Lecture 13

55. Stokes Theorem

Recall that Green's Theorem allows us to find the work (as a line integral) performed on a particle around a simple closed loop path C by evaluating a double integral over the interior R that is bounded by the loop:

$$\text{Green's Theorem: } \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(N_x - M_y\right) dA.$$

Green's Theorem is restricted to closed loop paths in R^2. What about a closed loop path in R^3? For such paths, we use **Stokes Theorem**, which extends Green's Theorem into R^3.

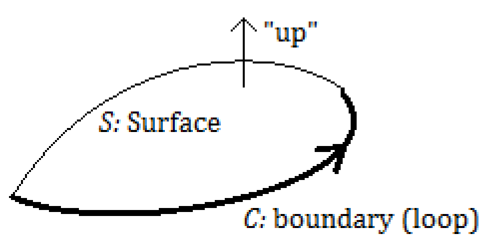
If $\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$ is a vector field and S is a simple oriented surface in R^3 with a boundary C, then Stokes Theorem is given by

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS.$$

Recall that curl $\mathbf{F}$ is defined by

$$\begin{aligned}\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \\ &= \left(P_y - N_z\right)\mathbf{i} - (P_x - M_z)\mathbf{j} + \left(N_x - M_y\right)\mathbf{k} \\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle.\end{aligned}$$

A positive orientation of the surface is stated such that the path C is traversed counterclockwise. However, in R^3, the notion of counterclockwise can be less intuitive. Thus, a positively-oriented surface is one where someone standing "up" would walk the loop with their left arm hanging over the surface.



In Green's Theorem, the surface *S* is the region *R* in the *xy*-plane, and "up" is in the positive *z* direction. Since $\mathbf{F} = \langle M, N, 0\rangle$ in R^2, then curl $\mathbf{F} = \langle 0,0, N_x - M_y\rangle$ so that Green's Theorem is a special case of Stokes Theorem when limited to R^2.

The integral $\iint_S$ (curl $\mathbf{F}$) $\cdot$ $\mathbf{n}\, dS$ needs to be expanded so that it can be useful. Suppose for now that the surface *S* is defined by $z = f(x, y)$. From this, we have that $\mathbf{n}$ is a normal vector to f by

$$\mathbf{n} = \frac{\langle f_x, f_y, -1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}} \text{ or } \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

Also, recall that $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA$. Thus, making substitutions, we have

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (\text{curl } \mathbf{F}) \cdot \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}} \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

This simplifies to

$$\iint_R (\text{curl } \mathbf{F}) \cdot \langle -f_x, -f_y, 1\rangle\, dA.$$

The vector $\langle -f_x, -f_y, 1\rangle$ is chosen depending on what direction "up" is stated. We may choose $\langle f_x, f_y, -1\rangle$ in certain cases. A useful tactic is to note that if a path *C* is stated first, we can choose *any* surface f that is bounded by that path. Obviously, we choose "easy" surfaces in such a case.

The usual routine is as follows:

You will be asked to find the value of a line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ around a simple loop path *C* in R^3. Path *C* may be stated explicitly, or may be implied by some surface *S* given by $z = f(x, y)$. You will also be given the vector field $\mathbf{F} = \langle M, N, P\rangle$.

1. Find curl **F**. For now, it will be in terms of *x*, *y* and *z*.
2. Determine $\langle -f_x, -f_y, 1\rangle$ or $\langle f_x, f_y, -1\rangle$, depending on the context. Usually, the first version is used because we can always declare that positive *z* is "up".
3. Find (curl $\mathbf{F}$) $\cdot$ $\langle -f_x, -f_y, 1\rangle$. If variable *z* remains, substitute with $z = f(x, y)$. You now have an expression in terms of *x* and *y*.
4. Determine the region of integration *R*, which will be the footprint cast by *S* onto the *xy*-plane.
5. Integrate the result in step (3) over region *R*.

Normal adjustments would be made, *e.g.* if the surface was stated as $x = f(y, z)$.

Example 55.1: Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z)=\langle xy, x+y+z, x^2\rangle$ and C is a circle of radius 1, centered at the origin, in the *xy*-plane, traverse counterclockwise where "up" is the positive *z* direction.

Solution: No surface *S* is specified, just a boundary path *C*. So let's try a couple different surfaces that have *C* as its boundary. First, we will let *S* be the interior of the circle in the *xy*-plane. That is, $z=f(x,y)=0$. Thus, $\mathbf{n}=\langle 0,0,1\rangle$.

Next, we find curl $\mathbf{F}$:

$$\text{curl }\mathbf{F}=\langle P_y-N_z, M_z-P_x, N_x-M_y\rangle=\langle -1,-2x,1-x\rangle.$$

Thus, $(\text{curl }\mathbf{F})\cdot\mathbf{n}=1-x$. This is integrated over the region inside the circle of radius 1, centered at the origin. We use polar coordinates, where $x=r\cos\theta$:

$$\begin{aligned}\iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS &= \iint_S (1-x)\,dA\\ &= \int_0^{2\pi}\int_0^1 (1-r\cos\theta)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1 (r-r^2\cos\theta)\,dr\,d\theta.\end{aligned}$$

We have

$$\int_0^1 (r-r^2\cos\theta)\,dr=\left[\frac{1}{2}r^2-\frac{1}{3}r^3\cos\theta\right]_0^1=\frac{1}{2}-\frac{1}{3}\cos\theta\,.$$

Then, we have

$$\int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{3}\cos\theta\right)\,d\theta=\left[\frac{1}{2}\theta-\frac{1}{3}\sin\theta\right]_0^{2\pi}=\pi.$$

Therefore, with *S* as the portion of the *xy*-plane inside the circle of radius 1 centered at the origin, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\pi.$$

Next, let's try a different surface: Let S be the paraboloid $z = f(x, y) = 1 - x^2 - y^2$ that lies above the xy-plane. Note that C is the same bounding curve. We find $\mathbf{n}$:

$$\mathbf{n} = \langle -f_x, -f_y, 1 \rangle = \langle -(-2x), -(-2y), 1 \rangle = \langle 2x, 2y, 1 \rangle.$$

The curl $\mathbf{F}$ has not changed. Thus,

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle -1, -2x, 1 - x \rangle \cdot \langle 2x, 2y, 1 \rangle = -3x - 4xy + 1.$$

The region of integration is the same—the interior of the circle of radius 1, centered at the origin. Once again, we use polar coordinates:

$$\begin{aligned}\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS &= \iint_S (-3x - 4xy + 1)\, dA \\ &= \int_0^{2\pi} \int_0^1 (-3r\cos\theta - 4(r\cos\theta)(r\sin\theta) + 1)\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (-3r^2\cos\theta - 4r^3\cos\theta\sin\theta + r)\, dr\, d\theta.\end{aligned}$$

The inside integral, evaluated with respect to r, is

$$\begin{aligned}\int_0^1 (-3r^2\cos\theta - 4r^3\cos\theta\sin\theta + r)\, dr\, d\theta &= \left[-r^3\cos\theta - r^4\cos\theta\sin\theta + \frac{1}{2}r^2\right]_0^1 \\ &= -\cos\theta - \cos\theta\sin\theta + \frac{1}{2}.\end{aligned}$$

Then this is integrated with respect to θ:

$$\int_0^{2\pi} \left(-\cos\theta - \cos\theta\sin\theta + \frac{1}{2}\right) d\theta = \left[-\sin\theta - \frac{1}{2}\sin^2\theta + \frac{1}{2}\theta\right]_0^{2\pi} = \pi.$$

Note that at $\theta = 0$ and 2π, the sine terms vanish. Thus, we get the same result, $\int_C \mathbf{F} \cdot d\mathbf{r} = \pi$.

Try this with another surface, for example, the hemisphere of radius 1, $z = \sqrt{1 - x^2 - y^2}$.

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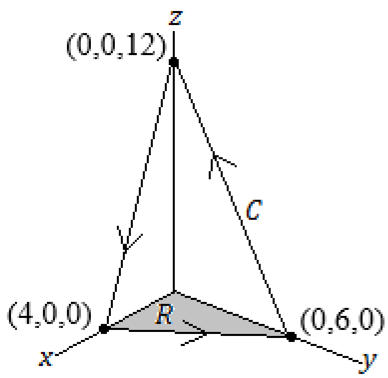
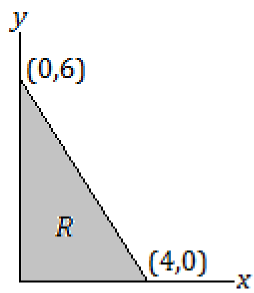
Example 55.2: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle x + y, zy, 3x\rangle$ and C is the triangle traversed from (4,0,0) to (0,6,0) to (0,0,12), back to (4,0,0). Assume "up" is in the direction of positive z.

Solution: Since no surface is specified, let's use a plane passing through the vertices of the triangle. Below is an image of the path C and the eventual region of integration R:





The plane is $\frac{x}{4} + \frac{y}{6} + \frac{z}{12} = 1$, or $3x + 2y + z = 12$ when fractions are cleared. We can read off a normal vector from the plane's equation: $\mathbf{n} = \langle 3,2,1\rangle$. This is a useful vector since it has a 1 in the z position, agreeing with the upward direction. We find curl $\mathbf{F}$, which is $\langle -y, -3, -1\rangle$. Thus,

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (-3y - 7)\, dA = \int_0^4 \int_0^{6-(3/2)x} (-3y - 7)\, dy\, dx.$$

The inside integral is

$$\begin{aligned}\int_0^{6-(3/2)x} (-3y - 7)\, dy &= \left[-\frac{3}{2}y^2 - 7y\right]_0^{6-(3/2)x} \\ &= -\frac{3}{2}\left(6 - \frac{3}{2}x\right)^2 - 7\left(6 - \frac{3}{2}x\right) \\ &= -\frac{27}{8}x^2 + \frac{75}{2}x - 96.\end{aligned}$$

The outside integral is

$$\int_0^4 \left(-\frac{27}{8}x^2 + \frac{75}{2}x - 96\right) dx = \left[-\frac{9}{8}x^3 + \frac{75}{4}x^2 - 96x\right]_0^4 = -156.$$

Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} = -156$. Let's verify this by finding the line integral along each segment of the triangle.

From (4,0,0) to (0,6,0), we have $\mathbf{r}(t) = \langle 4-4t, 6t, 0\rangle$ for $0 \le t \le 1$, so that $d\mathbf{r} = \langle -4,6,0\rangle$. Meanwhile,

$$\begin{aligned}\mathbf{F}(t) &= \langle x+y, zy, 3x\rangle \\ &= \langle (4-4t)+(6t), (0)(6t), 3(4-4t)\rangle \quad \begin{cases} x = 4-4t \\ y = 6t \\ z = 0 \end{cases}\end{aligned}$$

or after simplification, $\mathbf{F}(t) = \langle 4+2t, 0, 12-12t\rangle$. Thus, $\mathbf{F}\cdot d\mathbf{r} = -4(4+2t) = -16-8t$, and the line integral is

$$\int_0^1 (-16-8t)\,dt = [-16t-4t^2]_0^1 = -20.$$

From (0,6,0) to (0,0,12), we have $\mathbf{r}(t) = \langle 0, 6-6t, 12t\rangle$ for $0 \le t \le 1$, so that $d\mathbf{r} = \langle 0,-6,12\rangle$. Meanwhile, $\mathbf{F}(t) = \langle 6-6t, 72t-72t^2, 0\rangle$ after simplification.

Thus, $\mathbf{F}\cdot d\mathbf{r} = -6(72t-72t^2) = -432(t-t^2)$, and the line integral is

$$\int_0^1 -432(t-t^2)\,dt = -432\left[\frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1 = -72$$

From (0,0,12) to (4,0,0), we have $\mathbf{r}(t) = \langle 4t, 0, 12-12t\rangle$ for $0 \le t \le 1$. This gives $d\mathbf{r} = \langle 4,0,-12\rangle$. Also, $\mathbf{F}(t) = \langle 4t, 0, 12t\rangle$ after simplification. Therefore,

$$\mathbf{F}\cdot d\mathbf{r} = 4(4t) - 12(12t) = 16t - 144t = -128t.$$

Finally, the line integral is

$$\int_0^1 -128t\,dt = -128\left[\frac{1}{2}t^2\right]_0^1 = -64.$$

The sum of these three line integrals is $-20-72-64 = -156$, agreeing with the result found by Stokes Theorem.

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