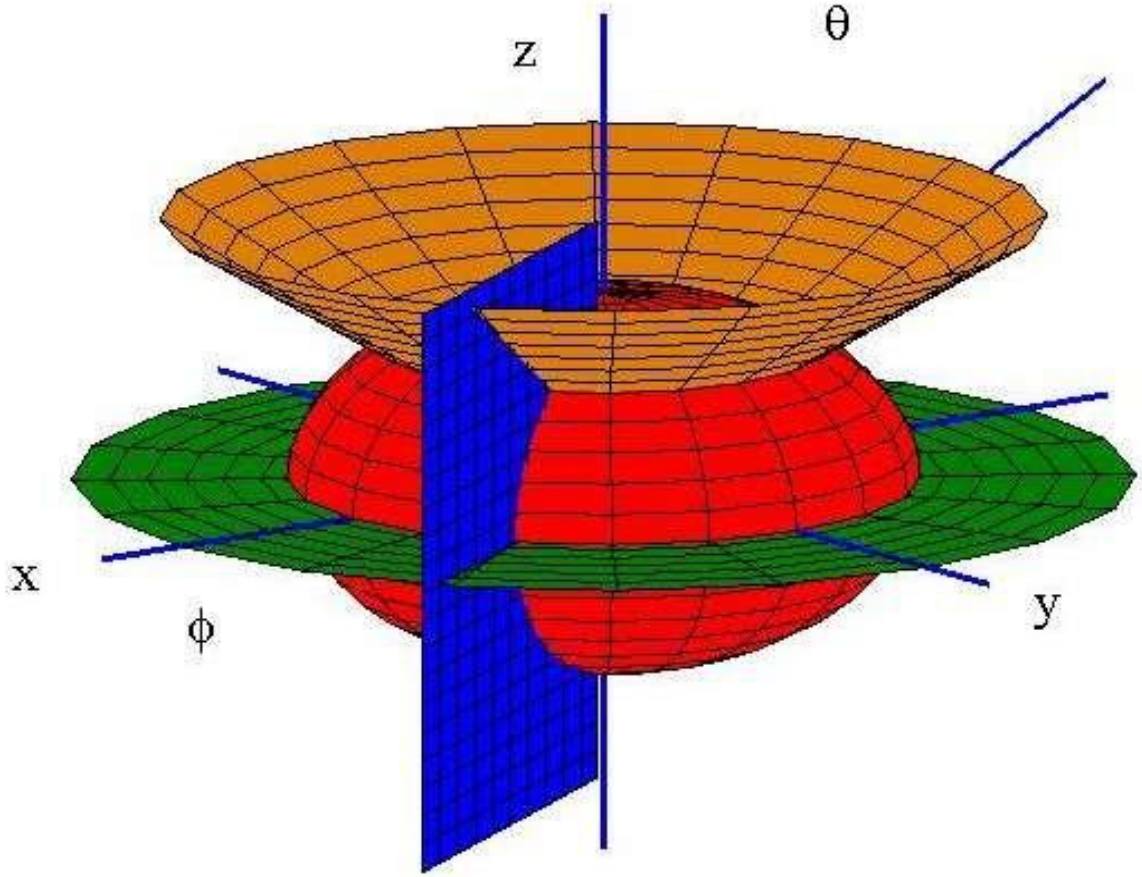

بسم الله الرحمن الرحيم

Electromagnetic Fields



About Me

- **Dr. Rafiq Mansoor**
- Education
  - B.Sc. Electrical Engineering
  - M.S Telecom and Networking
  - Ph.D. Engineering Management
- Experience
  - 07+ Years Teaching
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Course Outline

- Course as according to course outline
- Marks Distribution
  - Assignments – 5%
  - Quizzes – 5%
  - Class Participation – 5%
  - Complex Engineering Problem
  - Mid Term – 30%
  - Final – 50%
- Final Examination Question Paper

Course Breakup

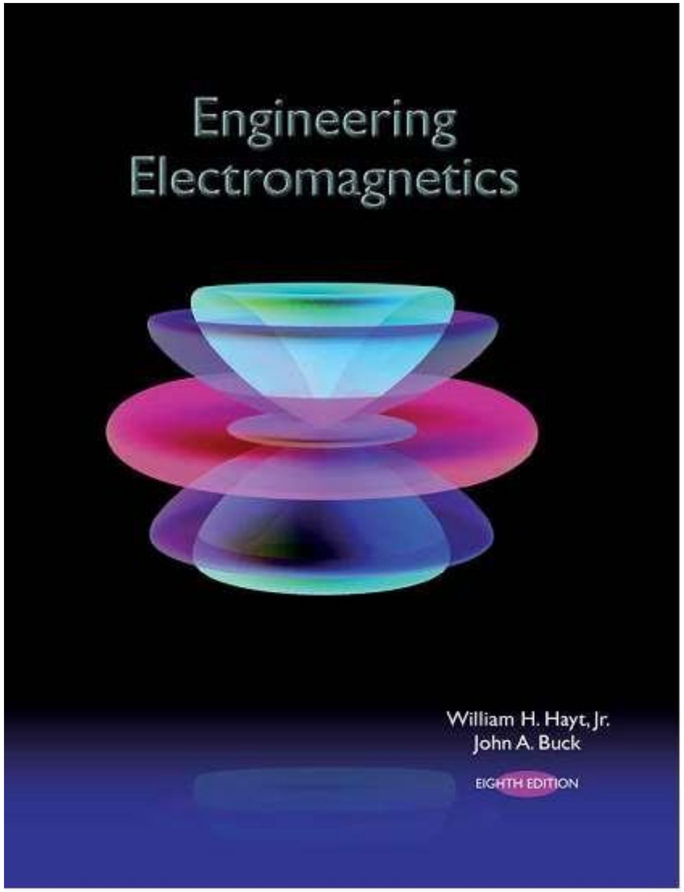
Chapter #	Final Exam Q. No	Chapter Heading
1	1	Vector Analysis
2	2	Coulomb's Law & Electric Field Intensity
3	3	Electric Flux Density, Gauss' Law, & Divergence
4	4	Energy & Potential
6	5,6	Dielectrics & Capacitance
8	7	The Steady Magnetic Field
9	8,9	Magnetic Forces, Materials & Inductance

Course Breakup

Chapter #	Final Exam Q. No	Chapter Heading
10	10	Time-Varying Fields & Maxwell's Equations
11	11	Transmission Lines
12	12	The Uniform Plane Wave

Text Book

- Engineering Electromagnetics – 7^{th} Edition
  - William H. Hayt, Jr. & John A. Buck



Brief History of Electromagnetics

The early stages:

the ancient Greeks and Chinese well aware of some electric and magnetic phenomena (*Plato* and *Socrates*, 600 BC)

Hans Christian Oersted (1777-1851) discovers the relation between current carrying wire and magnetic field

André Ampère (1775-1836) discovers the force between two current-carrying wires

Jean-Baptiste Biot (1774-1862) and *Félix Savart* (1791-1841) formulate the law of the force between current elements

Benjamin Franklin (1706-1790) and *Joseph Priestly* (1733-1804) postulate the inverse square law of electrostatics

Brief History of Electromagnetics, cont.

Coulomb (in 1785) proves experimentally the inverse square law for stationary electric charges

Alessandro Volta (1745-1827) investigates reactions between dissimilar metals and develops the first electric battery (1800)

Karl Friedrich Gauss (1777-1855) formulates the divergence theorem of electricity

EM Trivia

- What are the origins of the word *ELECTRICITY*?

  Electricity is a word coined from the Greek word *electron* (meaning amber). The word was allegedly suggested by William Gilbert, physician to Queen Elizabeth I, physicist and philosopher. The Greeks were the first to document the attraction of small particles (e.g., feathers, hair, dry leaves) to a piece of amber after rubbing it.

- What are the origins of the word *MAGNETISM*?

  According to Lucretius (98-55 BC), the term *magnet* is derived from *Magnesia on the Maeander*, the Greek name of a city in Asia Minor where iron ore was found.

- Who was the first to suggest that the Earth is a giant magnet?

  William Gilbert (1544 – 1603): He wrote "On the Magnet and Magnetic Bodies, and on the Great Magnet the Earth", published in 1600.

Brief History of Electromagnetics, cont.

Milestones of the classical science of electromagnetism

Michael Faraday (1791-1867) discovers in 1831 that time-changing magnetic field produces electric field. Similar observations are made by Joseph Henry (1797-1878).

James Clerk Maxwell (1831-1879) formulates the mathematical model of electromagnetism (classical electro-dynamics), "*A Treatise on Electricity and Magnetism*" (1873).

Brief History of Electromagnetics, cont.

Heinrich Rudolph Hertz (1857-1894) demonstrates in 1886 the first wireless EM wave link. In his memoirs on electrodynamics, he replaces all potentials by field strengths, and deduces Ohm's, Kirchhoff's and Coulomb's laws from Maxwell's equations.

Guglielmo Marconi (the father of radio) sends signals over large distances. In 1901, he performs the first transatlantic transmission from Poldhu in Cornwall, England, to Newfoundland, Canada.

Applications of Electrostatics

- copiers (xerography)
- batteries and battery chargers
- semiconductor device control
- air cleaners
- electro-painting
- ionizing plasma (e.g., fluorescent lights)
- electrostatic separation of ores, garbage, etc.
- charged-coupled device (CCD) cameras
- ink-jet printers
- electrophoresis (separation of charged colloidal particles used in medicine and biology)
- electrostatic motors
- cardiopulmonary resuscitation (CPR)
- cosmetics: electrolysis
- any other suggestions?

Applications of Magnetism

- compasses
- magnetic resonance imaging (MRI)
- electromagnets: switches, industrial transport, etc.
- maglev trains
- loudspeakers
- clasps
- any other suggestions?

Applications of Electromagnetism: “Wired”

“wired” and semiconductor technology

- cable communications (TV, data , telephony)
- digital and analog microelectronics
- power generation, power grids, power supply, power electronics
- electro-mechanical devices (motors, dynamos, relays, actuators, MEMS)

photonics and optics

- light generation, LEDs
- optical fibers, laser technology
- photonic and infrared imaging and surveillance

Applications of Electromagnetism: Wireless

Public and personal services

- radio and TV broadcasting
- cordless telephony
- cellular (mobile) telephony and data transfer
- wireless LANs (local area network) and *bluetooth* data transfer
- satellite communications (telephony, data, TV)
- global navigation/positioning systems (GPS)

Special services

- radars
- microwave relay links
- satellite systems (military/intelligence)
- radio astronomy
- biomedical engineering (imaging and treatment)
- military communications, guidance, surveillance, RCVs, etc.

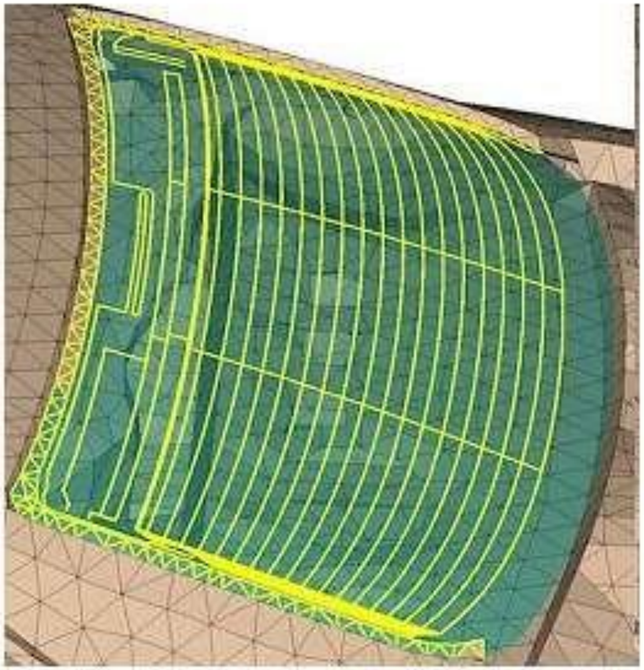
Applications of Electromagnetic Science: CAA and CAD

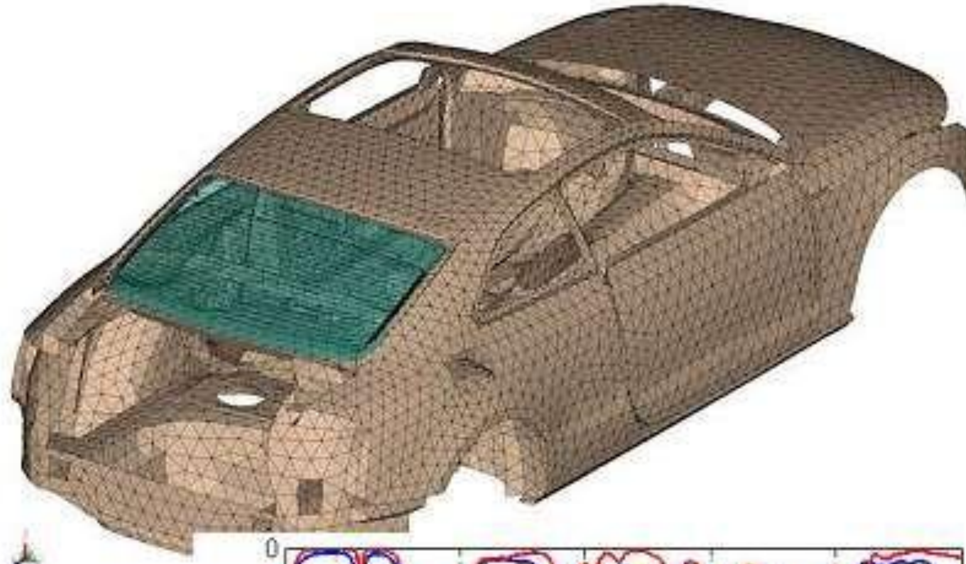
electromagnetic simulators are now required tools in R&D

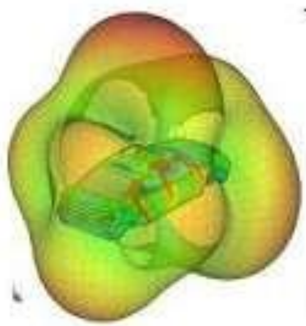
- high-frequency electroimagnetic simulators
- electro/magneto-static simulators
- electro-mechanical simulators
- electromagnetic/thermal simulators

Applications of Electromagnetism: CAA and CAD

windscreen antenna design [EMSS, FEKO]







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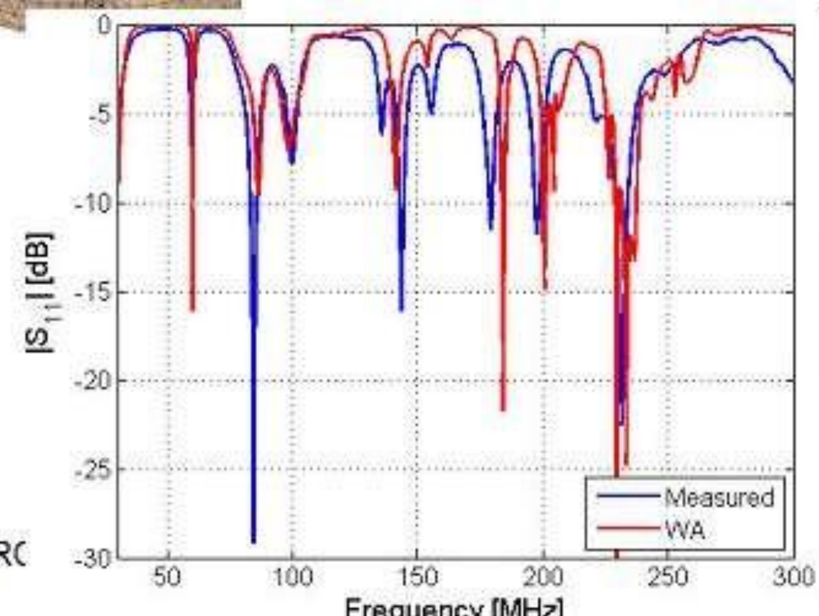
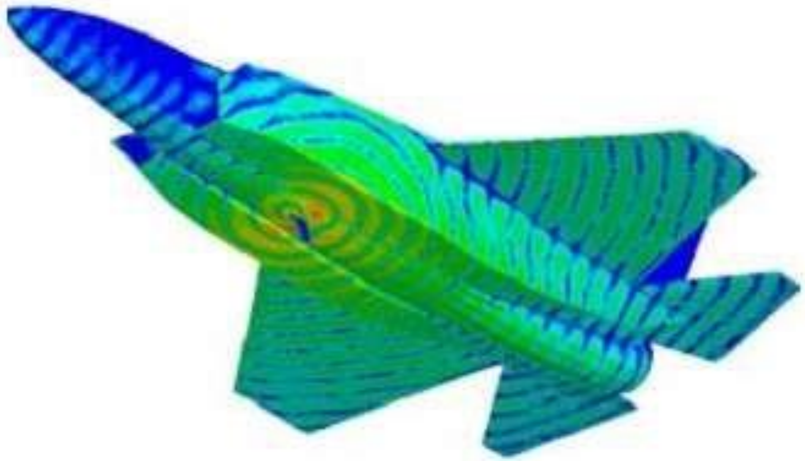
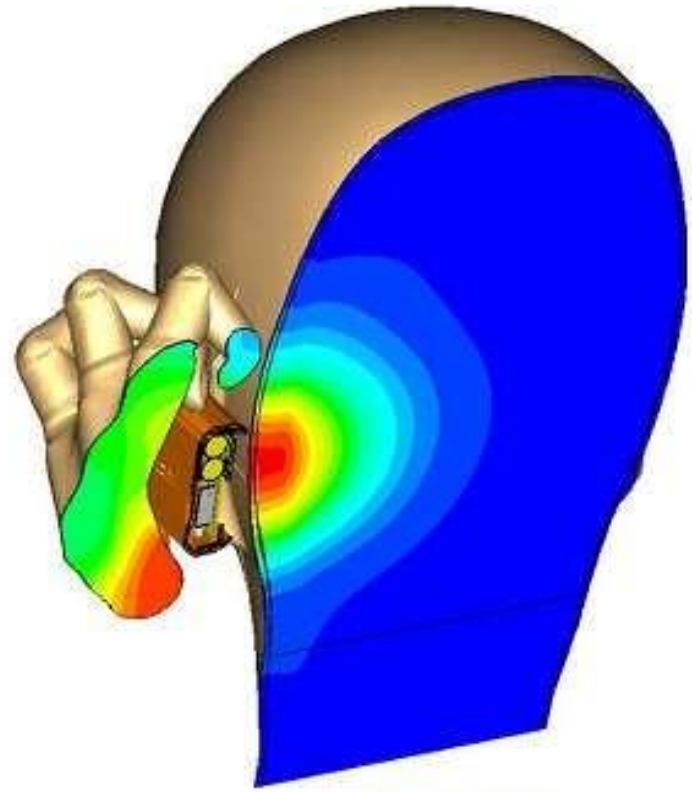


Applications of Electromagnetism: CAA and CAD



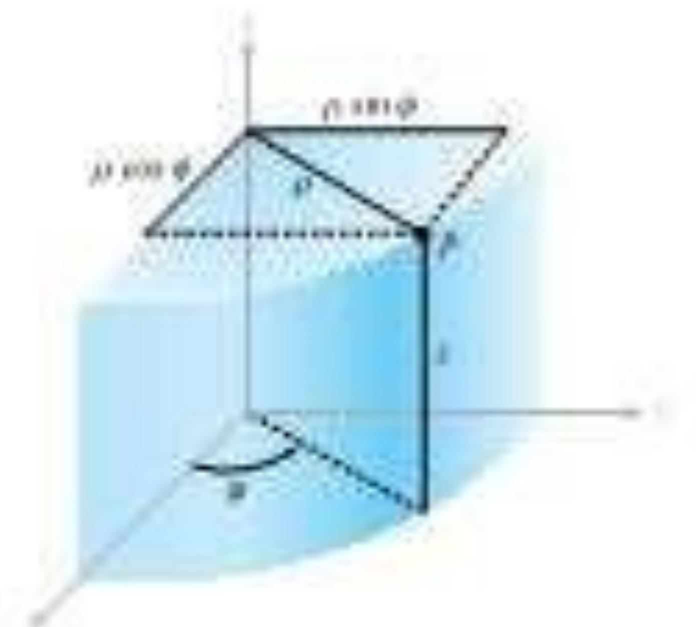
air-borne antenna
[Ansoft HFSS]

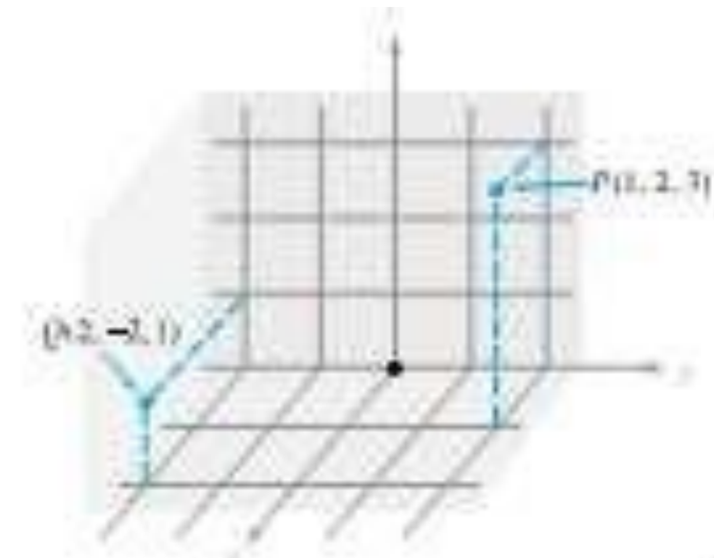


SAR distribution in human head
[CST Microwave Studio]

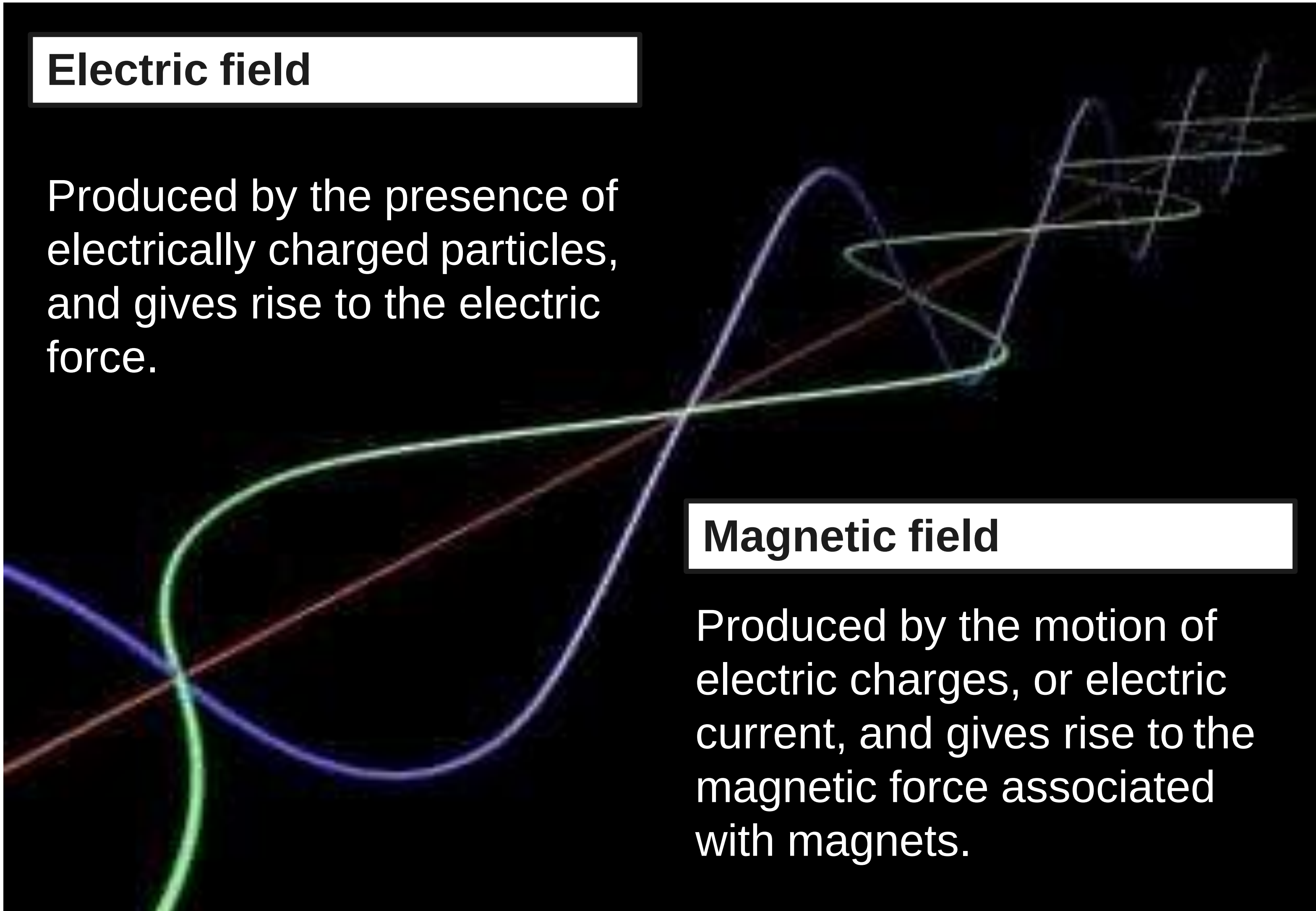
Chapter 1

Vector Analysis





Electromagnetic is the study of the effects of charges at rest and charges in motion



Scalars & Vectors

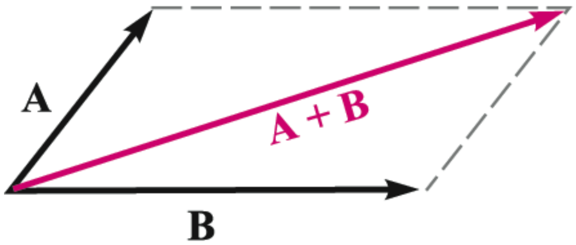
- Scalar
  - Refers to a Quantity whose value may be represented by a Single (Positive/Negative) real number.
    - Body falling a distance L in Time t
    - Temperature T at any point in a bowl of soup whose co-ordinates are x, y,z
    - L, t, T, z, y & z are all scalars
  - Mass, Density,Pressure, Volume, Volume resistivity, Voltage

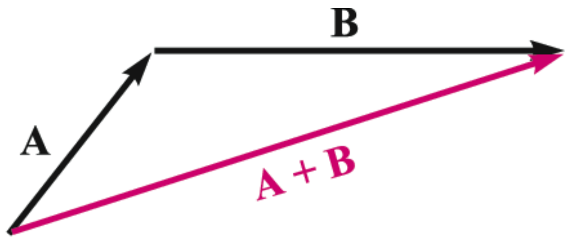
Scalars & Vectors

- Vector
  - A quantity who has both a magnitude and direction in space.
  - Force, Velocity, Acceleration & a straight line from positive to negative terminal of a storage battery

Vector Algebra

- Addition



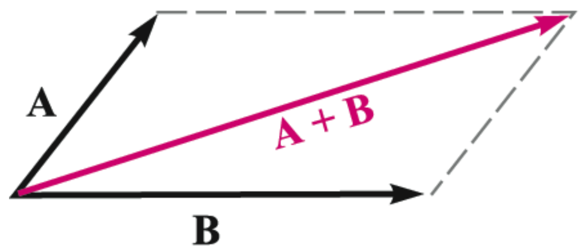


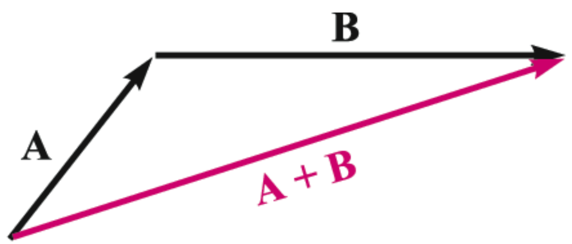
Associative Law: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Distributive Law: $(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B})$

Vector Algebra

- Coplanar vectors
  - Lying in a common plane





Vector Algebra

- Subtraction
  - $A - B = A + (-B)$
- Multiplication
  - Obeys Associative & Distributive laws
  - $(r + s)(A + B) = r(A + B) + s(A + B)$

    $= rA + rB + sA + sB$

Orthogonal Coordinate Systems

- A ***coordinate system*** defines a set of reference directions. In a 3D space, a coordinate system can be specified by the intersection of 3 surfaces at each and every point in space.
- The ***origin*** of the coordinate system is the reference point relative to which we locate every other point in space.

Orthogonal Coordinate Systems

- A ***position vector*** defines the position of a point in space relative to the origin.

  These three reference directions are referred to as ***coordinate directions or base vectors***, and are usually taken to be mutually perpendicular (*orthogonal*) . In this class, we use three coordinate systems:
  - Cartesian
  - cylindrical
  - Spherical

Rectangular Coordinate System

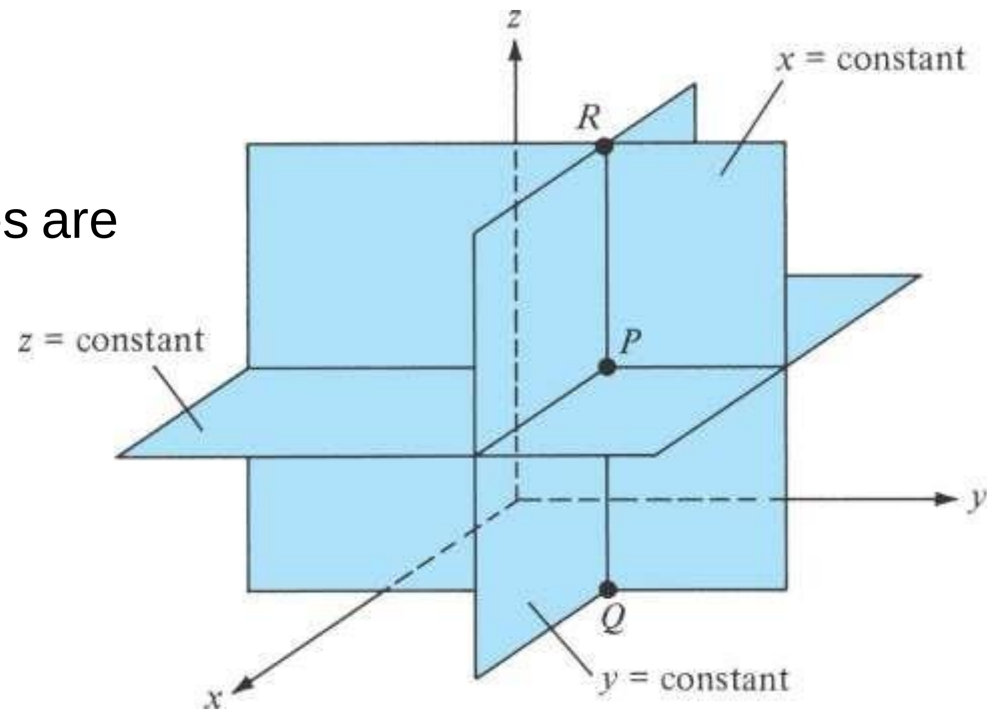
In Cartesian or rectangular coordinate system a point P is represented by coordinates (x,y,z) All the three coordinates represent the mutually perpendicular plane surfaces

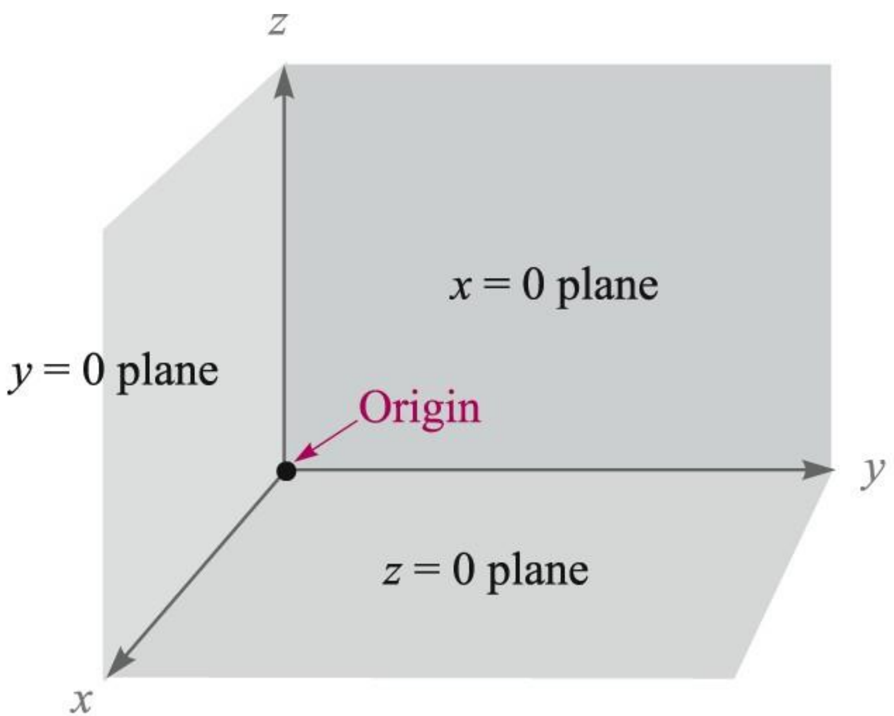
The range of coordinates are

$-\infty < x < \infty$

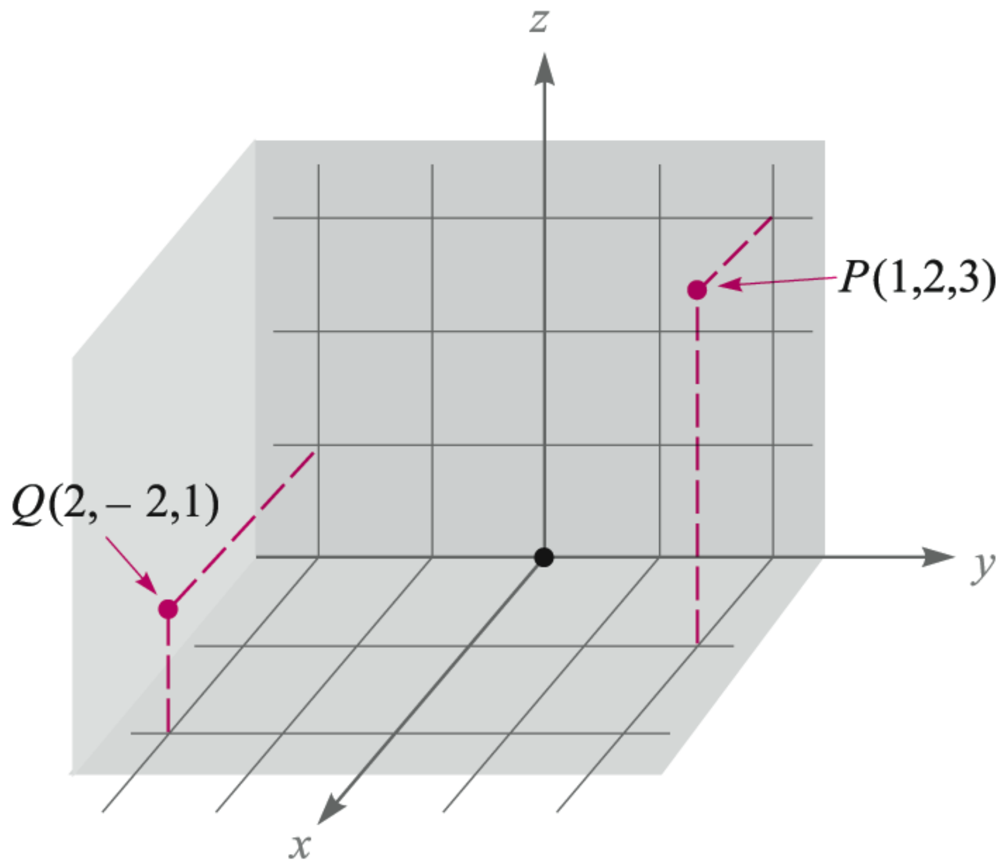
$-\infty < y < \infty$

$-\infty < z < \infty$

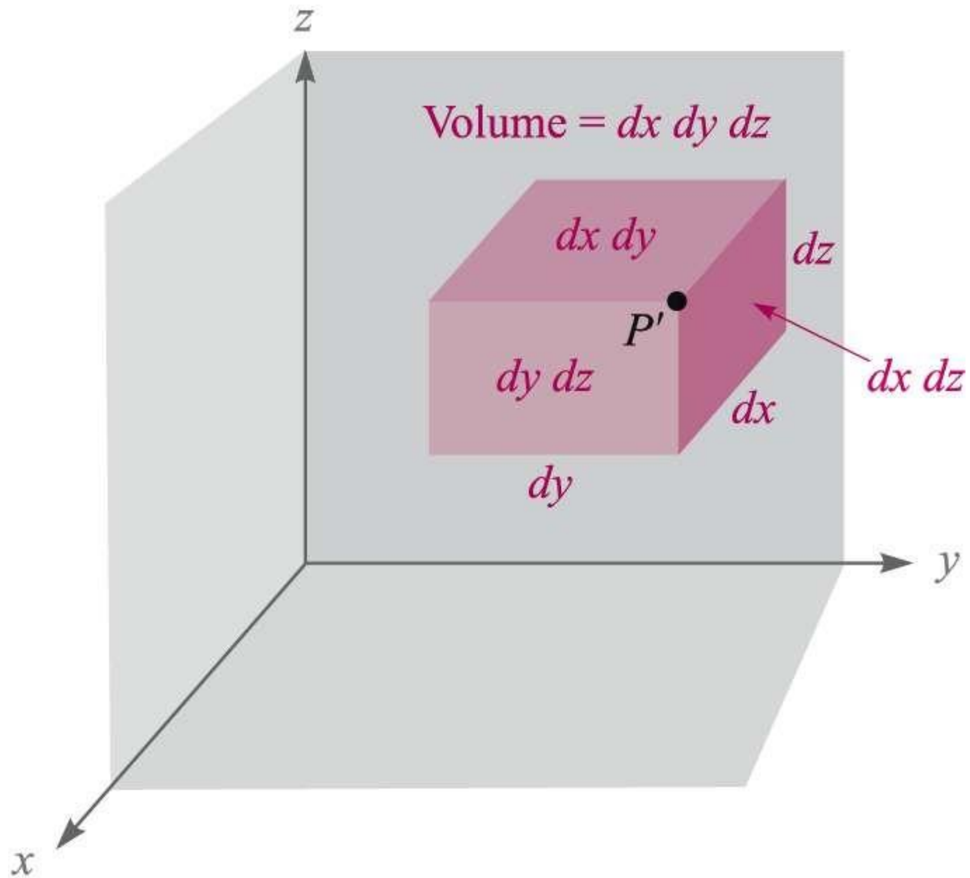



z
x = 0 plane
y = 0 plane
Origin
y
z = 0 plane
x

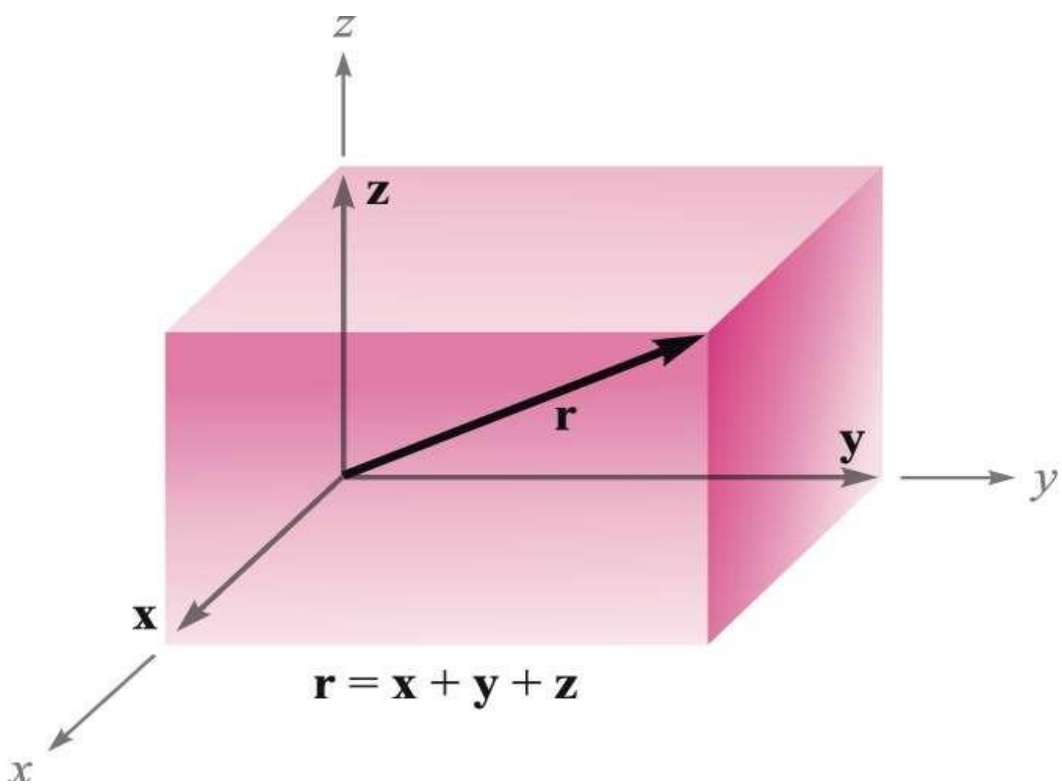
Point Locations in Rectangular Coordinates



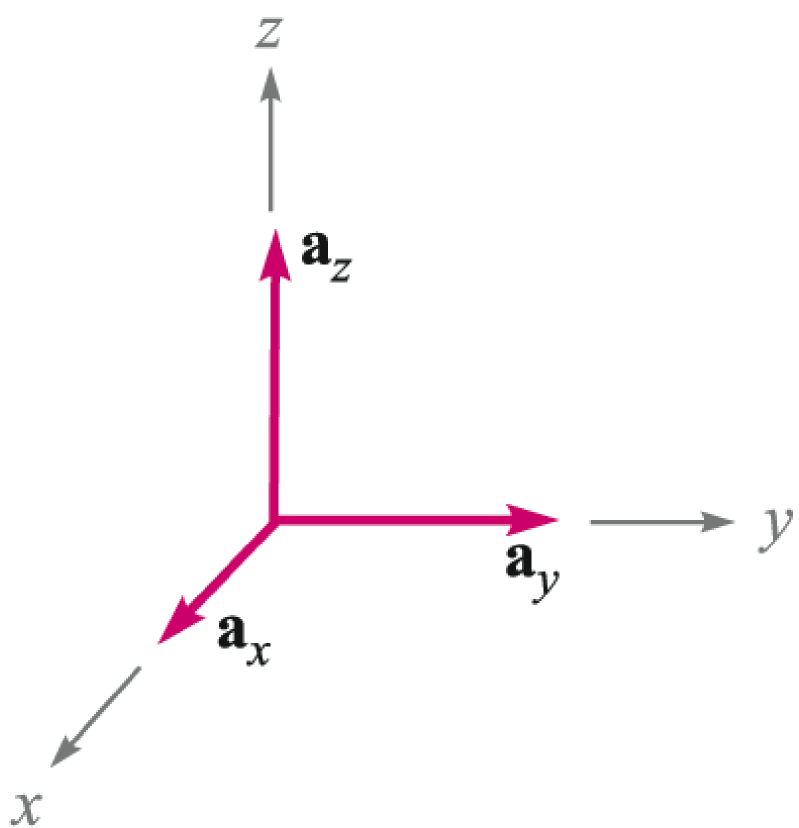
Differential Volume Element



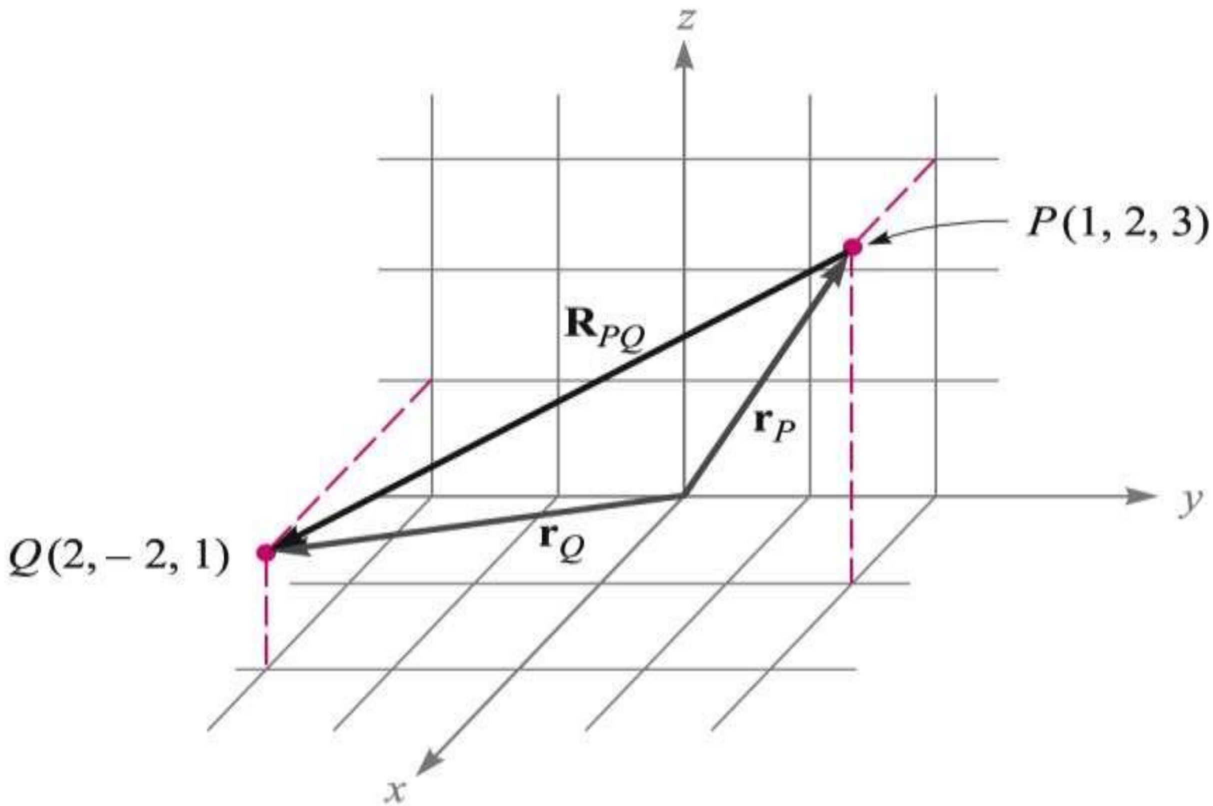
Orthogonal Vector Components



Orthogonal Unit Vectors



Vector Representation in Terms of Orthogonal Rectangular Components



$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

Vector Expressions in Rectangular Coordinates

General Vector, B: $$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$$

Magnitude of B: $$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Unit Vector in the Direction of B: $$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

Example

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

Vector Components and Unit Vectors

Example

- **Given points *M*(–1,2,1) and *N*(3,–3,0), find $\mathbf{R}_{MN}$ and $\mathbf{a}_{MN}$.**

$$\mathbf{R}_{MN} = (3\mathbf{a}_x - 3\mathbf{a}_y + 0\mathbf{a}_z) - (-1\mathbf{a}_x + 2\mathbf{a}_y + 1\mathbf{a}_z) \qquad = \underline{\underline{4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z}}$$

$$\mathbf{a}_{MN} = \frac{\mathbf{R}_{MN}}{|\mathbf{R}_{MN}|} = \frac{4\mathbf{a}_x - 5\mathbf{a}_y - 1\mathbf{a}_z}{\sqrt{4^2 + (-5)^2 + (-1)^2}} = \underline{\underline{0.617\mathbf{a}_x - 0.772\mathbf{a}_y - 0.154\mathbf{a}_z}}$$

Field

- Function, which specifies a particular quantity everywhere in the region
- Two types:
  - **Vector Field** has a direction feature pertaining to it e.g. Gravitational field in space and
  - **Scalar Field** has only magnitude e.g. Temperature

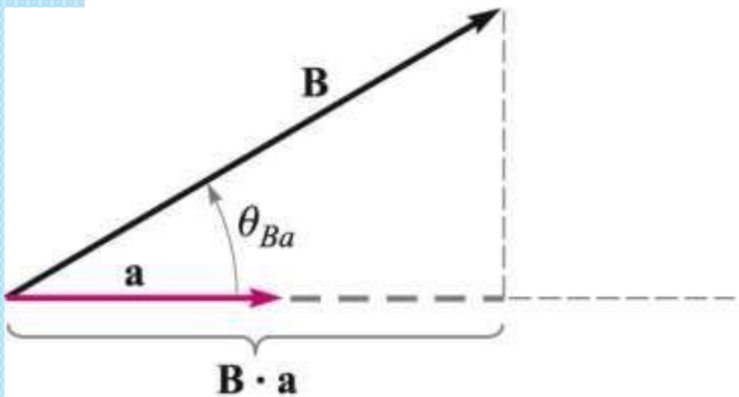
The Dot Product

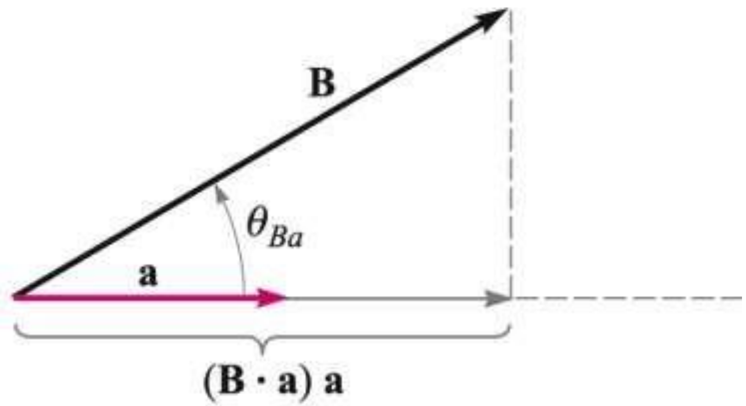
Given two vectors **A** and **B**, the *dot product*, or *scalar product*, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle between them,

$$\boxed{\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta_{AB}}$$

Commutative Law: $\boxed{\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}}$

Vector Projections Using the Dot Product





B • a gives the component of B in the horizontal direction

(B • a) a gives the *vector* component of B in the horizontal direction

Operational Use of the Dot Product

Given
$$\begin{cases} \mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \\ \mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z \end{cases}$$

Find
$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z}$$

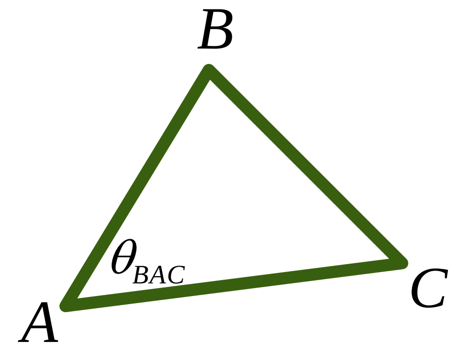
where we have used:
$$\begin{cases} \mathbf{a}_x\cdot\mathbf{a}_y = \mathbf{a}_y\cdot\mathbf{a}_z = \mathbf{a}_x\cdot\mathbf{a}_z = 0 \\ \mathbf{a}_x\cdot\mathbf{a}_x = \mathbf{a}_y\cdot\mathbf{a}_y = \mathbf{a}_z\cdot\mathbf{a}_z = 1 \end{cases}$$

Note also:
$$\boxed{\mathbf{A}\cdot\mathbf{A} = A^2 = |\mathbf{A}|^2}$$

Drill Problem

The three vertices of a triangle are located at *A*(6,–1,2), *B*(–2,3,–4), and *C*(–3,1,5). Find: (*a*) $\mathbf{R}_{AB}$; (*b*) $\mathbf{R}_{AC}$; (*c*) angle θ_{BAC} at vertex *A*; (*d*) the vector projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$.

Solution



$$\mathbf{R}_{AB} = (-2\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z) - (6\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z) = \underline{\underline{-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z}}$$

$$\mathbf{R}_{AC} = (-3\mathbf{a}_x + 1\mathbf{a}_y + 5\mathbf{a}_z) - (6\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z) = \underline{\underline{-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z}}$$

$$\mathbf{R}_{AB} \cdot \mathbf{R}_{AC} = |\mathbf{R}_{AB}||\mathbf{R}_{AC}|\cos\theta_{BAC}$$

$$\Rightarrow \cos\theta_{BAC} = \frac{\mathbf{R}_{AB} \cdot \mathbf{R}_{AC}}{|\mathbf{R}_{AB}||\mathbf{R}_{AC}|} = \frac{(-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot (-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z)}{\left|\sqrt{(-8)^2 + (4)^2 + (-6)^2}\right|\left|\sqrt{(-9)^2 + (2)^2 + (3)^2}\right|}$$

$$= \frac{62}{\left|\sqrt{116}\right|\left|\sqrt{94}\right|} = 0.594$$

$$\Rightarrow \theta_{BAC} = \cos_{-1}(0.594) = \underline{\underline{53.56^\circ}}$$

Continued …

$$\mathbf{R}_{AB} \text{ on } \mathbf{R}_{AC} = \left(\mathbf{R}_{AB} \cdot \mathbf{a}_{AC}\right)\mathbf{a}_{AC}$$

$$= \left((-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z) \frac{(-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z)}{\left|\sqrt{(-9)^2 + (2)^2 + (3)^2}\right|} \right) \frac{(-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z)}{\left|\sqrt{(-9)^2 + (2)^2 + (3)^2}\right|}$$

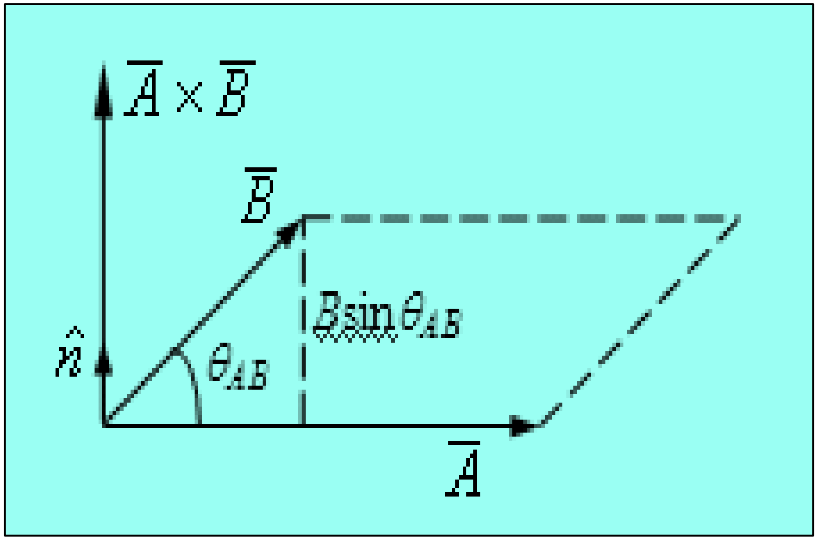
$$= \frac{62}{\sqrt{94}} \frac{(-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z)}{\sqrt{94}}$$

$$= \underline{\underline{-5.963\mathbf{a}_x + 1.319\mathbf{a}_y + 1.979\mathbf{a}_z}}$$

Cross Product

The cross product $\mathbf{A} \times \mathbf{B}$ is a vector; the magnitude of $\mathbf{A} \times \mathbf{B}$ is equal to the product of the magnitudes of $\mathbf{A}$, $\mathbf{B}$, and the sine of the smaller angle between $\mathbf{A}$ and $\mathbf{B}$; the direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$ and is along that one of the two possible perpendiculars which is in the direction of advance of a right-handed screw as $\mathbf{A}$ is turned into $\mathbf{B}$.

$$\boxed{\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\,|\mathbf{B}| \sin\theta_{AB}}$$



Operational Definition of the Cross Product in Rectangular Coordinates

Begin with:

$$\mathbf{A} \times \mathbf{B} = A_x B_x \mathbf{a}_x \times \mathbf{a}_x + A_x B_y \mathbf{a}_x \times \mathbf{a}_y + A_x B_z \mathbf{a}_x \times \mathbf{a}_z$$
$$+ A_y B_x \mathbf{a}_y \times \mathbf{a}_x + A_y B_y \mathbf{a}_y \times \mathbf{a}_y + A_y B_z \mathbf{a}_y \times \mathbf{a}_z$$
$$+ A_z B_x \mathbf{a}_z \times \mathbf{a}_x + A_z B_y \mathbf{a}_z \times \mathbf{a}_y + A_z B_z \mathbf{a}_z \times \mathbf{a}_z$$

where

Therefore:

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z$$

Or…

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

The Cross Product

- **Example**

Given A = 2a_x–3a_y+a_z and B = –4a_x–2a_y+5a_z, find A×B.

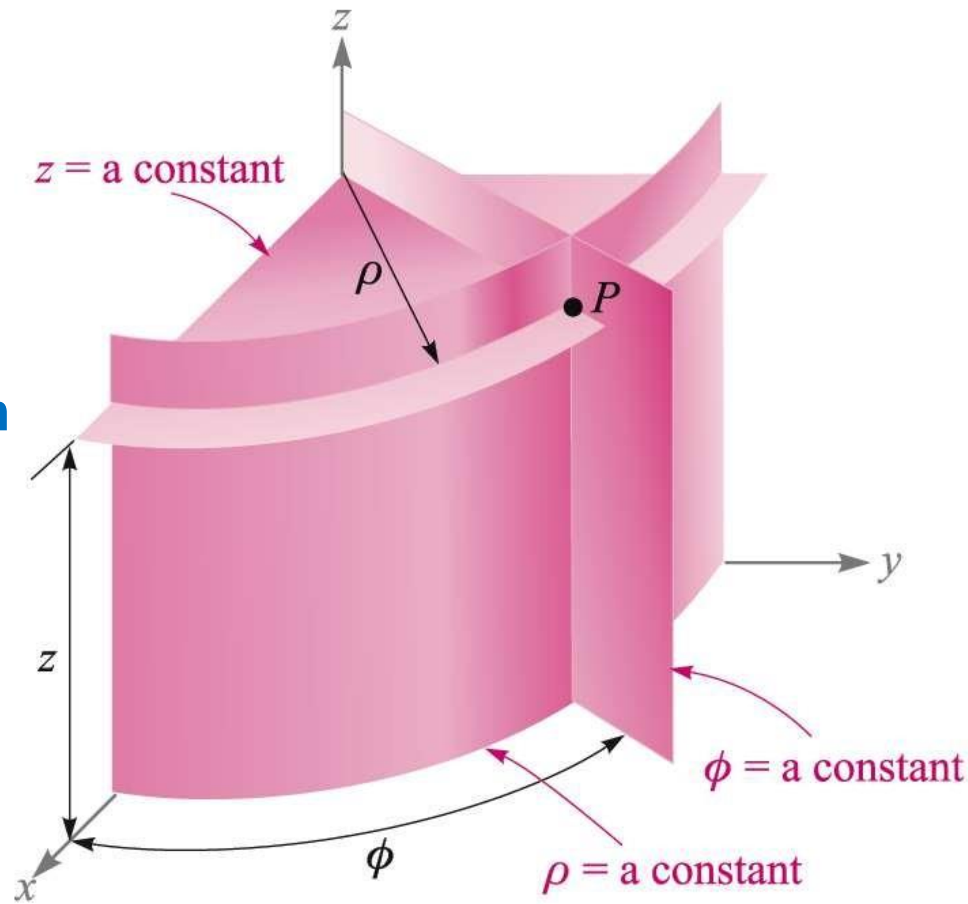
$$\mathbf{A}\times\mathbf{B} = (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z$$

$$= \big((-3)(5) - (1)(-2)\big)\mathbf{a}_x + \big((1)(-4) - (2)(5)\big)\mathbf{a}_y + \big((2)(-2) - (-3)(-4)\big)\mathbf{a}_z$$

$$= \underline{\underline{-13\mathbf{a}_x - 14\mathbf{a}_y - 16\mathbf{a}_z}}$$

Circular Cylindrical Coordinates

Point P has coordinates
Specified by P(ρ,ϕ,z)



The ρ coordinate represents a cylinder of radius ρ with z axis as its axis. The ø coordinate (the azimuthal angle) is measured from x axis in xy plane. Z is same as in Cartesian coordinates

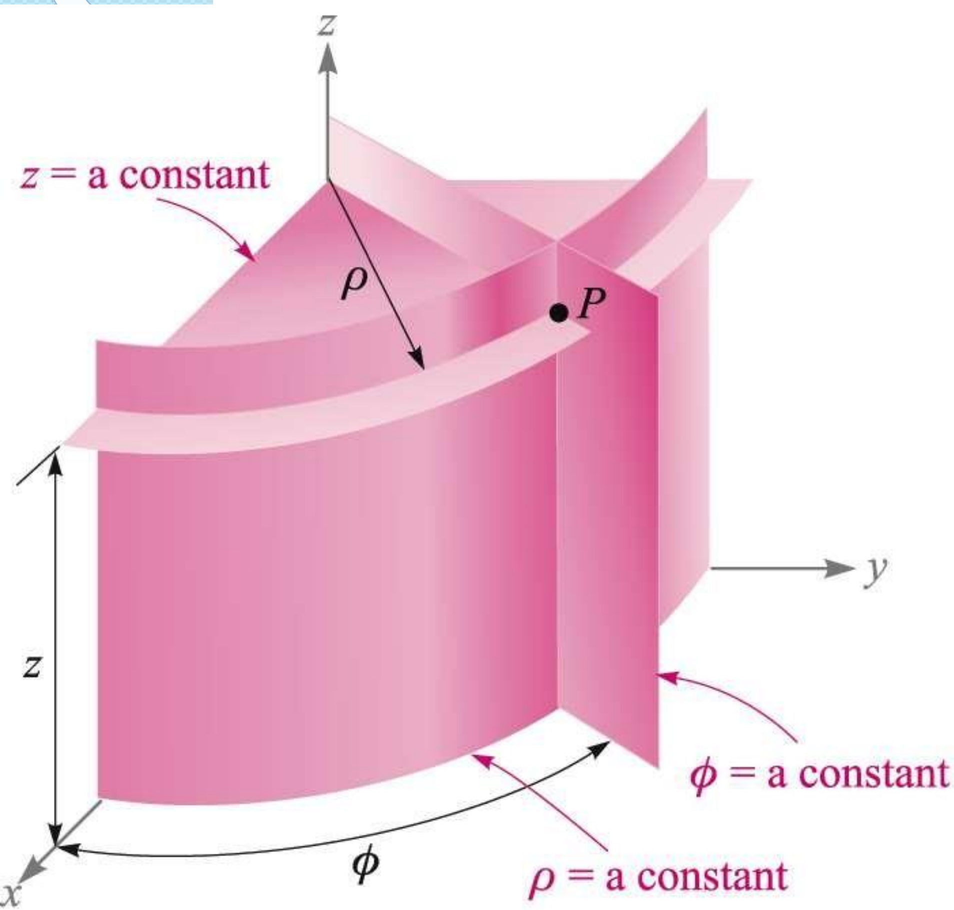
The range of coordinates are

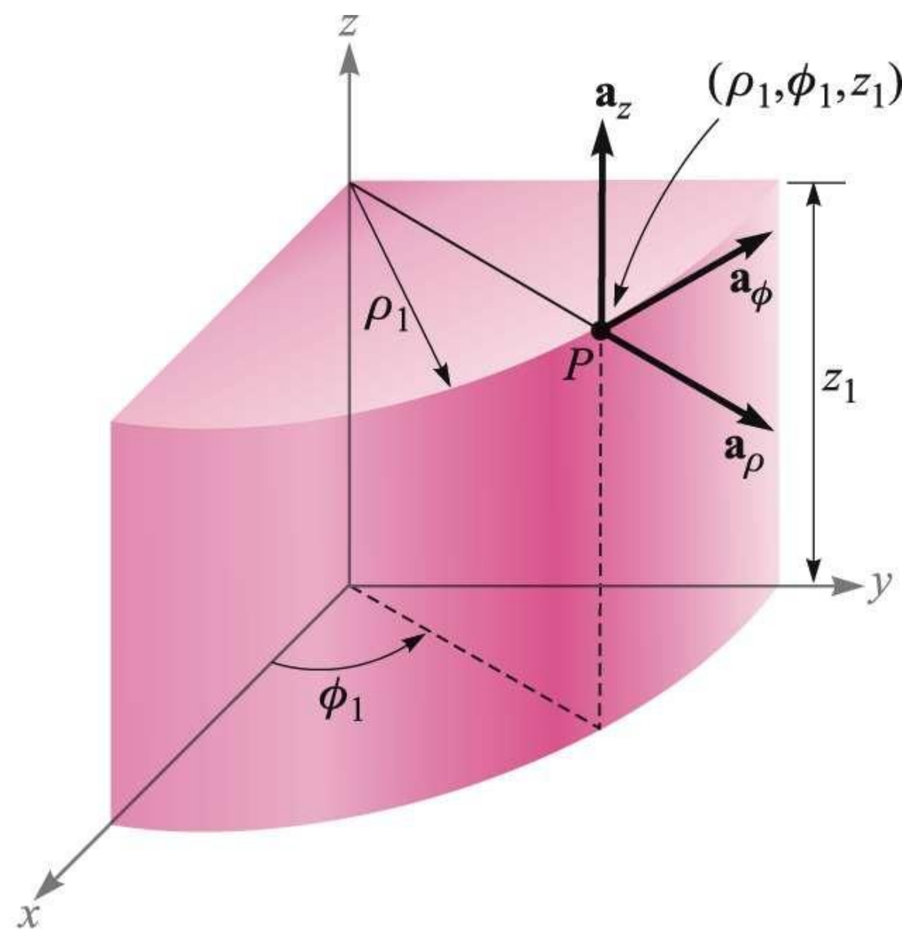
$0 \leq \rho < \infty$

$0 \leq ø < 2\pi$

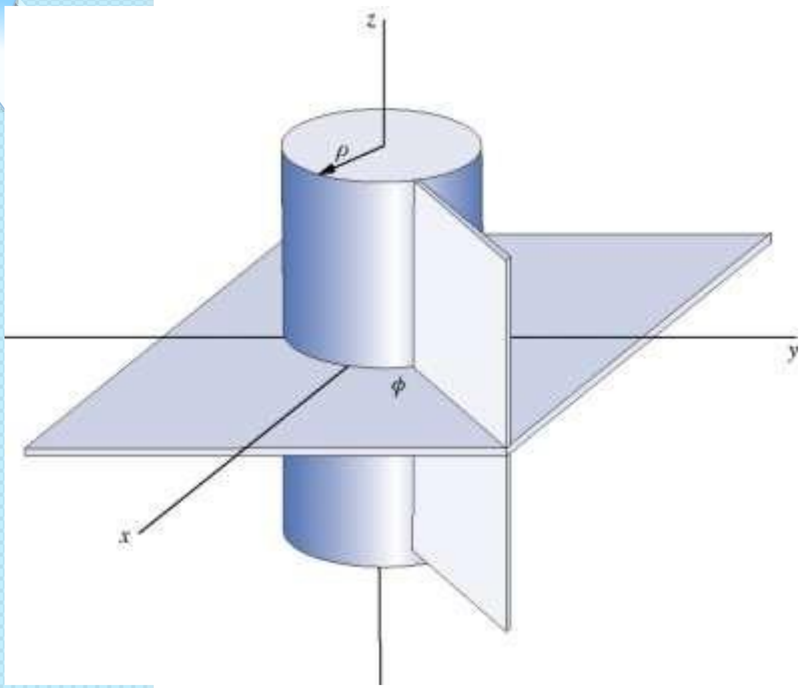
$-\infty < z < \infty$

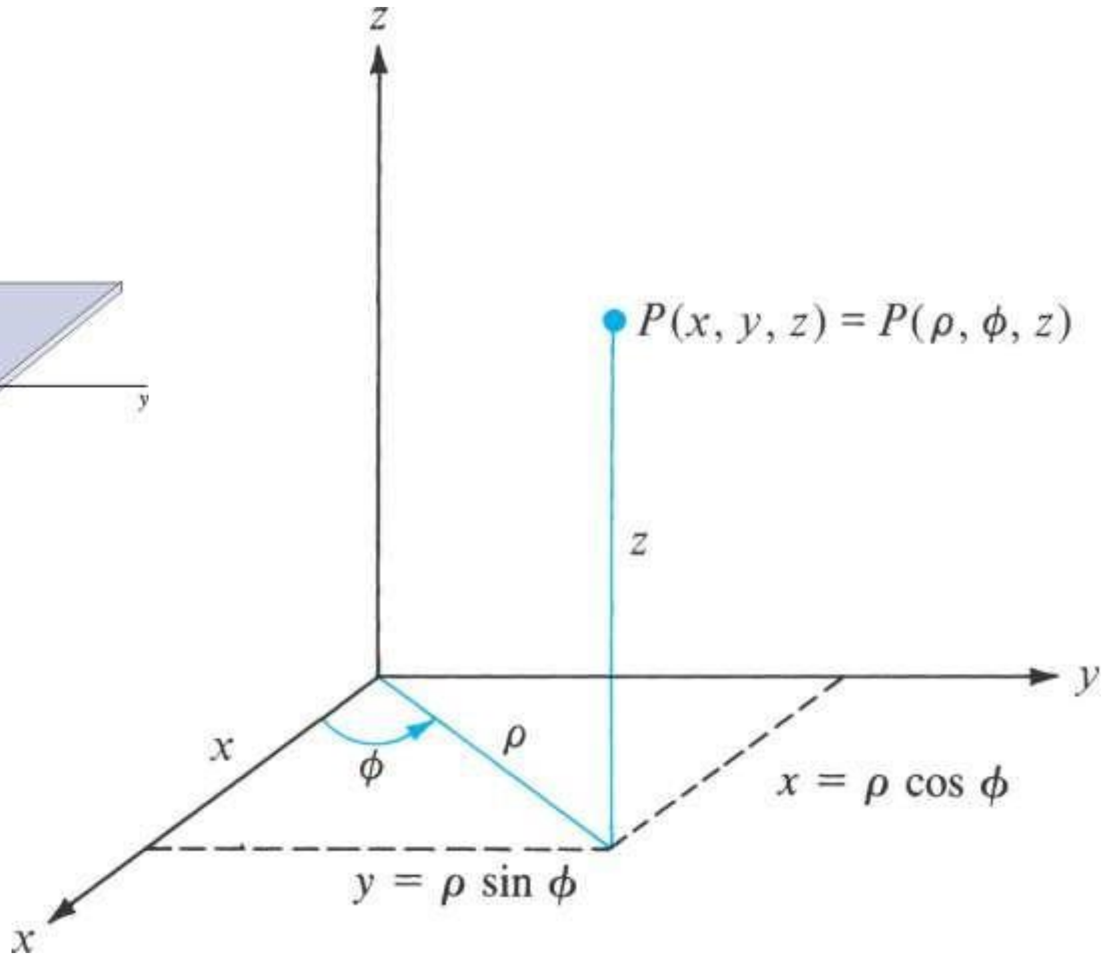
Orthogonal Unit Vectors in Cylindrical Coordinates



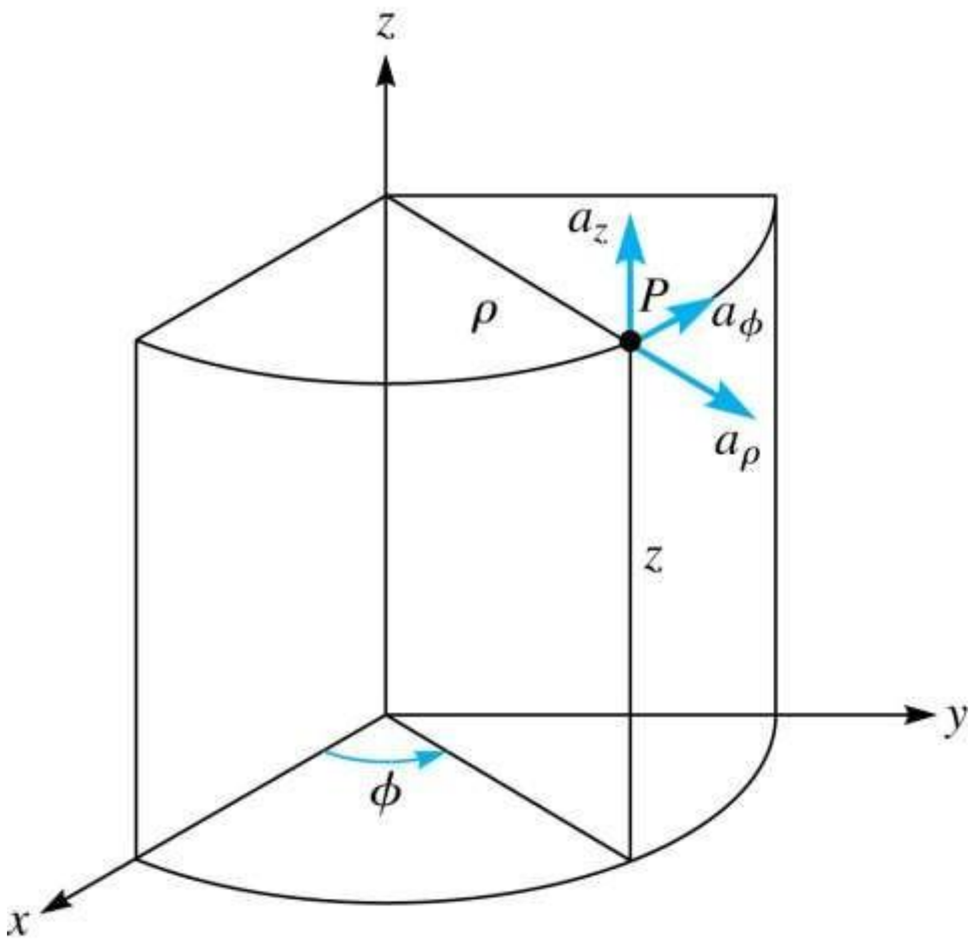


Relationship between (x, y, z) and (ρ, ϕ, z).

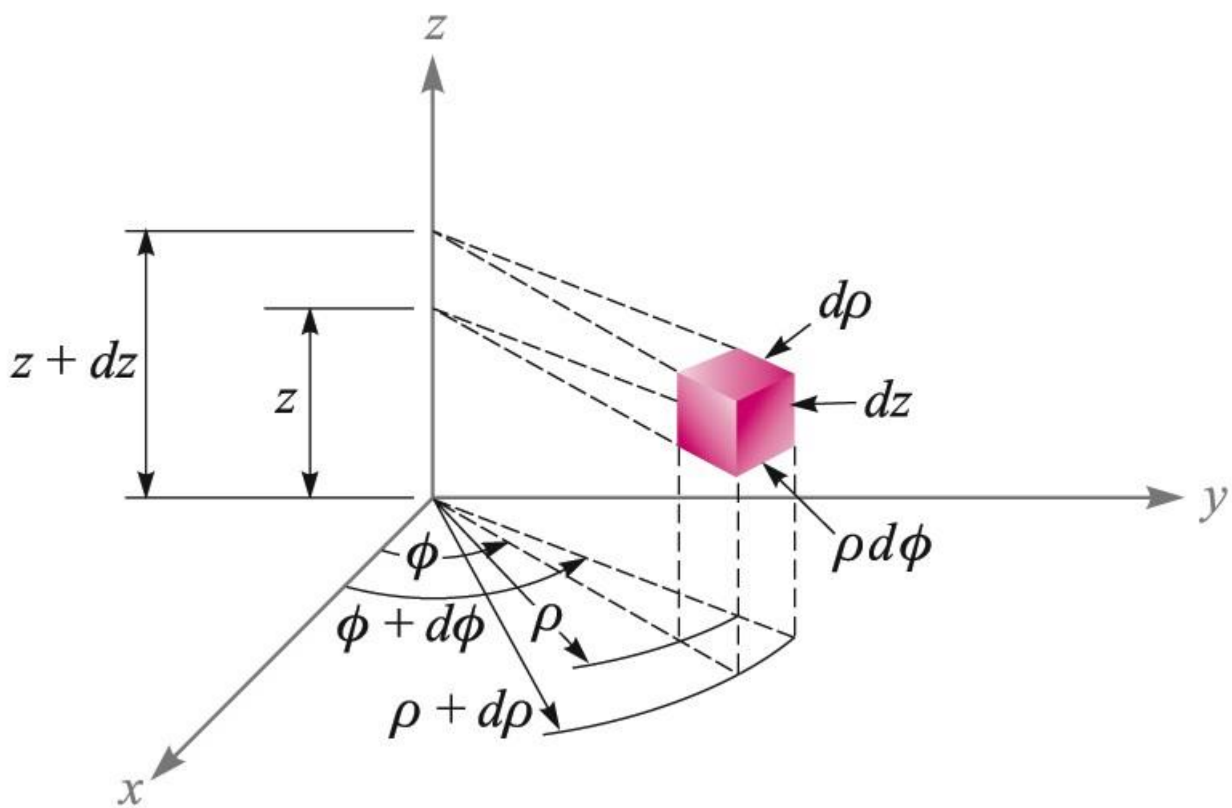




Point P and unit vectors in the cylindrical coordinate system.



Differential Volume in Cylindrical Coordinates



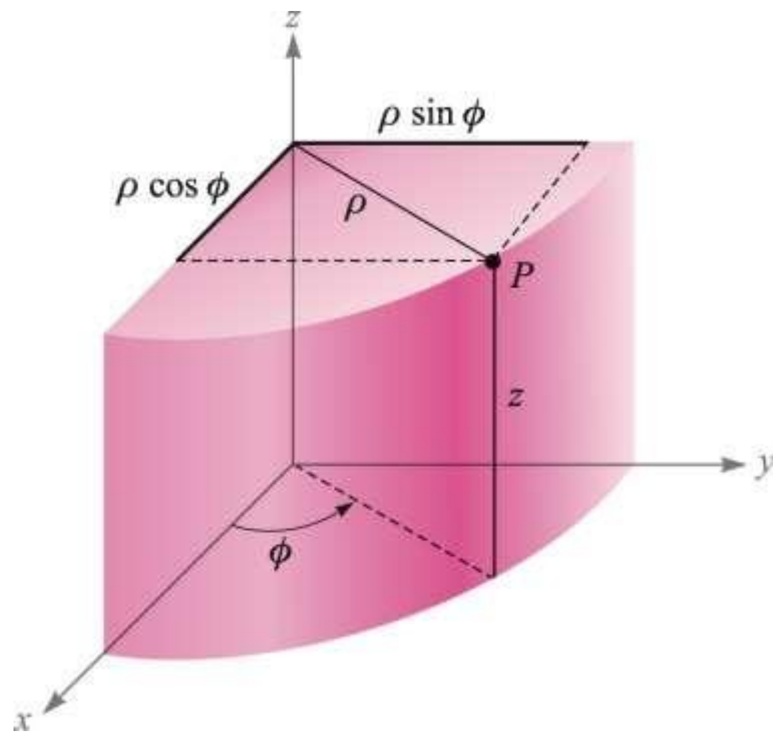
$$dv = \rho \, d\rho \, d\phi \, dz$$

Point Transformations in Cylindrical Coordinates

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$z = z$$



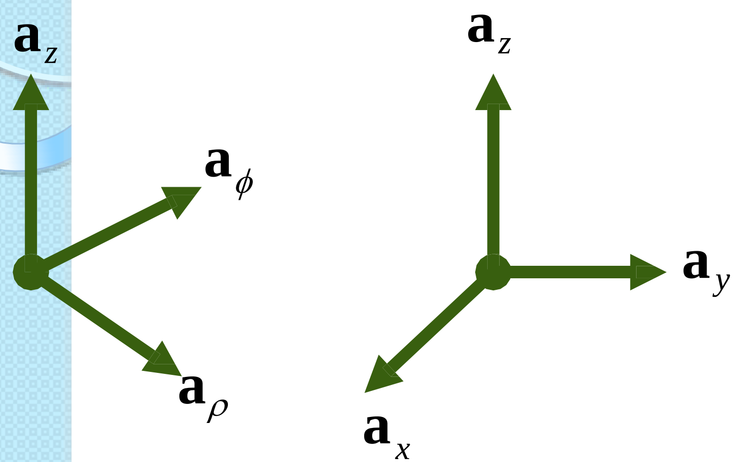
$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

The Cylindrical Coordinate System

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \overset{?}{\Rightarrow} \mathbf{A} = A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$



$$\begin{aligned} A_\rho &= \mathbf{A}\cdot\mathbf{a}_\rho \\ &= (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\rho \\ &= A_x\mathbf{a}_x\cdot\mathbf{a}_\rho + A_y\mathbf{a}_y\cdot\mathbf{a}_\rho + A_z\mathbf{a}_z\cdot\mathbf{a}_\rho \\ &= A_x\cos\phi + A_y\sin\phi \end{aligned}$$

$$\begin{aligned} A_\phi &= \mathbf{A}\cdot\mathbf{a}_\phi \\ &= (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\phi \\ &= A_x\mathbf{a}_x\cdot\mathbf{a}_\phi + A_y\mathbf{a}_y\cdot\mathbf{a}_\phi + A_z\mathbf{a}_z\cdot\mathbf{a}_\phi \\ &= -A_x\sin\phi + A_y\cos\phi \end{aligned}$$

Dot products of unit vectors in cylindrical and rectangular coordinate systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	1

$$\begin{aligned} A_z &= \mathbf{A}\cdot\mathbf{a}_z \\ &= (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_z \\ &= A_x\mathbf{a}_x\cdot\mathbf{a}_z + A_y\mathbf{a}_y\cdot\mathbf{a}_z + A_z\mathbf{a}_z\cdot\mathbf{a}_z \\ &= A_z \end{aligned}$$

Example

Transform the vector B into cylindrical coordinates:

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

Problem

Transform the vector B into cylindrical coordinates

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

Start with:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

Then:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y \cos\phi - x \sin\phi = \rho \sin\phi \cos\phi - \rho \cos\phi \sin\phi = 0$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

$$= -y \sin\phi - x \cos\phi = -\rho \sin^2\phi - \rho \cos^2\phi = -\rho$$

Finally:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y \cos\phi - x \sin\phi = \rho \sin\phi \cos\phi - \rho \cos\phi \sin\phi = 0$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

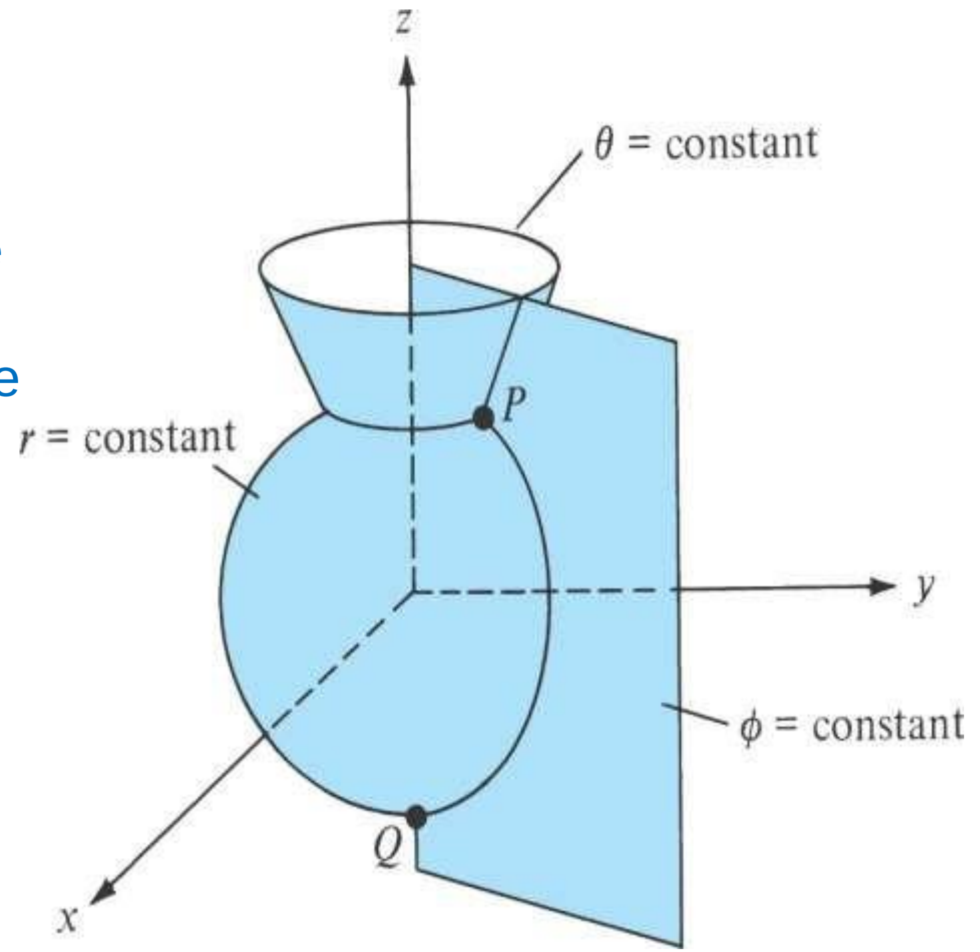
$$= -y \sin\phi - x \cos\phi = -\rho \sin^2\phi - \rho \cos^2\phi = -\rho$$

$$\mathbf{B} = -\rho \mathbf{a}_\phi + z\mathbf{a}_z$$

Spherical Coordinates

Point P has coordinates
Specified by P(r,θ,ϕ)

The r coordinate represents a sphere of radius r centered at origin . The θ coordinate represents the angle made by the cone with z-axis. The ø coordinate is the same as cylindrical coordinate.

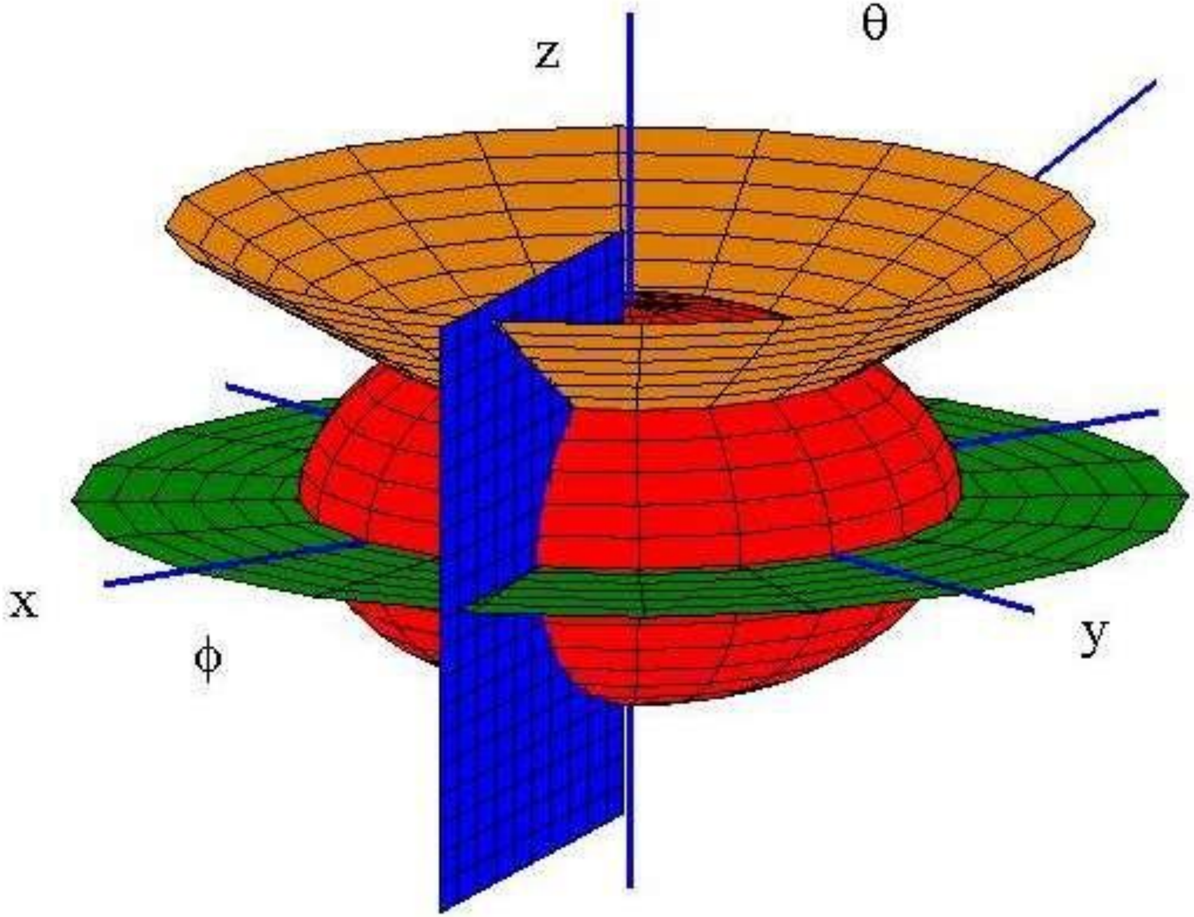


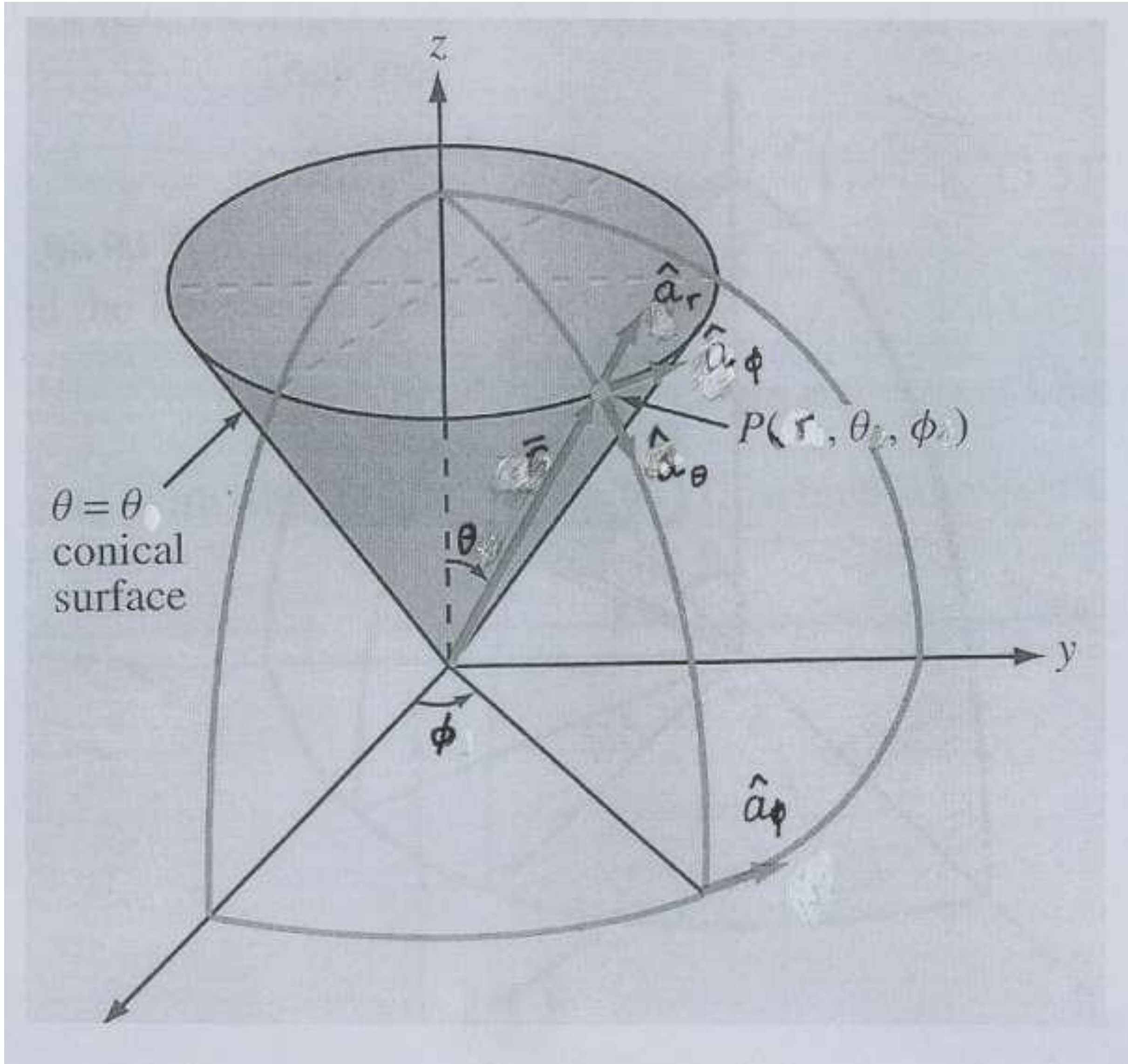
The range of coordinates are

$0 \leq r < \infty$

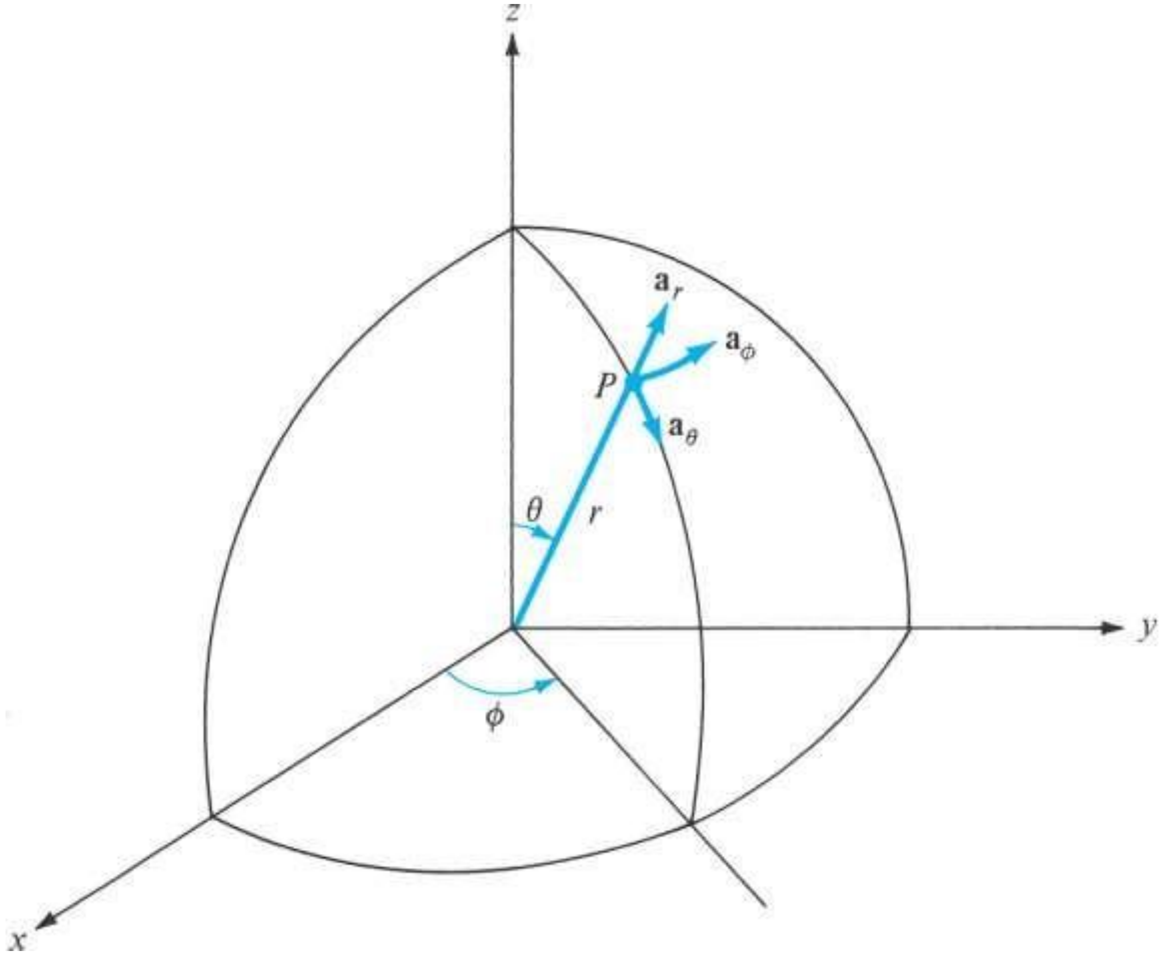
$0 \leq \theta \leq \pi$

$0 \leq ø < 2\pi$

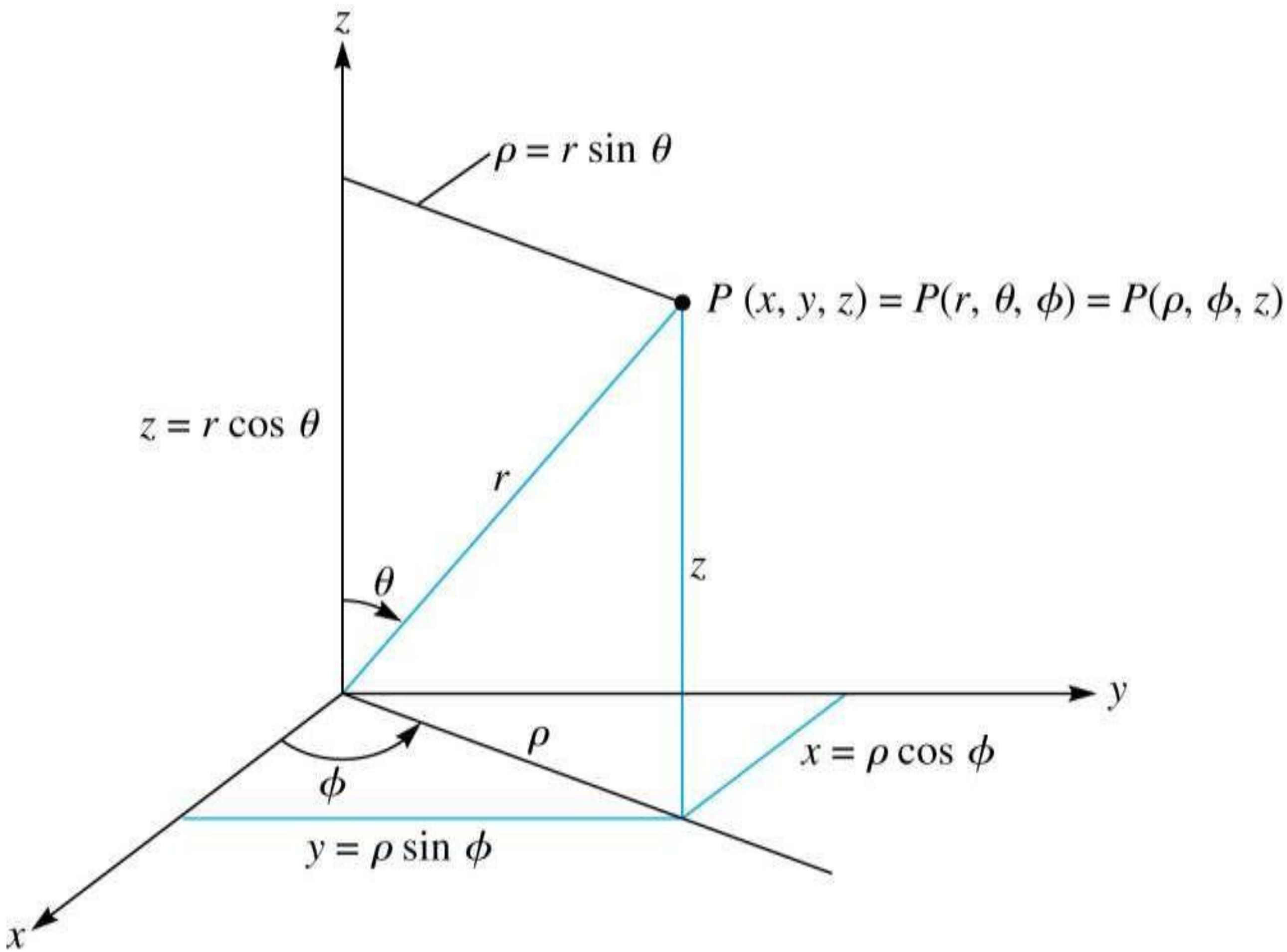

θ
z
x
φ
y


z
$\hat{a}_r$
$\hat{a}_\phi$
$P(r, \theta, \phi)$
$\bar{r}$
$\hat{a}_\theta$
$\theta = \theta$
conical
surface
θ
y
ϕ
$\hat{a}_\phi$

Point P and unit vectors in spherical coordinates.



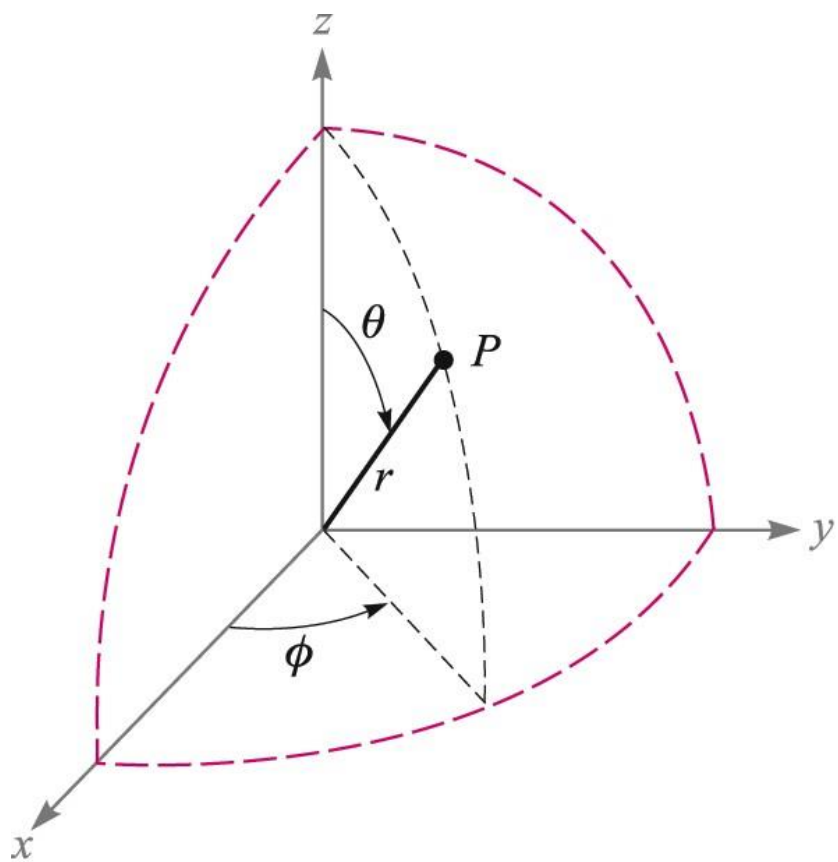
Relationships between space variables (x, y, z), (r, θ, ϕ), and (ρ, ϕ, z).



$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

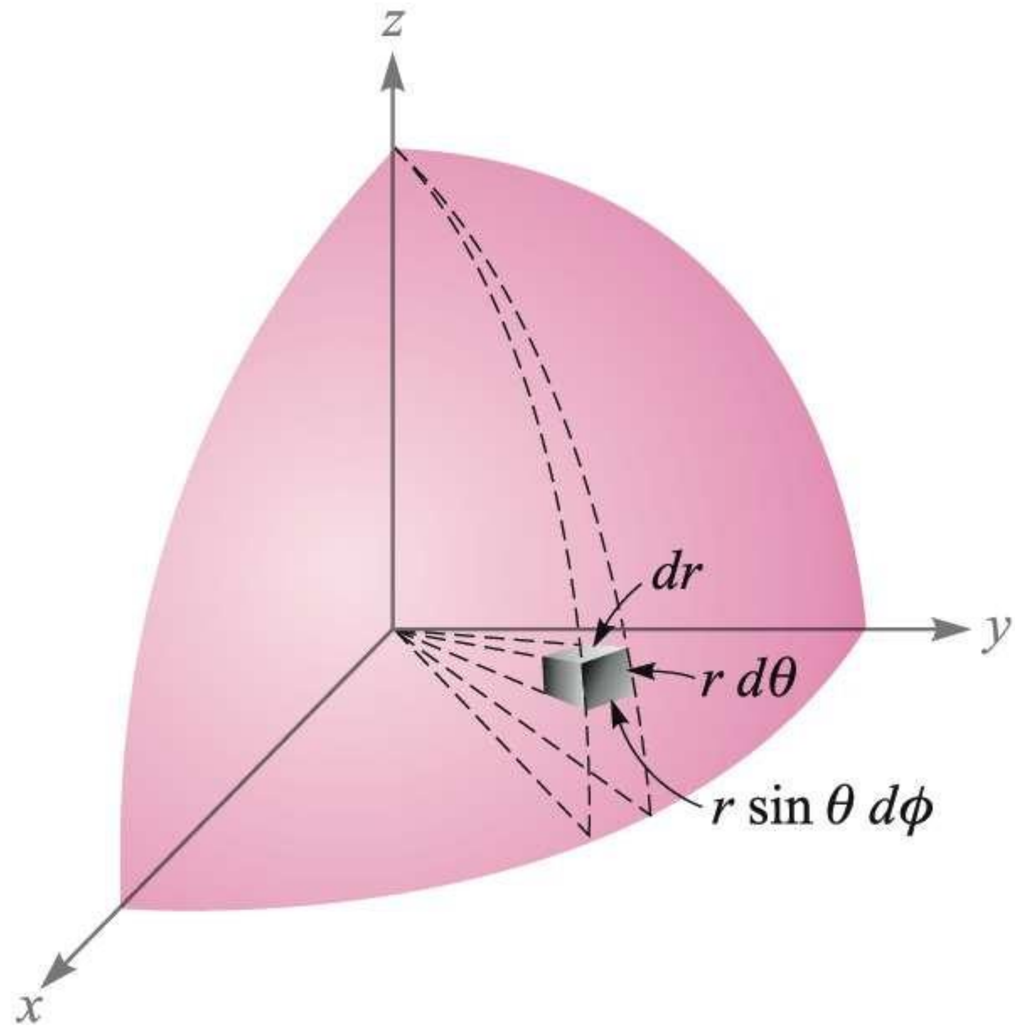
$$y = r\sin\theta\sin\phi \qquad \theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

$$z = r\cos\theta \qquad \phi = \tan^{-1}\frac{y}{x}$$



Differential Volume in Spherical Coordinates

$dv = r^2 \sin\theta \, dr d\theta \, d\phi$



Dot Products of Unit Vectors in the Spherical and Rectangular Coordinate Systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

The Spherical Coordinate System

Example

Given the two points, $C(-3,2,1)$ and $D(r = 5, \theta = 20°, \Phi = -70°)$, find: (*a*) the spherical coordinates of *C*; (*b*) the rectangular coordinates of *D.*

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-3)^2 + (2)^2 + (1)^2} = 3.742$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \cos^{-1} \frac{1}{3.742} = 74.50°$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{2}{-3} = -33.69° + 180° = 146.31°$$

$$\therefore C(r = 3.742, \theta = 74.50°, \phi = 146.31°)$$

$$\therefore D(x = 0.585, y = -1.607, z = 4.698)$$

Example: Vector Component Transformation

Transform the field,

$\mathbf{G} = (xz/y)\mathbf{a}_x$, into spherical coordinates and components

$$G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi$$

$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$

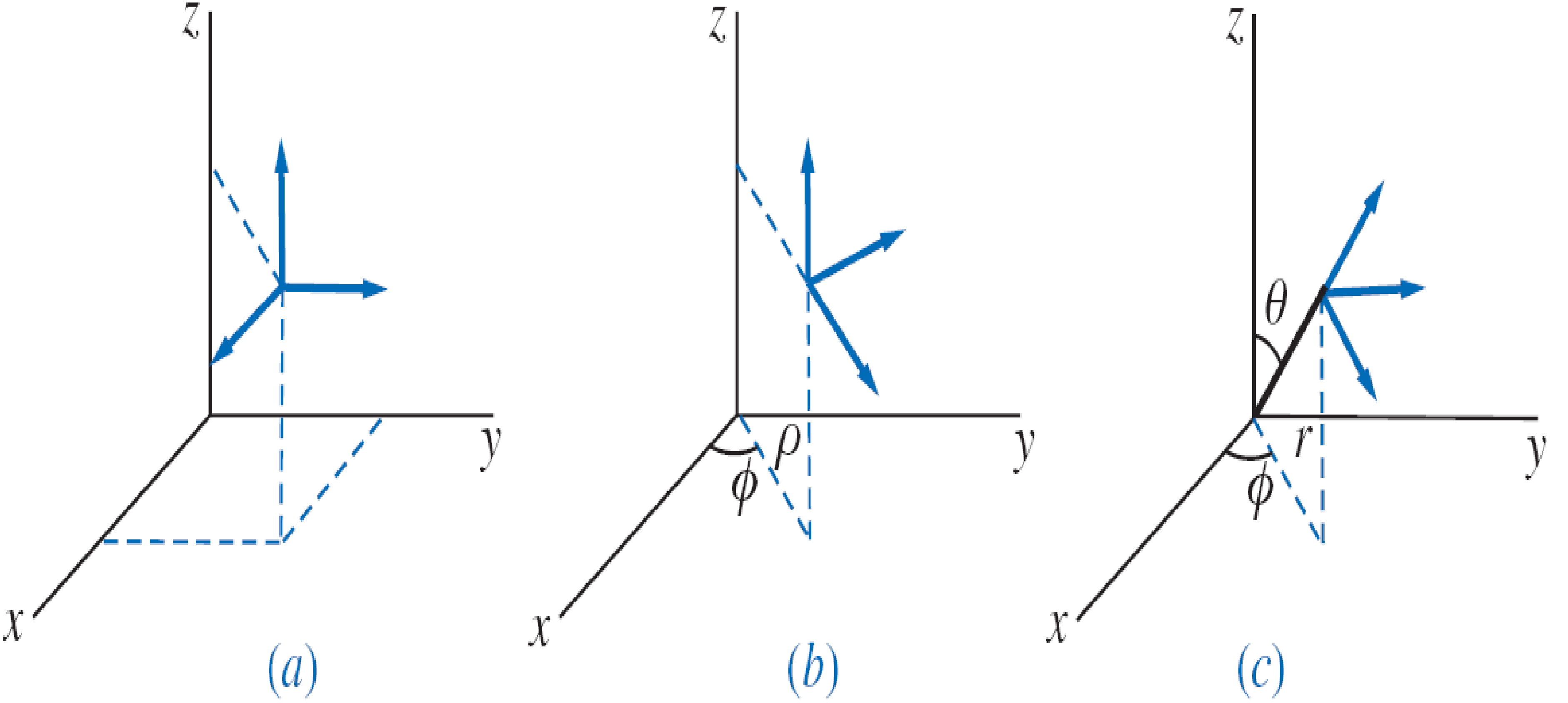
$$G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi$$

$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi)$$

$$= -r\cos\theta\cos\phi$$

$$\mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$


z
y
x
(a)
z
y
x
ρ
ϕ
(b)
z
y
x
θ
r
ϕ
(c)