Lecture 10:
Line Integral and Potential of Point Charge

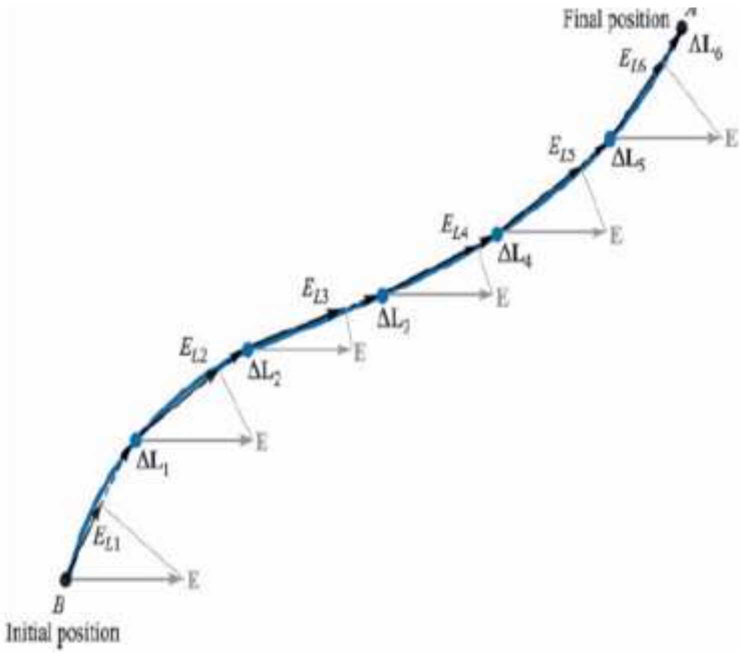
- Related YouTube Video Link:
  1) https://www.youtube.com/watch?v=C7jjaqt7E-A
  2) https://www.youtube.com/watch?v=uVDriEyIRg0
  3) https://www.youtube.com/watch?v=wBJvbww6EiY
  4) https://www.youtube.com/watch?v=p8OSoburdt0
  5) https://www.youtube.com/watch?v=Mltwn2G8Ors

- Read 4.1 till 4.4 of the Given Book

Lecture Notes on next Page

The Line Integral

- The integral expression of previous equation is an example of a line integral, taking the form of integral along a prescribed path.



- Without using vector notation, we should have to write:

$$W = -Q\int_{\text{init}}^{\text{final}} E_L dL$$

  - ***E_L*: component of E along *d*L**

- The work involved in moving a charge Q from B to A is approximately:

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

- If we assume that the electric field is uniform,

$$\mathbf{E}_1 = \mathbf{E}_2 = \cdots = \mathbf{E}_6$$

$$W = -Q\mathbf{E} \cdot \underbrace{(\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \cdots + \Delta\mathbf{L}_6)}_{\mathbf{L}_{BA}}$$

- Therefore,

$$W = -Q\mathbf{E} \cdot \mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$

- Since the summation can be interpreted as a line integral, the exact result for the uniform field can be obtained as:

$$W = -Q\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

$$W = -Q\mathbf{E} \cdot \int_B^A d\mathbf{L} \qquad \text{(uniform } \mathbf{E}\text{)}$$

$$W = -Q\mathbf{E} \cdot \mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$

  - For the case of uniform E, W does not depend on the particular path selected along which the charge is carried

- Example

Given the nonuniform field E = yax + xay +2az, determine the work expended in carrying 2 C from B(1,0,1) to A(0.8,0.6,1) along the shorter arc of the circle x2 + y2 = 1, z = 1.

$$d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$$

Differential path, rectangular coordinate

$$W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L}$$

$$= -Q\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z)\cdot(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

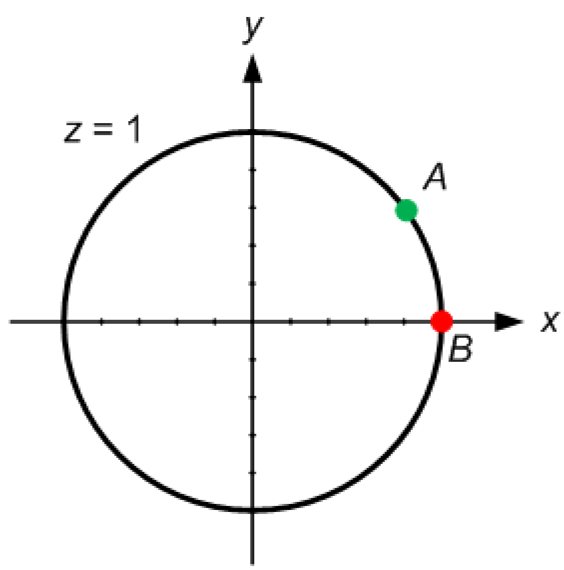
$$= -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 2\int_1^1 2\,dz$$

Circle equation:

$$x^2 + y^2 = 1$$

$$x = \sqrt{1-y^2}$$

$$y = \sqrt{1-x^2}$$



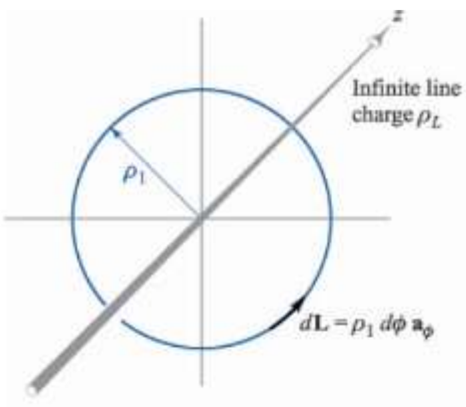
$$W = -2\int_1^{0.8} \sqrt{1-x^2}\,dx - 2\int_0^{0.6} \sqrt{1-y^2}\,dy - 2\int_1^1 2\,dz$$

$$= -2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_1^{0.8} - 2\left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1} y\right]_0^{0.6}$$

$$= \underline{\underline{-0.962\ \text{J}}}$$

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}$$

Work and Path Near an Infinite Line Charge



$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\varepsilon_0 \rho}\mathbf{a}_\rho$$

$$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z$$

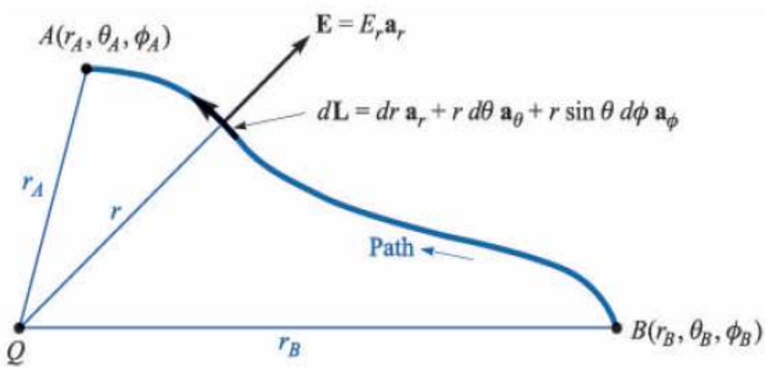
$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\varepsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1 d\phi\mathbf{a}_\phi$$

$$= -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\varepsilon_0} d\phi\mathbf{a}_\rho \cdot \mathbf{a}_\phi$$

$$= 0$$

The Potential Field of a Point Charge

- In previous section we found an expression for the potential difference between two points located at $r = r_A$ and $r = r_B$ in the field of a point charge Q placed at the origin:



$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B$$

$$V_{AB} = -\int_{r_B}^{r_A} E_r dr$$

- Any initial and final values of θ or Φ will not affect the answer. As long as the radial distance between r_A and r_B is constant, any complicated path between two points will not change the results.

- This is because although d**L** has r, θ, and Φ components, the electric field **E** only has the radial r component.

- The potential difference between two points in the field of a point charge depends only on the distance of each point from the charge.

- Thus, the simplest way to define a zero reference for potential in this case is to let $V = 0$ at infinity.

- As the point $r = r_B$ recedes to infinity, the potential at r_A becomes:

$$V_{AB} = V_A - V_B$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A} - \frac{Q}{4\pi\varepsilon_0}\frac{1}{\infty}$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A} = V_A$$

- Generally,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

- Physically, $Q/4\pi\varepsilon_0 r$ joules of work must be done in carrying 1 coulomb charge from infinity to any point in a distance of r meters from the charge Q.

- We can also choose any point as a zero reference:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C_1$$

with C_1 may be selected so that $V = 0$ at any desired value of r

Equipotential Surface

- Equipotential surface is a surface composed of all those points having the same value of potential.
- No work is involved in moving a charge around on an equipotential surface.
- The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.
- The equipotential surfaces in the potential field of a line charge are cylindrical surfaces axed at the line charge.
- The equipotential surfaces in the potential field of a sheet of charge are surfaces parallel with the sheet of charge.