Fourier series →

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$
$$= a_0 + a_1 \cos x + a_2 \cos 2x + \text{———}$$
$$+ b_1 \sin x + b_2 \sin 2x + \text{———}$$

We assume that fourier series converges and has a continuous fn f(x) as its sum of interval $[-\pi, \pi]$ so,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$
$$-\pi \leq x \leq \pi$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left[(a_n \cos nx + b_n \sin nx)\,dx\right]$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\,dx$$

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$$

or $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

To determine a_n, multiply (x) on both sides by $\cos mx$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]\cos mx\,dx$$

$$= a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx +$$

$$\sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx$$

Now $\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = 0$ for all n and m

$$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \begin{cases} 0 & \text{for } n \neq m \\ \pi & \text{for } n = m \end{cases}$$

$$\int_{-\pi}^{\pi} f(x)\cos mx = a_m\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

a_0, a_n, b_n are fourier co-efficients of f

Q $f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ 1 & \text{if } 0 \leq x < \pi \end{cases}$

$f(x+2\pi) = f(x)$

Sol. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx$$

$$= 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx$$

$$= \frac{1}{n\pi}\left[\sin nx\right]_0^{\pi}$$

$$= \frac{1}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$= 0$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx$$

$$= -\frac{1}{n\pi}\left[\cos nx\right]_0^{\pi}$$

$$= -\frac{1}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Fourier series can be given as :-

$$= \frac{1}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots$$
$$b_1\sin x + b_2\sin 2x + b_3\sin 3x$$

$$= \frac{1}{2} + 0 + 0 + 0 + \dots + \frac{2}{\pi}\sin x + 0\sin 2x$$
$$+ \frac{2}{3\pi}\sin 3x + \dots$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin$$

Q $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

Sol:- $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$

$= \frac{1}{\pi} \int_0^{\pi} 1 dx + \frac{1}{\pi} \int_{-\pi}^{0} (-1) dx$

$= 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$= \frac{1}{\pi} \int_0^{\pi} \cos nx \, dx + \frac{1}{\pi} \int_{-\pi}^{0} -\cos nx \, dx$

$= \frac{1}{n\pi} \left[[\sin nx]_0^{\pi} - [\sin nx]_{-\pi}^{0} \right]$

$= \frac{1}{n\pi} \left\{ (\sin n\pi - \sin 0) - (\sin 0 - \sin n(-\pi)) \right]$

$= \frac{1}{n\pi} [0 - 0] = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$= \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx + \frac{1}{\pi} \int_{-\pi}^{0} \sin nx \, dx$

$= \frac{1}{n\pi} \left[(-\cos n\pi + \cos 0) + (\cos 0 - \cos \right.$

$= \frac{1}{n\pi} \left[-(-1) + 1 + 1 - (-1) \right]$

$= \frac{1}{n\pi} [1 + 1 + 1 + 1] = \frac{4}{n\pi}$

$f(x) = \frac{4}{1\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x +$