Convolution Sum Representation Of LTI Systems: (Proof):- V.V.V Imp

The response of a linear system to a discrete time signal $x[n]$ is the superposition of the scaled responses of the system to each of the shifted impulses.

Consider the response of a linear system to an arbitrary input $x[n]$. Let $h_k[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n-k]$. Then according to the superposition property for a linear system, the response $y[n]$ of a linear system to the input $x[n]$ is simply the weighted linear combination of these basic responses.

i.e

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h_k[n] \longrightarrow (1)$$

Let a signal $x[n]$ is applied as the input to a linear system whose responses are given by:-

If the linear system is also time invariant, these responses to time-shifted unit impulses are all time shifted versions of each other. Since, $\delta[n-k]$ is a time-shifted version of $\delta[n]$, the response $h_k[n]$ is a time-shifted version of $h_0[n]$ i.e.

$$h_k[n] = h_0[n-k].$$

For notational convenience ; $h[n] = h_0[n]$.

Hence, for an LTI system, eq (1) becomes:-

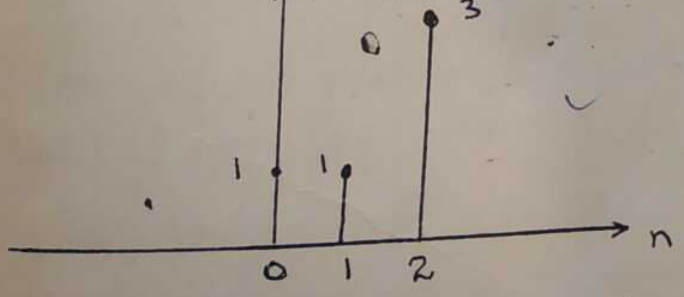
$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]}$$

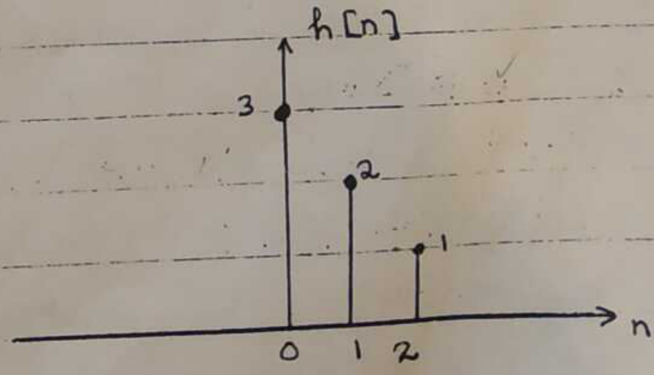
This result is known as the convolution sum or superposition sum & the operation on the right hand side is known as the convolution of the sequences $x[n]$ and $h[n]$. This can also be represented as:

$$\boxed{y[n] = x[n] * h[n].}$$

Example:-

Given that:-





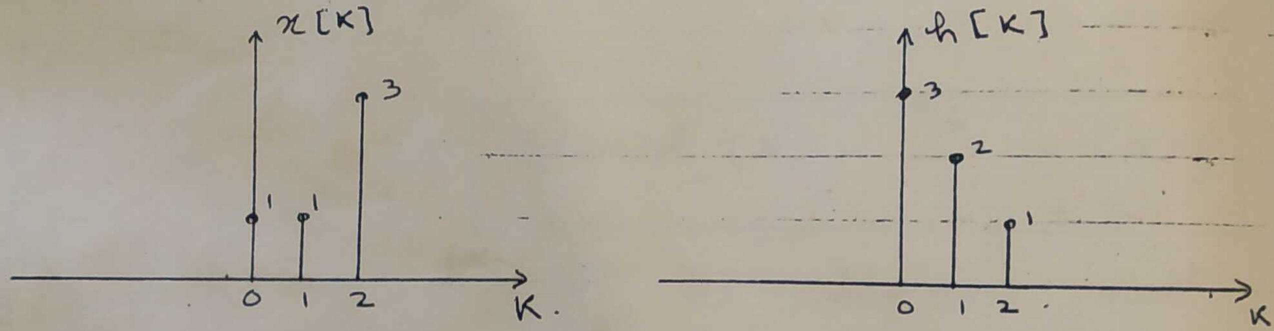
Find $y[n]$ using convolution summation.

Solution:
As we know that the formula for convolution summation is given by.

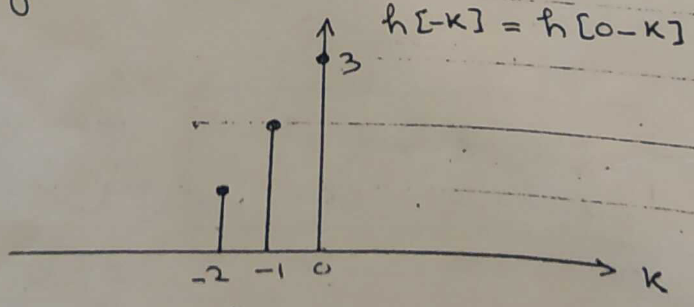
$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$

Step #1:
Replace "K" for "n" in the given signal & impulse response.



Step #2:
Reflect the signal (i.e impulse response) $h[K]$ to get $h[-K]$.



Step # 3:
For the interval.

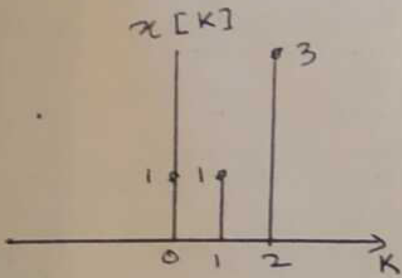
$-\infty < n < 0$

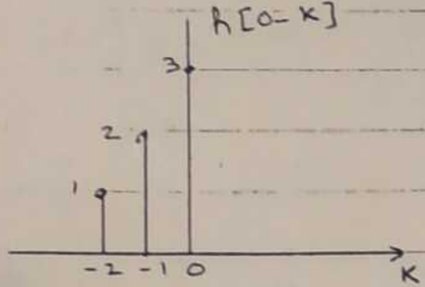
$h(n-K)$ is a value between $-\infty$ and 0. So when $h(n-K)$ is multiplied by $x[K]$, the output is zero. i.e

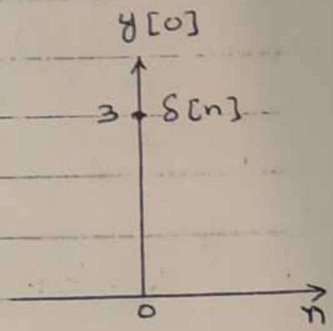
$$\boxed{y[n] = 0}$$

For $n \geq 0$.

At $n = 0$:-







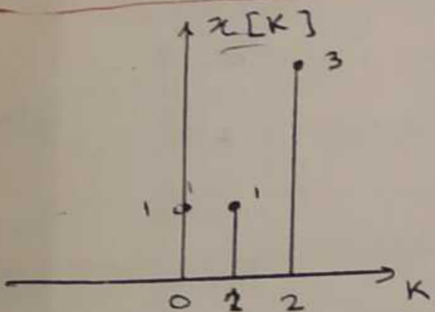
i.e

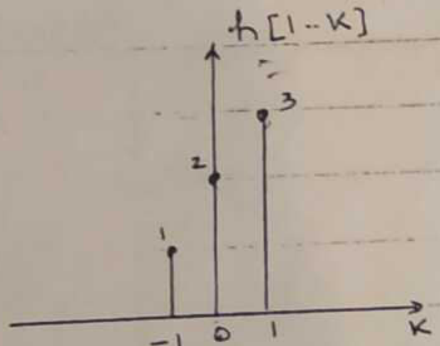
$$y[0] = \sum_{K=-\infty}^{\infty} x[K]\, h[0-K]$$

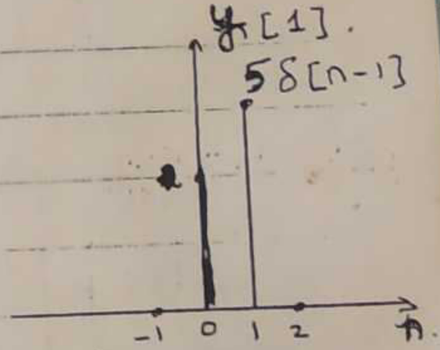
$= (1)(3)$

$\boxed{y[0] = 3}$ $\Rightarrow$ $y[0] = 3\,\delta[n] \longrightarrow (1)$

At $n = 1$







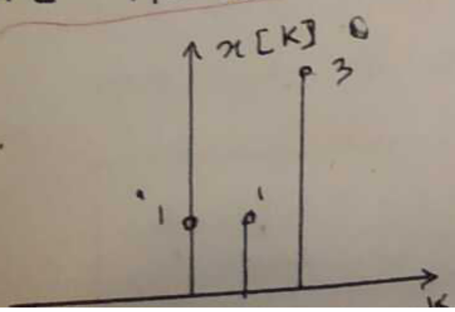
i-e

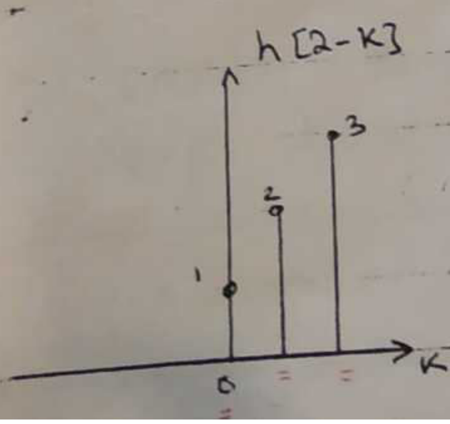
$$y[1] = \sum_{K=-\infty}^{\infty} x[K]\, h[1-K]$$

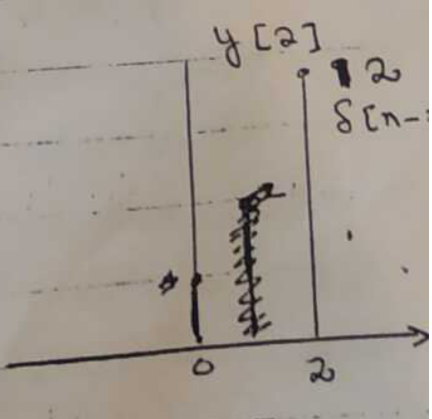
$= (2)(1) + (1)(3)$

$\boxed{y[1] = 5}$ $\Rightarrow$ $y[1] = \cancel{\ \ } + 5\,\delta[n-1] \rightarrow$

At $n = 2$:





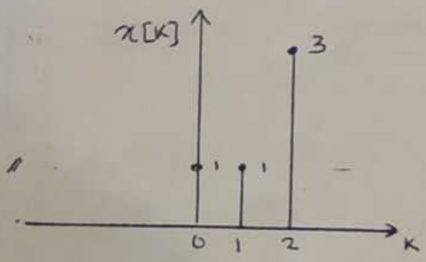


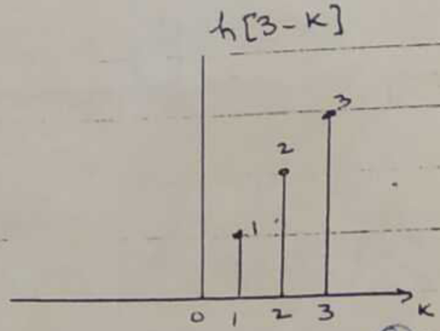
$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\, h[2-k].$$

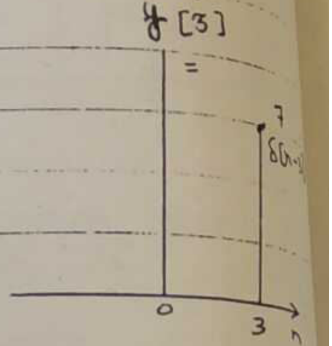
$$y[2] = 1 + 2 + 9$$

$$\boxed{y[2] = 12} \quad \Rightarrow \quad y[2] = 12\,\delta[n-2].$$

At $n = 3$:







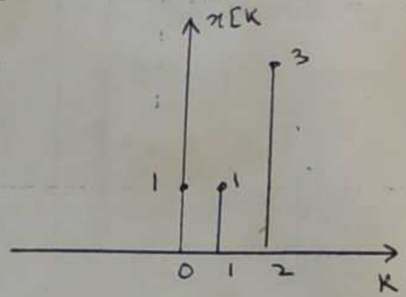
$0\times1 + 1\times1 + 2\times3 + 3\times0 = 6+1 = 7$

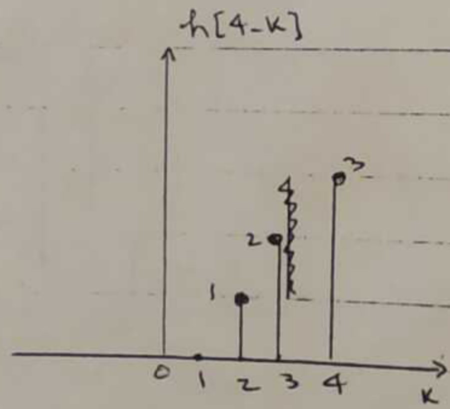
i.e

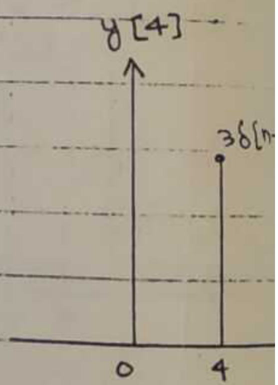
$$y[3] = (1)(1) + (3)(2) = 7$$

$$= 7\,\delta[n-3].$$

At $n = 4$:







$$y[4] = 3\,\delta[n-4]$$ $1\times0 + 1\times0 + 3\times1 +$

For $n > 4$

There is no overlaping of the signal $x[k]$ & $h[n-k]$. Hence.

$$y[n] = 0.$$

Overall output $y[n]$

Overall output $y[n]$ can be written as:

$$\boxed{y[n] = 3\,\delta[n] + 5\,\delta[n-1] + 12\,\delta[n-2] + 7\,\delta[n-3] + 3\,\delta[n-4]}$$

rraphically: