

MAGNETIC FIELDS DUE TO CURRENTS

29-1 WHAT IS PHYSICS?

One basic observation of physics is that a moving charged particle produces a magnetic field around itself. Thus a current of moving charged particles produces a magnetic field around the current. This feature of *electromagnetism*, which is the combined study of electric and magnetic effects, came as a surprise to the people who discovered it. Surprise or not, this feature has become enormously important in everyday life because it is the basis of countless electromagnetic devices. For example, a magnetic field is produced in maglev trains and other devices used to lift heavy loads.

Our first step in this chapter is to find the magnetic field due to the current in a very small section of current-carrying wire. Then we shall find the magnetic field due to the entire wire for several different arrangements of the wire.

29-2 Calculating the Magnetic Field Due to a Current

Figure 29-1 shows a wire of arbitrary shape carrying a current i . We want to find the magnetic field \vec{B} at a nearby point P . We first mentally divide the wire into differential elements ds and then define for each element a length vector $d\vec{s}$ that has length ds and whose direction is the direction of the current in ds . We can then define a differential *current-length element* to be $i d\vec{s}$; we wish to calculate the field $d\vec{B}$ produced at P by a typical current-length element. From experiment we find that magnetic fields, like electric fields, can be superimposed to find a net field. Thus, we can calculate the net field \vec{B} at P by summing, via integration, the

This element of current creates a magnetic field at P , into the page.

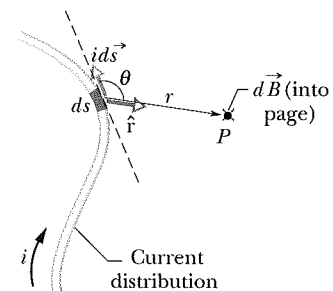


Fig. 29-1 A current-length element $i d\vec{s}$ produces a differential magnetic field $d\vec{B}$ at point P . The green \times (the tail of an arrow) at the dot for point P indicates that $d\vec{B}$ is directed *into* the page there.

contributions $d\vec{B}$ from all the current-length elements. However, this summation is more challenging than the process associated with electric fields because of a complexity; whereas a charge element dq producing an electric field is a scalar, a current-length element $i d\vec{s}$ producing a magnetic field is a vector, being the product of a scalar and a vector.

The magnitude of the field $d\vec{B}$ produced at point P at distance r by a current-length element $i d\vec{s}$ turns out to be

$$dB = \frac{\mu_0}{4\pi} \frac{i ds \sin \theta}{r^2}, \quad (29-1)$$

where θ is the angle between the directions of $d\vec{s}$ and \hat{r} , a unit vector that points from ds toward P . Symbol μ_0 is a constant, called the *permeability constant*, whose value is defined to be exactly

$$\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A} \approx 1.26 \times 10^{-6} \text{ T} \cdot \text{m/A}. \quad (29-2)$$

The direction of $d\vec{B}$, shown as being into the page in Fig. 29-1, is that of the cross product $d\vec{s} \times \hat{r}$. We can therefore write Eq. 29-1 in vector form as

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i d\vec{s} \times \hat{r}}{r^2} \quad (\text{Biot-Savart law}). \quad (29-3)$$

This vector equation and its scalar form, Eq. 29-1, are known as the **law of Biot and Savart** (rhymes with “Leo and bazaar”). The law, which is experimentally deduced, is an inverse-square law. We shall use this law to calculate the net magnetic field \vec{B} produced at a point by various distributions of current.

Magnetic Field Due to a Current in a Long Straight Wire

Shortly we shall use the law of Biot and Savart to prove that the magnitude of the magnetic field at a perpendicular distance R from a long (infinite) straight wire carrying a current i is given by

$$B = \frac{\mu_0 i}{2\pi R} \quad (\text{long straight wire}). \quad (29-4)$$

The field magnitude B in Eq. 29-4 depends only on the current and the perpendicular distance R of the point from the wire. We shall show in our derivation that the field lines of \vec{B} form concentric circles around the wire, as Fig. 29-2 shows and as the iron filings in Fig. 29-3 suggest. The increase in the spacing of the lines in Fig. 29-2 with increasing distance from the wire represents the $1/R$ decrease in the magnitude of \vec{B} predicted by Eq. 29-4. The lengths of the two vectors \vec{B} in the figure also show the $1/R$ decrease.

The magnetic field vector at any point is tangent to a circle.

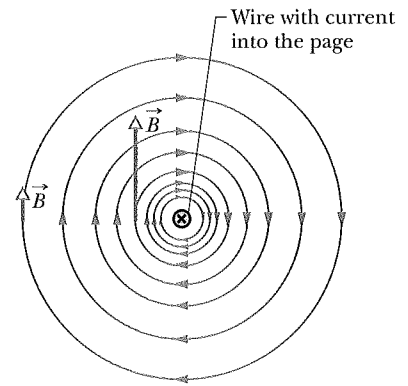


Fig. 29-2 The magnetic field lines produced by a current in a long straight wire form concentric circles around the wire. Here the current is into the page, as indicated by the \times .

Fig. 29-3 Iron filings that have been sprinkled onto cardboard collect in concentric circles when current is sent through the central wire. The alignment, which is along magnetic field lines, is caused by the magnetic field produced by the current. (Courtesy Education Development Center)

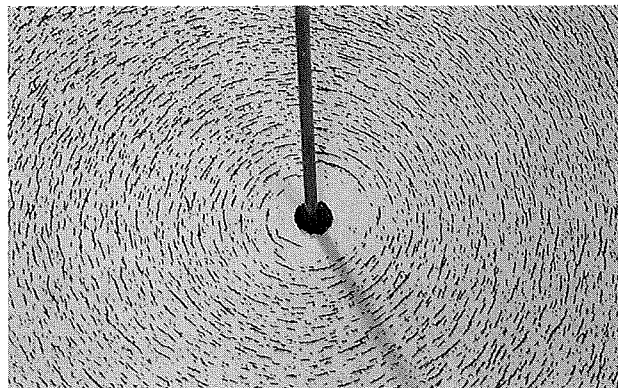
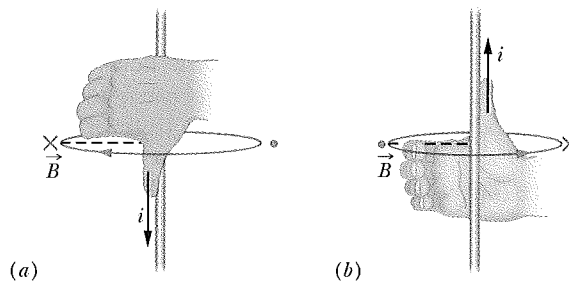


Fig. 29-4 A right-hand rule gives the direction of the magnetic field due to a current in a wire. (a) The situation of Fig. 29-2, seen from the side. The magnetic field \vec{B} at any point to the left of the wire is perpendicular to the dashed radial line and directed into the page, in the direction of the fingertips, as indicated by the \times . (b) If the current is reversed, \vec{B} at any point to the left is still perpendicular to the dashed radial line but now is directed out of the page, as indicated by the dot.



The thumb is in the current's direction. The fingers reveal the field vector's direction, which is tangent to a circle.

Here is a simple right-hand rule for finding the direction of the magnetic field set up by a current-length element, such as a section of a long wire:

Right-hand rule: Grasp the element in your right hand with your extended thumb pointing in the direction of the current. Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.

The result of applying this right-hand rule to the current in the straight wire of Fig. 29-2 is shown in a side view in Fig. 29-4a. To determine the direction of the magnetic field \vec{B} set up at any particular point by this current, mentally wrap your right hand around the wire with your thumb in the direction of the current. Let your fingertips pass through the point; their direction is then the direction of the magnetic field at that point. In the view of Fig. 29-2, \vec{B} at any point is *tangent to a magnetic field line*; in the view of Fig. 29-4, it is *perpendicular to a dashed radial line connecting the point and the current*.

Proof of Equation 29-4

Figure 29-5, which is just like Fig. 29-1 except that now the wire is straight and of infinite length, illustrates the task at hand. We seek the field \vec{B} at point P , a perpendicular distance R from the wire. The magnitude of the differential magnetic field produced at P by the current-length element $i d\vec{s}$ located a distance r from P is given by Eq. 29-1:

$$dB = \frac{\mu_0}{4\pi} \frac{i ds \sin \theta}{r^2}.$$

The direction of $d\vec{B}$ in Fig. 29-5 is that of the vector $d\vec{s} \times \hat{r}$ —namely, directly into the page.

Note that $d\vec{B}$ at point P has this same direction for all the current-length elements into which the wire can be divided. Thus, we can find the magnitude of the magnetic field produced at P by the current-length elements in the upper half of the infinitely long wire by integrating dB in Eq. 29-1 from 0 to ∞ .

Now consider a current-length element in the lower half of the wire, one that is as far below P as $d\vec{s}$ is above P . By Eq. 29-3, the magnetic field produced at P by this current-length element has the same magnitude and direction as that from element $i d\vec{s}$ in Fig. 29-5. Further, the magnetic field produced by the lower half of the wire is exactly the same as that produced by the upper half. To find the magnitude of the *total* magnetic field \vec{B} at P , we need only multiply the result of our integration by 2. We get

$$B = 2 \int_0^\infty dB = \frac{\mu_0 i}{2\pi} \int_0^\infty \frac{\sin \theta ds}{r^2}. \quad (29-5)$$

The variables θ , s , and r in this equation are not independent; Fig. 29-5 shows that they are related by

$$r = \sqrt{s^2 + R^2}$$

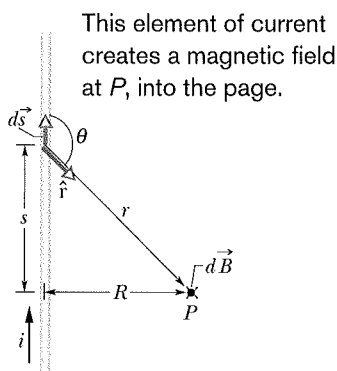


Fig. 29-5 Calculating the magnetic field produced by a current i in a long straight wire. The field $d\vec{B}$ at P associated with the current-length element $i d\vec{s}$ is directed into the page, as shown.

and
$$\sin \theta = \sin(\pi - \theta) = \frac{R}{\sqrt{s^2 + R^2}}.$$

With these substitutions and integral 19 in Appendix E, Eq. 29-5 becomes

$$\begin{aligned} B &= \frac{\mu_0 i}{2\pi} \int_0^\infty \frac{R ds}{(s^2 + R^2)^{3/2}} \\ &= \frac{\mu_0 i}{2\pi R} \left[\frac{s}{(s^2 + R^2)^{1/2}} \right]_0^\infty = \frac{\mu_0 i}{2\pi R}, \end{aligned} \quad (29-6)$$

as we wanted. Note that the magnetic field at P due to either the lower half or the upper half of the infinite wire in Fig. 29-5 is half this value; that is,

$$B = \frac{\mu_0 i}{4\pi R} \quad (\text{semi-infinite straight wire}). \quad (29-7)$$

Magnetic Field Due to a Current in a Circular Arc of Wire

To find the magnetic field produced at a point by a current in a curved wire, we would again use Eq. 29-1 to write the magnitude of the field produced by a single current-length element, and we would again integrate to find the net field produced by all the current-length elements. That integration can be difficult, depending on the shape of the wire; it is fairly straightforward, however, when the wire is a circular arc and the point is the center of curvature.

Figure 29-6a shows such an arc-shaped wire with central angle ϕ , radius R , and center C , carrying current i . At C , each current-length element $i d\vec{s}$ of the wire produces a magnetic field of magnitude dB given by Eq. 29-1. Moreover, as Fig. 29-6b shows, no matter where the element is located on the wire, the angle θ between the vectors $d\vec{s}$ and \hat{r} is 90° ; also, $r = R$. Thus, by substituting R for r and 90° for θ in Eq. 29-1, we obtain

$$dB = \frac{\mu_0}{4\pi} \frac{i ds \sin 90^\circ}{R^2} = \frac{\mu_0}{4\pi} \frac{i ds}{R^2}. \quad (29-8)$$

The field at C due to each current-length element in the arc has this magnitude.

An application of the right-hand rule anywhere along the wire (as in Fig. 29-6c) will show that all the differential fields $d\vec{B}$ have the same direction at C —directly out of the page. Thus, the total field at C is simply the sum (via integration) of all the differential fields $d\vec{B}$. We use the identity $ds = R d\phi$ to change the variable of integration from ds to $d\phi$ and obtain, from Eq. 29-8,

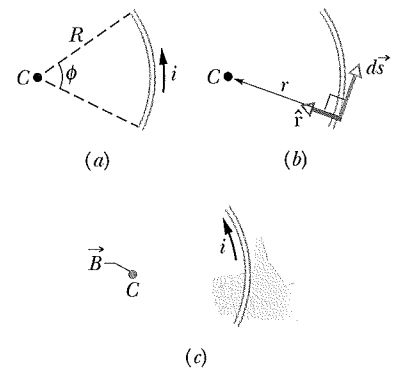
$$B = \int dB = \int_0^\phi \frac{\mu_0}{4\pi} \frac{iR d\phi}{R^2} = \frac{\mu_0 i}{4\pi R} \int_0^\phi d\phi.$$

Integrating, we find that

$$B = \frac{\mu_0 i \phi}{4\pi R} \quad (\text{at center of circular arc}). \quad (29-9)$$

Note that this equation gives us the magnetic field *only* at the center of curvature of a circular arc of current. When you insert data into the equation, you must be careful to express ϕ in radians rather than degrees. For example, to find the magnitude of the magnetic field at the center of a full circle of current, you would substitute 2π rad for ϕ in Eq. 29-9, finding

$$B = \frac{\mu_0 i (2\pi)}{4\pi R} = \frac{\mu_0 i}{2R} \quad (\text{at center of full circle}). \quad (29-10)$$



The right-hand rule reveals the field's direction at the center.

Fig. 29-6 (a) A wire in the shape of a circular arc with center C carries current i . (b) For any element of wire along the arc, the angle between the directions of $d\vec{s}$ and \hat{r} is 90° . (c) Determining the direction of the magnetic field at the center C due to the current in the wire; the field is out of the page, in the direction of the fingertips, as indicated by the colored dot at C .

The field due to a at the position of b creates a force on b .

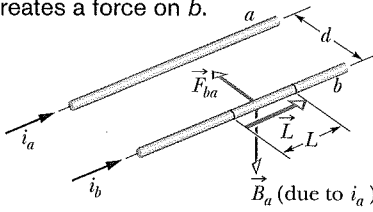


Fig. 29-9 Two parallel wires carrying currents in the same direction attract each other. \vec{B}_a is the magnetic field at wire b produced by the current in wire a . \vec{F}_{ba} is the resulting force acting on wire b because it carries current in \vec{B}_a .

29-3 Force Between Two Parallel Currents

Two long parallel wires carrying currents exert forces on each other. Figure 29-9 shows two such wires, separated by a distance d and carrying currents i_a and i_b . Let us analyze the forces on these wires due to each other.

We seek first the force on wire b in Fig. 29-9 due to the current in wire a . That current produces a magnetic field \vec{B}_a , and it is this magnetic field that actually causes the force we seek. To find the force, then, we need the magnitude and direction of the field \vec{B}_a at the site of wire b . The magnitude of \vec{B}_a at every point of wire b is, from Eq. 29-4,

$$B_a = \frac{\mu_0 i_a}{2\pi d}. \tag{29-11}$$

The curled–straight right-hand rule tells us that the direction of \vec{B}_a at wire b is down, as Fig. 29-9 shows.

Now that we have the field, we can find the force it produces on wire b . Equation 28-26 tells us that the force \vec{F}_{ba} on a length L of wire b due to the external magnetic field \vec{B}_a is

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a, \tag{29-12}$$

where \vec{L} is the length vector of the wire. In Fig. 29-9, vectors \vec{L} and \vec{B}_a are perpendicular to each other, and so with Eq. 29-11, we can write

$$F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d}. \tag{29-13}$$

The direction of \vec{F}_{ba} is the direction of the cross product $\vec{L} \times \vec{B}_a$. Applying the right-hand rule for cross products to \vec{L} and \vec{B}_a in Fig. 29-9, we see that \vec{F}_{ba} is directly toward wire a , as shown.

The general procedure for finding the force on a current-carrying wire is this:

To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

We could now use this procedure to compute the force on wire a due to the current in wire b . We would find that the force is directly toward wire b ; hence, the two wires with parallel currents attract each other. Similarly, if the two currents were antiparallel, we could show that the two wires repel each other. Thus,

Parallel currents attract each other, and antiparallel currents repel each other.

The force acting between currents in parallel wires is the basis for the definition of the ampere, which is one of the seven SI base units. The definition, adopted in 1946, is this: The ampere is that constant current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in vacuum, would produce on each of these conductors a force of magnitude 2×10^{-7} newton per meter of wire length.

Rail Gun

One application of the physics of Eq. 29-13 is a rail gun. In this device, a magnetic force accelerates a projectile to a high speed in a short time. The basics of a rail gun are shown in Fig. 29-10a. A large current is sent out along one of two parallel conducting rails, across a conducting “fuse” (such as a narrow piece of copper)

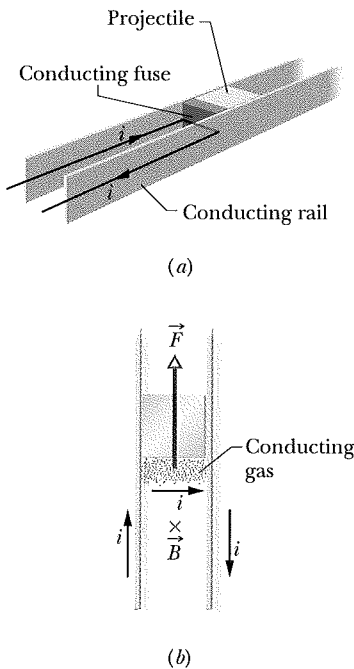


Fig. 29-10 (a) A rail gun, as a current i is set up in it. The current rapidly causes the conducting fuse to vaporize. (b) The current produces a magnetic field \vec{B} between the rails, and the field causes a force \vec{F} to act on the conducting gas, which is part of the current path. The gas propels the projectile along the rails, launching it.

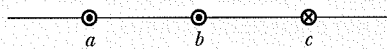
between the rails, and then back to the current source along the second rail. The projectile to be fired lies on the far side of the fuse and fits loosely between the rails. Immediately after the current begins, the fuse element melts and vaporizes, creating a conducting gas between the rails where the fuse had been.

The curled–straight right-hand rule of Fig. 29-4 reveals that the currents in the rails of Fig. 29-10a produce magnetic fields that are directed downward between the rails. The net magnetic field \vec{B} exerts a force \vec{F} on the gas due to the current i through the gas (Fig. 29-10b). With Eq. 29-12 and the right-hand rule for cross products, we find that \vec{F} points outward along the rails. As the gas is forced outward along the rails, it pushes the projectile, accelerating it by as much as 5×10^6g , and then launches it with a speed of 10 km/s, all within 1 ms. Some-day rail guns may be used to launch materials into space from mining operations on the Moon or an asteroid.



CHECKPOINT 1

The figure here shows three long, straight, parallel, equally spaced wires with identical currents either into or out of the page. Rank the wires according to the magnitude of the force on each due to the currents in the other two wires, greatest first.



29-4 Ampere's Law

We can find the net electric field due to *any* distribution of charges by first writing the differential electric field $d\vec{E}$ due to a charge element and then summing the contributions of $d\vec{E}$ from all the elements. However, if the distribution is complicated, we may have to use a computer. Recall, however, that if the distribution has planar, cylindrical, or spherical symmetry, we can apply Gauss' law to find the net electric field with considerably less effort.

Similarly, we can find the net magnetic field due to *any* distribution of currents by first writing the differential magnetic field $d\vec{B}$ (Eq. 29-3) due to a current-length element and then summing the contributions of $d\vec{B}$ from all the elements. Again we may have to use a computer for a complicated distribution. However, if the distribution has some symmetry, we may be able to apply **Ampere's law** to find the magnetic field with considerably less effort. This law, which can be derived from the Biot–Savart law, has traditionally been credited to André-Marie Ampère (1775–1836), for whom the SI unit of current is named. However, the law actually was advanced by English physicist James Clerk Maxwell.

Ampere's law is

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} \quad (\text{Ampere's law}). \quad (29-14)$$

The loop on the integral sign means that the scalar (dot) product $\vec{B} \cdot d\vec{s}$ is to be integrated around a *closed* loop, called an *Amperian loop*. The current i_{enc} is the *net* current encircled by that closed loop.

To see the meaning of the scalar product $\vec{B} \cdot d\vec{s}$ and its integral, let us first apply Ampere's law to the general situation of Fig. 29-11. The figure shows cross sections of three long straight wires that carry currents i_1 , i_2 , and i_3 either directly into or directly out of the page. An arbitrary Amperian loop lying in the plane of the page encircles two of the currents but not the third. The counterclockwise direction marked on the loop indicates the arbitrarily chosen direction of integration for Eq. 29-14.

To apply Ampere's law, we mentally divide the loop into differential vector elements $d\vec{s}$ that are everywhere directed along the tangent to the loop in the

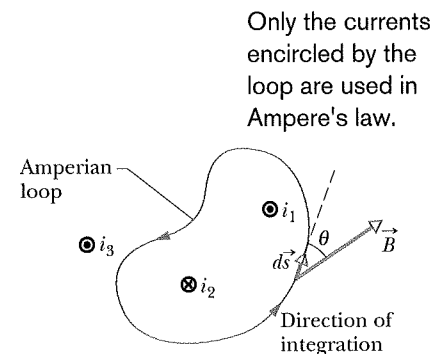


Fig. 29-11 Ampere's law applied to an arbitrary Amperian loop that encircles two long straight wires but excludes a third wire. Note the directions of the currents.