(2) Buckingham's π - Theorm :-

Since Rayleigh's Method becomes laborious if variables are more than fundamental dimensions (MLT) so the difficulty is overcome by Buckingham's π - Theorm which states that:
"If there are n variables (independent and dependent) in a physical phenomenon and if these variables contain "m" fundamental dimensions then the variables are arranged into (n-m) dimension less terms called π - terms".

Let $X_1, X_2, X_3 \text{-----} X_4, X_n$ are the variables involved in a physical problem - Let X_1 be the dependent variable and $X_2, X_3, X_4 \text{---}$ X_n are the independent variables on which X_1 depends. Mathematically it can be written

as; $$X_1 = f(X_2, X_3, X_4, X_n)$$
$$f(X_1 X_2, X_3, X_4, X_n)$$

Above equation is dimensionally homogenous If it contain "n" variables and if there are m fundamental dimensions then it can be written in terms of dimensions groups called π - terms which are equal to (n-m)

Hence; $$f(\pi_1, \pi_2, \pi_3 \text{-----} \pi_{n-m}) = 0$$

Properties of π terms:-

* Each π term is dimension less and is independent of system of units.
* Division or multiplication by a Constant doesnot change the character of the π terms.
* Each π term contain $m+1$ variables, where m is the number of fundamental dimensions also called Repeating Variables.

Let in the above case X_2, X_3, X_4 are repeating variables and if fundamental dimensions $m = 3$, then each π term is written as.

$$\pi_1 = X_2^{a1}, X_3^{b1}, X_4^{c1} \cdot X_1$$

$$\pi_2 = X_2^{a2} X_3^{b2} X_4^{c2} \; X_5$$

$$\vdots$$

$$\pi_{n-m} = X_2^{a(n-m)} X_3^{b(n-m)} X_4^{c(n-m)} \cdot X_n$$

Methods of Selecting Repeating Variables:-

The number of repeating Variables are equal to number of fundamental dimensions of the problem. The choice of repeating Variables is governed by following considerations;

* As far as possible, dependent variable should not be selected as repeating variables.

* The repeating variables should be chosen in such a way that one variable contains geometric property, other contains flow property and third contains fluid property.
* The repeating variables together must contain all three fundamental dimensions (ie) MLT

<u>Note</u>:- In most of hydraulic Engineering and Fluid Mechanics problems, the choice of repeating variables may be

(i) d, v, ρ
(ii) l, v, ρ
(iii) d, v, μ

<u>Problem</u>:- The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of aircraft l, velocity v, air viscosity μ, air density ρ, and bulk density of air K. Express the functional relationship b/w the variables and the resisting force.

<u>Solution</u>:- $R = f(l, v, \mu, \rho, K)$

$$\Rightarrow f(R, l, v, \mu, \rho, K) = 0$$

Total No of Variables $= 6 = n$

No of fundamental dimension $m = 3$

No of dimension less π terms $n - m = 3$

Thus $f(\pi_1, \pi_2, \pi_3) = 0$.

No of Repeating Variables $= m = 3$

Repeating variables $= l, v, \rho$

Thus π - terms are written as

$$\pi_1 = l^{a1} V^{b1} \rho^{c1} R$$

$$\pi_2 = l^{a2} V^{b2} \rho^{c2} \mu$$

$$\pi_3 = l^{a3} V^{b3} \rho^{c3} K$$

Now each pi term is solved by the principal of dimensional homogeneity.

π_1 term $\quad M^0 L^0 T^0 = L^{a1} (LT^{-1})^{b1} (ML^{-3})^{c1} MLT^{-2}$

equating the powers of MLT on both sides

Power of M $\quad 0 = c_1 + 1 \Rightarrow \boxed{c_1 = -1} \rightarrow$ (i)

power of L $\quad 0 = a_1 + b_1 - 3c_1 + 1 \rightarrow$ (iii)

power of T $\quad 0 = -b_1 - 2 \Rightarrow \boxed{b_1 = -2} \rightarrow$ (ii)

put (i) and (ii) in (iii)

$0 = a_1 - 2 - 3(-1) + 1$

$0 = a_1 + 1 + 1$

$\boxed{a_1 = -2}$

$$\pi_1 = l^{-2} V^{-2} \rho^{-1} R$$

$$\boxed{\pi_1 = \frac{R}{\rho l^2 V^2}}$$

π_2 term $= M^0 L^0 T^0 = L^{a_2} (LT^{-1})^{b_2} (ML^{-3})^{c_2} ML^{-1}T^{-1}$

Equating the powers of MLT on both sides.

Power of M: $0 = c_2 + 1 \Rightarrow \boxed{c_2 = -1}$

power of L: $0 = a_2 + b_2 - 3c_2 - 1 \longrightarrow (x)$

power of T: $0 = -b_2 - 1 \Rightarrow \boxed{b_2 = -1}$

put c_2 and b_2 in eq (x)

$0 = a_2 - 1 - 3(-1) - 1$

$0 = a_2 - 1 + 3 - 1$

$0 = a_2 + 1$

$\boxed{a_2 = -1}$

$$\pi_2 = l^{a_2} V^{b_2} \rho^{c_2} \mu = l^{-1} V^{-1} \rho^{-1} \mu$$

$$\boxed{\pi_2 = \frac{\mu}{\rho l V}}$$

π_3 - term $\Rightarrow M^0 L^0 T^0 = L^{a_3} (LT^{-1})^{b_3} (ML^{-3})^{c_3} ML^{-1}T^{-2}$

Equating the powers of MLT on both sides

power of M : $0 = c_3 + 1 \Rightarrow \boxed{c_3 = -1}$

power of L : $0 = a_3 + b_3 - 3c_3 - 1$

power of T: $0 = -b_3 - 2 \Rightarrow \boxed{b_3 = -2}$

$$0 = a_3 - 2 - 3(-1) - 1$$

$$0 = a_3 - 2 + 3 - 1 \Rightarrow \boxed{a_3 = 0}$$

$$\pi_3 = l^{a_3} V^{b_3} \rho^{c_3} K$$

$$\pi_3 = l^0 V^{-2} \rho^{-1} K$$

$$\boxed{\pi_3 = \frac{K}{\rho V^2}}$$

Hence;

$$f(\pi_1, \pi_2, \pi_3) = f\left(\frac{R}{\rho l^2 V^2}, \frac{\mu}{l V \rho}, \frac{K}{V^2 \rho}\right) = 0$$

or

$$\frac{R}{\rho l^2 V^2} = \phi\left[\frac{\mu}{l V \rho}, \frac{K}{V^2 \rho}\right]$$

$$\boxed{R = \rho l^2 V^2 \phi\left[\frac{\mu}{l V \rho}, \frac{K}{V^2 \rho}\right]}$$