*, Fig. 9–17*a*.
- In this case the angles θ_{s_1} and θ_{s_2} give the orientation of the planes that contain these components, Fig. 9–17*d*. The angle $2\theta_{s_1}$ is shown in Fig. 9–17*a* and can be determined using trigonometry. Here the rotation happens to be clockwise, from *CA* to *CE*, and so θ_{s_1} must be clockwise on the element, Fig. 9–17*d*.*

sis:

Stresses on Arbitrary Plane.

- The normal and shear stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on a specified plane or x' axis, defined by the angle θ, Fig. 9–17*e*, can be obtained from the circle using trigonometry to determine the coordinates of point *P*, Fig. 9–17*a*.
- To locate *P*, the known angle θ (in this case counterclockwise), Fig. 9–17*e*, must be measured on the circle in the *same direction* 2θ (counterclockwise), *from* the radial reference line *CA* *to* the radial line *CP*, Fig. 9–17*a*.*

*If the τ axis were constructed *positive upwards*, then the angle 2θ on the circle would be measured in the *opposite direction* to the orientation θ of the x' axis.

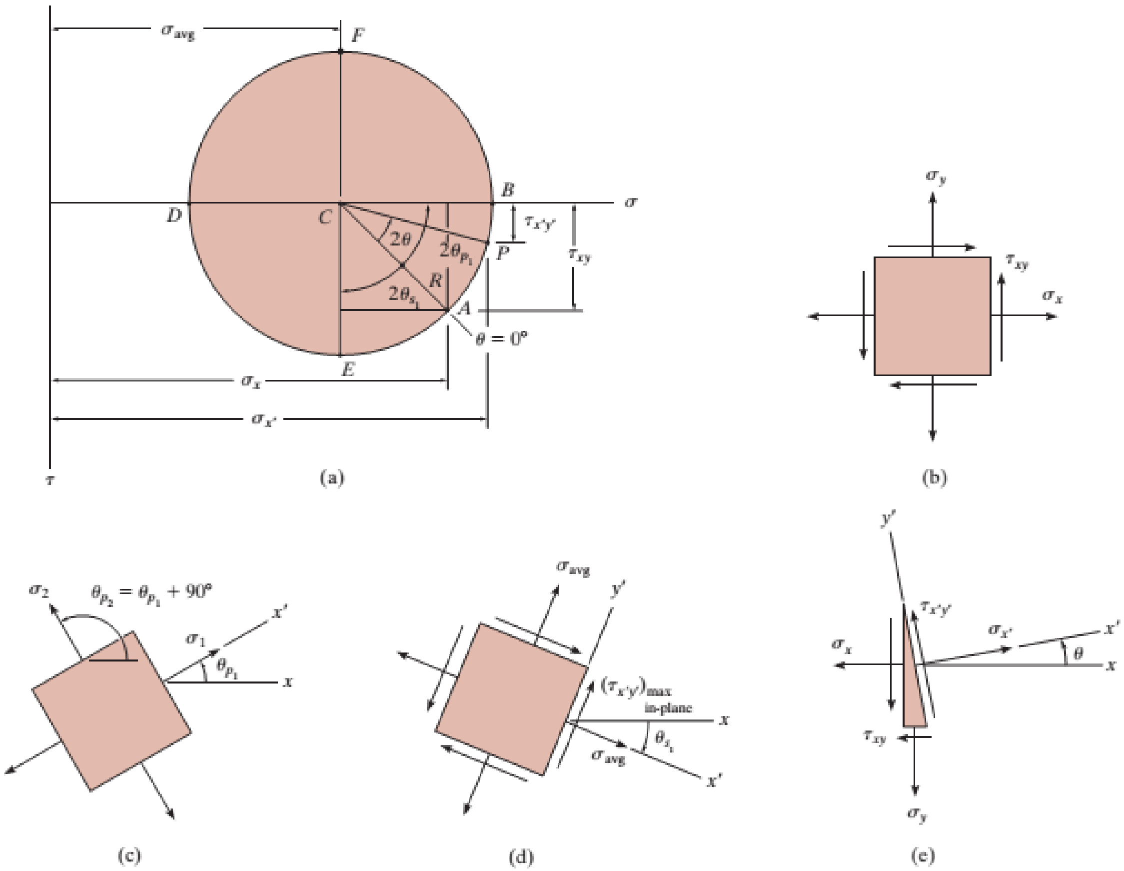
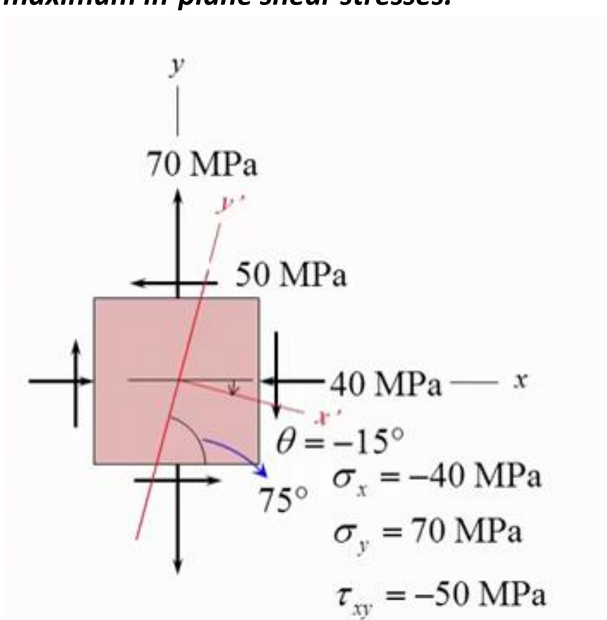


Fig. 9–17

Example: Determine the stresses for the rotated orientation, the principle stresses and the maximum in-plane shear stresses.



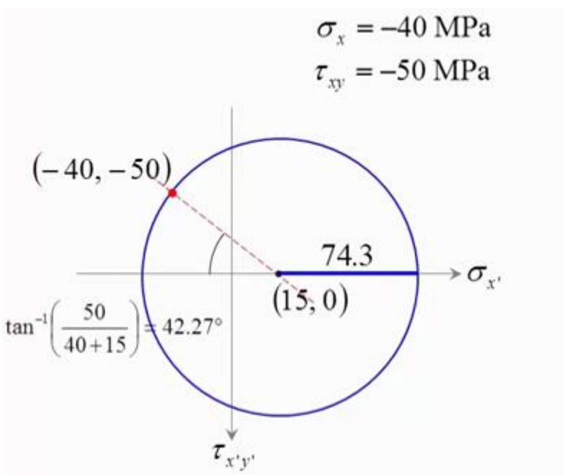
We need to determine the new state of stress associated with new orientation as shown. Also we need to determine the principle normal stresses and maximum in-plane shear stresses . But this time we are going to solbe thos problem usimhg Mohr's cirels . We need to first calculate the Avg normal shar ahteesd.

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = 15 \text{ MPa}$$

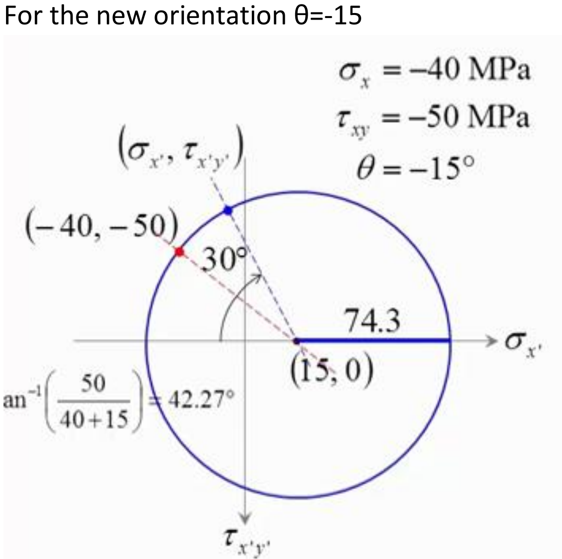
We also need to find the radius of Mohr's Circle.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 74.3 \text{ MPa}$$

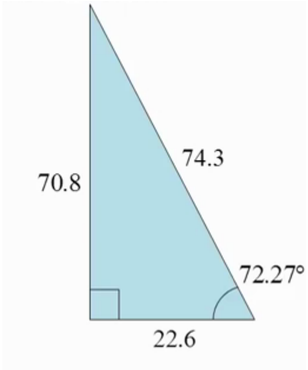
So in a coordinate system draw a circle with center at (15,0) .The R=74.3.Also plot the original values of stresses



For the new orientation θ=-15

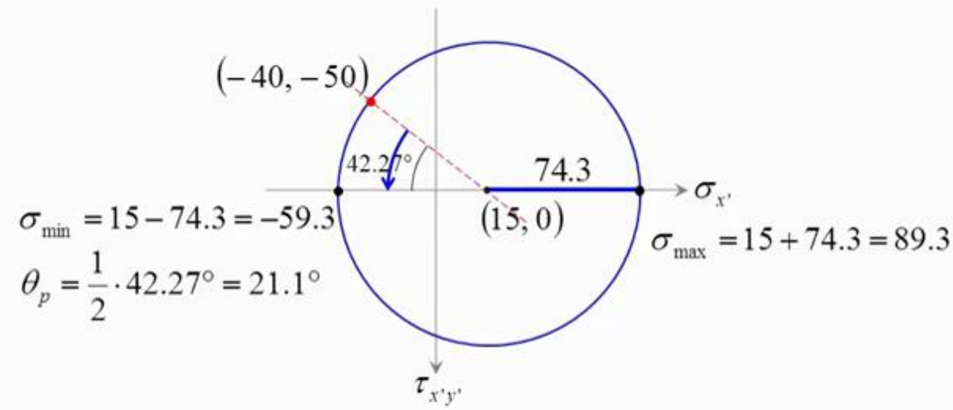


The coordinates can be found by simple trigonometry

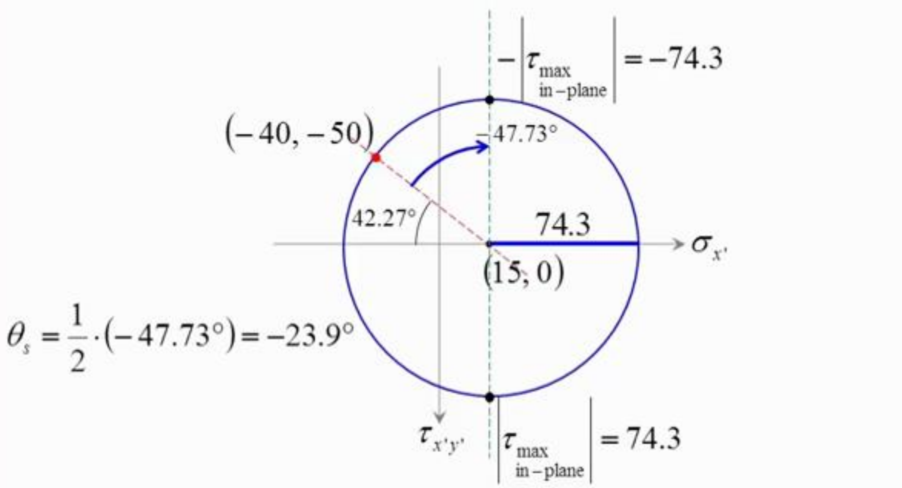


$$\sigma_{x'} = -(22.6 - 15) = -7.6\ \text{(MPa)} \quad \tau_{x'y'} = -70.8\text{(MPa)}$$

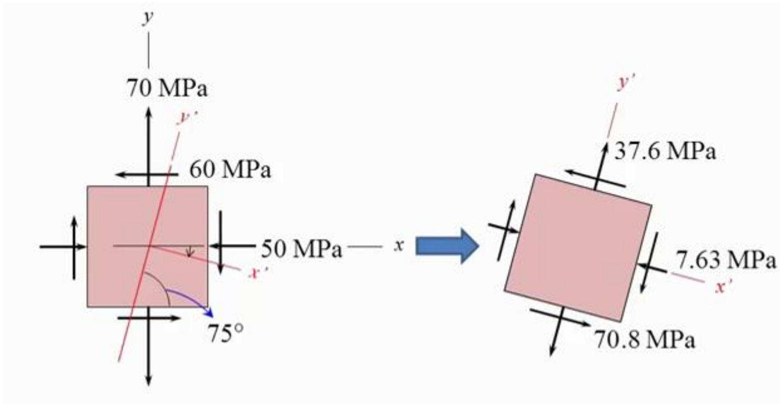
Principal stresses:



Maximum in-plane shear stress



These result can be visualized. The state of stress at new orientation is (theta =-15)



The orientation of principal stresses and maximum in plane shear stresses is as.