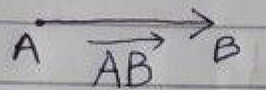
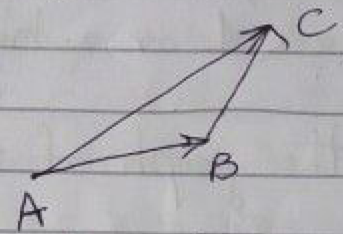
Vectors in two dimensions :



Two v. are equal if they have the same magnitude and direction.



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

(14.1) The two-dim vector space V_2 is the set of all ordered pairs $\langle x, y \rangle$ of real numbers, called vector subject to the following axioms.

i) If $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$ are vectors then $a+b = \langle a_1+b_1, a_2+b_2 \rangle$

ii) If $a = \langle a_1, a_2 \rangle$ and c is a scalar, then $ca = \langle ca_1, ca_2 \rangle$

Example If $a = \langle -1, 3 \rangle$ and $b = \langle 4, 2 \rangle$, i) find $a+b$, $\frac{-3}{2}b$ and $2a+3b$

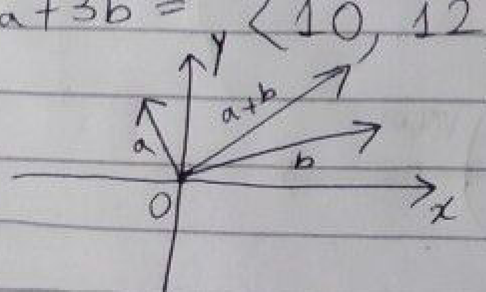
ii) Represent $a, b, a+b, \frac{-3}{2}b$ geometrically

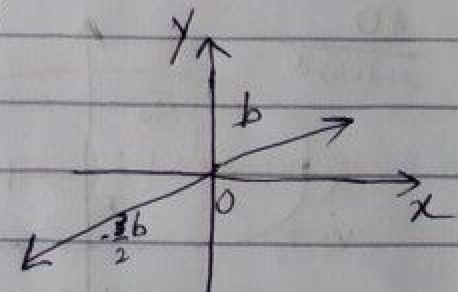
Sol:

$a+b = \langle -1, 3 \rangle + \langle 4, 2 \rangle = \langle -1+4, 3+2 \rangle = \langle 3, 5 \rangle$

$\frac{-3}{2}b = \langle -6, -3 \rangle$

$2a+3b = \langle 10, 12 \rangle$



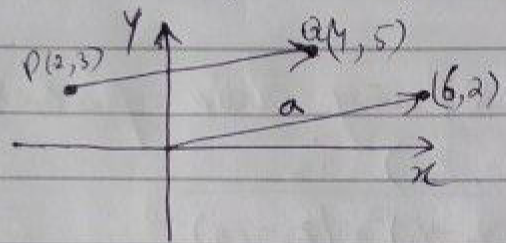


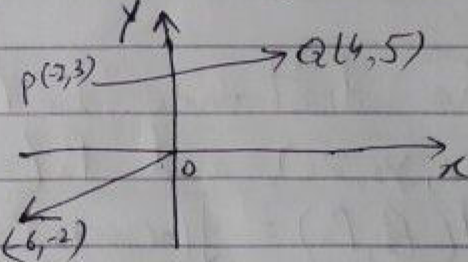
(14.3) The mag $|a|$ of a vector $a = \langle a_1, a_2 \rangle$ is $|a| = \sqrt{a_1^2 +$

Q Given points $P(-2,3)$ and $Q(4,5)$, find vectors a and b such that they have geo rep $\overrightarrow{PQ}$ and $\overrightarrow{QP}$. Sketch $\overrightarrow{PQ}, \overrightarrow{QP}$ and postn vectors corresponding to a and b.

Sol:

$$a = \langle 4-(-2), 5-3 \rangle = \langle 6, 2 \rangle$$
$$b = \langle -2-4, 3-5 \rangle = \langle -6, -2 \rangle$$





(14.5) i) $a+b = b+a$, ii) $a+(b+c) = (a+b)+c$

iii) $a+0 = a$, iv) $a+(-a) = 0$.

(14.11) If $a = \langle a_1, a_2 \rangle$ is any vector in V_2, then

$$a = a_1 i + a_2 j$$

Q If $a = 5i+j$, $b = 4i-7j$, express $3a-2b$ as linear combination of i and j.

Sol:

$$3a-2b = 3(5i+j) - 2(4i-7j)$$
$$= 15i + 3j - 8i + 14j$$
$$= 15i - 8i + 3j + 14j$$
$$= 7i + 7j$$

Q Find a unit vector having the same direction as $3i - 4j$

Sol: $|a| = \sqrt{9+16} = \sqrt{25} = 5$

$$u = \frac{1}{5}(3i-4j) = \frac{3}{5}i - \frac{4}{5}j$$

Vectors in 3-D :-

$$a = \langle a_1, a_2, a_3 \rangle = a_1 i + a_2 j + a_3 k$$

(14.19) The dot product $a \cdot b$ of $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ is $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

(14.23) If θ is the angle between vectors a and b, then $\cos\theta = \dfrac{a \cdot b}{|a||b|}$

Q Find the angle between $a = \langle 4, -3, 1 \rangle$ and $b = \langle -1, -2, 3 \rangle$

Sol:-
$$\cos\theta = \frac{(4)(-1) + (-3)(-2) + (1)(3)}{\sqrt{16+9+1}\sqrt{1+4+4}}$$

$$\cos\theta = \frac{4}{3\sqrt{26}} \Rightarrow \theta = 74.84 \approx 1.31 \text{ rad}$$

Vector product :-

(14.30) The v.p $a \times b$ of $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ is

$$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

(14.31) $$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$\boxed{\text{Assgn\#5: } a \times b \text{ if } a = \langle 2, -1, 6 \text{ and } b = \langle -3, 5, 1}$

Taylor formula:-

Let f have $n+1$ derivatives throughout an interval containing a. If x is any number in the interval, then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

where z is a number between a and x.

Q1 If $f(x) = \ln x$, find Taylor's formula with the remainder for $n=3$ and $a=1$

Sol:- If $n=3$, we need the first 4 der of f.

$f(x) = \ln x \quad , \quad f(1) = 0$

$f'(x) = 1/x \quad , \quad f'(1) = 1$

$f''(x) = -1/x^2 \quad , \quad f''(1) = -1$

$f'''(x) = 2x^{-3} \quad , \quad f'''(1) = 2$

$f''''(x) = -6x^{-4} \quad , \quad f''''(z) = -6z^{-4}$

So,

$$\ln x = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6z^{-4}}{4!}(x-1)^4$$

Where z is between 1 and x, we get

$$\ln x = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4z^{+4}}(x-1)^4$$

Q2 $f(x) = \sin x$ for $n=4$ and $a = \pi/2$

Sol:- If $n=4$, we need first 5 der. of f

$f(x) = \sin x \quad , \quad f(\pi/2) = 1$

$f'(x) = \cos x \quad , \quad f'(\pi/2) = 0$

$f''(x) = -\sin x \quad , \quad f^2(\pi/2) = -1$

$f^3(x) = -\cos x \quad , \quad f^3(\pi/2) = 0$

$$f^{(4)}(x) = \sin x \quad , \quad f^{4}(\pi/2) = 1$$
$$f^{5}(x) = \cos x \quad , \quad f^{5}(\cancel{\pi/2}) = \cos z$$

So,

$$\sin x = 1 + \frac{0}{1!}(x-\pi/2) - \frac{1}{2!}(x-\pi/2)^2 + \frac{0}{3!}(x-\pi/2)^3 + \frac{1}{4!}(x-\pi/2)^4$$
$$+ \frac{\cos z}{5!}(x-\pi/2)^5$$

$$\sin x = 1 - \frac{1}{2}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 + \frac{\cos z}{5!}(x-\pi/2)^5$$

<u>Maclaurin's formula:</u>

Let f have $n+1$ derivatives throughout an interval containing 0. If x is any number in the interval, then

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)}{2!}x^2 + \cdots \cdots \frac{f^{(n+1)}z}{(n+1)!}x^{n+1}$$

where z is between 0 and x

Q1 Find Maclaurin's formula for $f(x) = e^x$ and $n = 5$

sol.: If $n = 5$, we need the first 6 der of f

$$f(x) = e^x \quad , \quad f(0) = 1$$
$$f'(x) = e^x \quad , \quad f'(0) = 1$$
$$f''(x) = e^x \quad , \quad f''(0) = 1$$
$$f^{3}(x) = e^x \quad , \quad f^{3}(0) = 1$$
$$f^{4}(x) = e^x \quad , \quad f^{4}(0) = 1$$
$$f^{5}(x) = e^x \quad , \quad f^{5}(0) = 1$$
$$f^{6}(x) = e^x \quad , \quad f^{6}(z) = e^z$$

So,

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{e^z}{6!}x^6$$

Q2 Find Maclaurin's formula for $f(x) = \cos x$ and $n = 8$

Sol. If $n = 8$, we need the first $(8+1)$ der. of f

$$f(x) = \cos x, \quad f(0) = 1$$
$$f'(x) = -\sin x, \quad f'(0) = 0$$
$$f''(x) = -\cos x, \quad f''(0) = -1$$
$$f^3(x) = \sin x, \quad f^3(0) = 0$$
$$f^4(x) = \cos x, \quad f^4(0) = 1$$
$$f^5(x) = -\sin x, \quad f^5(0) = 0$$
$$f^6(x) = -\cos x, \quad f^6(0) = -1$$
$$f^7(x) = \sin x, \quad f^7(0) = 0$$
$$f^8(x) = \cos x, \quad f^8(0) = 1$$
$$\cancel{f^9(x)} \; f^9(x) = -\sin x, \quad f^9(z) = -\sin z$$

So,

$$\cos x = 1 + \frac{0}{1!}x + \frac{(-1)}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5$$
$$+ \frac{1}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1}{8!}x^8 + \frac{(-\sin z)}{9!}x^9$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \frac{(-1)}{6!}x^6 + \frac{1}{8!}x^8 + \frac{-\sin z}{9!}$$

Power Series:

Let x be a variable. A power series in x is a series of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

where each a_i is a real number.

Partial derivatives:

$$f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = \frac{\partial f}{\partial y}$$

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$$

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y)$$

Q If $f(x,y) = x^3y^2 - 2x^2y + 3x$

a) $f_x(x,y)$ and $f_y(x,y)$ b) $f_x(2,-1)$ and $f_y(2,-1)$

Sol: Now $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$

$$= \frac{\partial}{\partial x}\left[x^3y^2 - 2x^2y + 3x\right]$$

$$f_x(x,y) = 3x^2y^2 - 4xy + 3$$

Also, $f_y(x,y) = \frac{\partial}{\partial y}\left[x^3y^2 - 2x^2y + 3x\right]$

$$f_y(x,y) = 2x^3y - 2x^2$$

b) $f_x(2,-1) = 3(2)^2(-1) - 4(2)(-1) + 3 = 23$

$f_y(2,-1) = 2(2)^3(-1) - 2(2)^2 = -24$

i) $\frac{\partial}{\partial x}(uv) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x}$

ii) $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

iii) $\frac{\partial}{\partial x}(u^n) = nu^{n-1}\frac{\partial u}{\partial x}$

Q Give $\frac{\partial w}{\partial y}$ if $w = xy^2e^{xy}$

Sol-:

$$\frac{\partial w}{\partial y} = xy^2\frac{\partial}{\partial y}(e^{xy}) + e^{xy}\frac{\partial}{\partial y}(xy^2)$$

$$= xy^2(xe^{xy}) + e^{xy}(2xy)$$

$$= x^2y^2e^{xy} + 2xye^{xy}$$

$$\frac{\partial w}{\partial y} = (xy+2)\,xye^{xy}$$

If f is a function of two variables x and y, then f_x and f_y are also functions of two variables. These are called the second partial derivatives of f.

i) $\frac{\partial}{\partial x}f_x = (f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

ii) $\partial/\partial y\, f_x = (f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \partial^2 f/\partial y\partial x$

iii) $\partial/\partial x\, f_y = (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \partial^2 f/\partial x\partial y$

iv) $\partial/\partial y\, f_y = (f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \partial^2 f/\partial y^2$

f_{xy} and f_{yx} are mixed second partial derivatives of f

Q Find the second partial der of f if

$$f(x,y) = x^3y^2 - 2x^2y + 3x$$

$$\cancel{f(y,x) = 2x^3y - 2x^2}$$

Sol:- $f_{xx}(x,y) = \frac{\partial}{\partial x}(x^3y^2 - 2x^2y + 3x)$

$$= 3x^2y^2 - 4xy + 3$$

$$= \frac{\partial}{\partial x}(3x^2y^2 - 4xy + 3)$$

$$f_{xx}(x,y) = 6xy^2 - 4y$$

Now,

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(3x^2y^2 - 4xy + 3)$$

$$f_{xy}(x,y) = 6x^2y - 4x$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(2x^3y - 2x^2)$$

$$f_{yx}(x,y) = 6x^2y - 4x$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}(2x^3y - 2x^2) = 2x^3$$

Chain rule :-

If $w = f(u,v)$, $u = g(x,y)$ and $v = k(x,y)$ where f is differentiable and g and k have continuous first partial derivatives, then

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

Q If $w = u^3 + v^2$, $u = xy^2$ and $v = x^2 \sin y$, use the chain rule to find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$

Sol:-

$$w = u^3 + v^2$$

$$w = (xy^2)^3 + (x^2 \sin y)^2$$

Now,

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial u}(u^3+v^2)\frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial v}(u^3+v^2)\frac{\partial}{\partial x}(x^2 \sin y)$$

$$= 3u^2(y^2) + 2v(2x \sin y)$$

$$= 3(xy^2)^2 y^2 + 2(x^2 \sin y)(2x \sin y)$$

$$= 3x^2y^4y^2 + 4x^3 \sin^2 y$$

$$\frac{\partial w}{\partial x} = 3x^2y^6 + 4x^3 \sin^2 y$$

(Assgn#2)
$w = u \sin v$
$u = x^2 + y^2$
$v = xy$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial u}(u^3+v^2)\frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial v}(u^3+v^2)\frac{\partial}{\partial y}(x^2 \sin y)$$

$$= 3u^2(2xy) + 2v(x^2 \cos y)$$

$$= 3(xy^2)^2(2xy) + 2(x^2 \sin y)(x^2 \cos y)$$

$$= 6xy(x^2y^4) + 2x^4 \sin y \cos y$$

$$\frac{\partial w}{\partial y} = 6x^3y^5 + 2x^4 \sin y \cos y$$