Lecture # 09

Discrete Structure

Set Identities

- Let A and B be subsets of a universal set U
- Recall the following Definitions

$$A \cup B = \{ x \in U \mid x \in A \text{ or } x \in B\}$$

$$A \cap B = \{ x \in U \mid x \in A \text{ and } x \in B\}$$

$$A - B = \{ x \in U \mid x \in A \text{ and } x \notin B\}$$

$$A^c = \{ x \in U \mid x \notin A\}$$

1. Idempotent Laws

   a. $A \cup A = A$ b. $A \cap A = A$

2. Commutative Laws

   a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$

3. Associative Laws

   a. $A \cup (B \cup C) = (A \cup B) \cup C$

   b. $A \cap (B \cap C) = (A \cap B) \cap C$

Set Identities (Cont.)

4. Distributive Laws

   a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$

   b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. Identity Laws

   a. $A \cup \varnothing = A$ b. $A \cap \varnothing = \varnothing$

   c. $A \cup U = U$ d. $A \cap U = A$

6. Complement Laws

   a. $A \cup A^c = U$ b. $A \cap A^c = \varnothing$

   c. $U^c = \varnothing$ d. $\varnothing^c = U$

7. Double Complement Law

   $(A^c)^c = A$

8. DeMorgan's Laws

   a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$

Set Identities (Cont.)

9. Alternative Representation for Set Difference

   $A - B = A \cap B^c$

10. Subset Laws

    a. $A \cup B \subseteq C$ *iff* $A \subseteq C$ and $B \subseteq C$

    b. $C \subseteq A \cap B$ *iff* $C \subseteq A$ and $C \subseteq B$

11. Absorption Laws

    a. $A \cup (A \cap B) = A$ b. $A \cap (A \cup B) = A$

1. Prove that $A \subseteq A \cup B$

Proof:

Let A, B and C be subsets of a universal set U

Let x be an arbitrary element of A, that is $x \in A$.

$\Rightarrow \quad x \in A$ or $x \in B$

$\Rightarrow \quad x \in A \cup B$

But x is an arbitrary element of A.

$\therefore \quad A \subseteq A \cup B$ (proved)

2. Prove that $A - B \subseteq A$

Proof:

Let $x \in A - B$

$\Rightarrow$ $x \in A$ and $x \notin B$ (by definition of $A - B$)

$\Rightarrow$ $x \in A$ (in particular)

But x is an arbitrary element of $A - B$

$\therefore$ $A - B \subseteq A$ (proved)

3. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Proof:

Suppose that $A \subseteq B$ and $B \subseteq C$

Consider $x \in A$

$\Rightarrow$ $x \in B$ (as $A \subseteq B$)

$\Rightarrow$ $x \in C$ (as $B \subseteq C$)

But x is an arbitrary element of A

$\therefore$ $A \subseteq C$ (proved)

4. Prove that $A \subseteq B$ iff $B^c \subseteq A^c$

Proof:

Suppose $A \subseteq B$ $\{$To prove $B^c \subseteq A^c\}$

Let $x \in B^c$

$\Rightarrow$ $x \notin B$ (by definition of B^c)

$\Rightarrow$ $x \notin A$

$\Rightarrow$ $x \in A^c$ (by definition of A^c)

The contrapositive of "if $x \in A$ then $x \in B$" is "if $x \notin B$ then $x \notin A$"

Thus if $x \notin B$ then $x \notin A$ it means that $A \subseteq B$

But x is an arbitrary element of B^c

$\therefore$ $B^c \subseteq A^c$

4. Prove that $A \subseteq B$ iff $B^c \subseteq A^c$ (Cont.)

Conversely,

Suppose $B^c \subseteq A^c$ {To prove $A \subseteq B$}

Let $x \in A$

$\Rightarrow$ $x \notin A^c$ (by definition of A^c)

$\Rightarrow$ $x \notin B^c$ ($\therefore$ $B^c \subseteq A^c$)

$\Rightarrow$ $x \in B$ (by definition of B^c)

But x is an arbitrary element of A

$\therefore$ $A \subseteq B$ (proved)

5. Prove that $A - B = A \cap B^c$

Proof:

Let $x \in A - B$

$\Rightarrow$ $x \in A$ and $x \notin B$ (definition of set difference)

$\Rightarrow$ $x \in A$ and $x \in B^c$ (definition of complement)

$\Rightarrow$ $x \in A \cap B^c$ (definition of intersection)

But x is an arbitrary element of A – B so we can write

$\therefore$ $A - B \subseteq A \cap B^c$............(1)

5. Prove that $A - B = A \cap B^c$ (Cont.)

Conversely, let $y \in A \cap B^c$

$\Rightarrow$ $y \in A$ and $y \in B^c$ (definition of intersection)

$\Rightarrow$ $y \in A$ and $y \notin B$ (definition of complement)

$\Rightarrow$ $y \in A - B$ (definition of set difference)

But y is an arbitrary element of $A \cap B^c$

$\therefore$ $A \cap B^c \subseteq A - B$............ (2)

From (1) and (2) it follows that

$A - B = A \cap B^c$ (as required)

6. Prove the DeMorgan's Law: $(A \cup B)^c = A^c \cap B^c$

Proof:

Let $x \in (A \cup B)^c$

$\Rightarrow$	$x \notin A \cup B$	(definition of complement)
	$x \notin A$ and $x \notin B$	(DeMorgan's Law of Logic)
$\Rightarrow$	$x \in A^c$ and $x \in B^c$	(definition of complement)
$\Rightarrow$	$x \in A^c \cap B^c$	(definition of intersection)

But x is an arbitrary element of $(A \cup B)^c$

$\therefore$ $(A \cup B)^c \subseteq A^c \cap B^c$.........(1)

6. Prove the DeMorgan's Law: $(A \cup B)^c = A^c \cap B^c$ (Cont.)

Conversely, let $y \in A^c \cap B^c$

$\Rightarrow$	$y \in A^c$ and $y \in B^c$	(definition of intersection)
$\Rightarrow$	$y \notin A$ and $y \notin B$	(definition of complement)
$\Rightarrow$	$y \notin A \cup B$	(DeMorgan's Law of Logic)
$\Rightarrow$	$y \in (A \cup B)^c$	(definition of complement)

But y is an arbitrary element of $A^c \cap B^c$

$\therefore A^c \cap B^c \subseteq (A \cup B)^c$.................(2)

From (1) and (2) we have

$$(A \cup B)^c = A^c \cap B^c$$

Which is the DeMorgan's Law

7. Prove the associative law: $A \cap (B \cap C) = (A \cap B) \cap C$

Proof:

Consider $x \in A \cap (B \cap C)$

$\Rightarrow$ $x \in A$ and $x \in B \cap C$ (definition of intersection)

$\Rightarrow$ $x \in A$ and $x \in B$ and $x \in C$ (definition of intersection)

$\Rightarrow$ $x \in A \cap B$ and $x \in C$ (definition of intersection)

$\Rightarrow$ $x \in (A \cap B) \cap C$ (definition of intersection)

But x is an arbitrary element of $A \cap (B \cap C)$

$\therefore A \cap (B \cap C) \subseteq (A \cap B) \cap C$……(1)

7. Prove the associative law: $A \cap (B \cap C) = (A \cap B) \cap C$ (Cont.)

Conversely, let $y \in (A \cap B) \cap C$

$\Rightarrow$ $y \in A \cap B$ and $y \in C$ (definition of intersection)

$\Rightarrow$ $y \in A$ and $y \in B$ and $y \in C$ (definition of intersection)

$\Rightarrow$ $y \in A$ and $y \in B \cap C$ (definition of intersection)

$\Rightarrow$ $y \in A \cap (B \cap C)$ (definition of intersection)

But y is an arbitrary element of $(A \cap B) \cap C$

$\therefore (A \cap B) \cap C \subseteq A \cap (B \cap C)$........(2)

From (1) & (2), we conclude that

$A \cap (B \cap C) = (A \cap B) \cap C$ (proved)

8. Prove that $A \cap B = A$ when $A \subseteq B$

Proof: Let $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B$

$\Rightarrow$ $x \in A$ (in particular)

Hence $A \cap B \subseteq A$..............(1)

Conversely, let $x \in A$

Then $x \in B$ (since $A \subseteq B$)

Now $x \in A$ and $x \in B$, therefore $x \in A \cap B$

Hence, $A \subseteq A \cap B$..............(2)

From (1) and (2) it follows that

$A = A \cap B$ (proved)

9. Prove that $A \cup B = B$ when $A \subseteq B$

Proof: Suppose that $A \subseteq B$. Consider $x \in A \cup B$.

CASE I (when $x \in A$)

Since $A \subseteq B$, $x \in A \Rightarrow x \in B$

CASE II (when $x \notin A$)

Since $x \in A \cup B$, we have $x \in B$

Thus $x \in B$ in both the cases, and we have

$A \cup B \subseteq B$……………(1)

Conversely, let $x \in B$. Then clearly, $x \in A \cup B$

Hence $B \subseteq A \cup B$……………(2)

Combining (1) and (2), we deduce that

$A \cup B = B$ (proved)

Using Set Identities

Prove that (A – B) ∪ (A ∩ B) = A, where A and B are subsets of U

Proof:

LHS = $(A - B) \cup (A \cap B)$

= $(A \cap B^c) \cup (A \cap B)$ (Alternative representation for set difference)

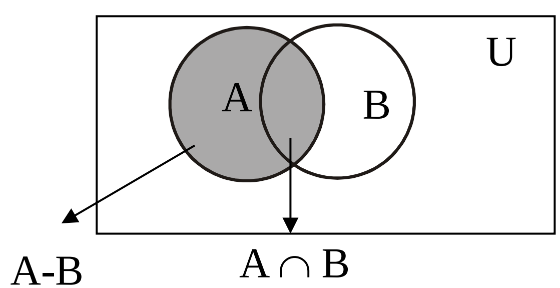
= $A \cap (B^c \cup B)$ Distributive Law

= $A \cap U$ Complement Law

= A Identity Law

= RHS (proved)

The above equality can be easily seen by Venn diagram below



Using Set Identities (Cont.)

Prove that $A - (A - B) = A \cap B$

Proof:

LHS = $A - (A - B)$

$= A - (A \cap B^c)$	Alternative representation for set difference
$= A \cap (A \cap B^c)^c$	Alternative representation for set difference
$= A \cap (A^c \cup (B^c)^c)$	DeMorgan's Law
$= A \cap (A^c \cup B)$	Double Complement Law
$= (A \cap A^c) \cup (A \cap B)$	Distributive Law
$= \emptyset \cup (A \cap B)$	Complement Law
$= A \cap B$	Identity Law
= RHS	(proved)

Using Set Identities (Cont.)

Prove that $(A - B) - C = (A - C) - B$

Proof:

LHS = $(A - B) - C$	
= $(A \cap B^c) - C$	Alternative representation of set difference
= $(A \cap B^c) \cap C^c$	Alternative representation of set difference
= $A \cap (B^c \cap C^c)$	Associative Law
= $A \cap (C^c \cap B^c)$	Commutative Law
= $(A \cap C^c) \cap B^c$	Associative Law
= $(A - C) \cap B^c$	Alternative representation of set difference
= $(A - C) - B$	Alternative representation of set difference
= RHS	(proved)

Using Set Identities (Cont.)

Simplify $(B^c \cup (B^c - A))^c$

Solution:

$(B^c \cup (B^c - A))^c$	$= (B^c \cup (B^c \cap A^c))^c$	Alternative representation for set difference
	$= (B^c)^c \cap (B^c \cap A^c)^c$	DeMorgan's Law
	$= B \cap ((B^c)^c \cup (A^c)^c)$	DeMorgan's Law
	$= B \cap (B \cup A)$	Double Complement Law
	$= B$	Absorption Law

Proving Set Identities Using Membership Table

Prove that $A - (A - B) = A \cap B$

Solution:

A	B	A-B	A-(A-B)	A∩B
1	1	0	1	1
1	0	1	0	0
0	1	0	0	0
0	0	0	0	0

The last two columns are same so $A - (A - B) = A \cap B$

Proving Set Identities Using Membership Table (Cont.)

Prove that $(A \cap B)^c = A^c \cup B^c$, using Membership Table

Solution:

A	B	$A \cap B$	$(A \cap B)^c$	A^c	B^c	$A^c \cup B^c$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Entries in the last columns are same thus $(A \cap B)^c = A^c \cup B^c$

Proving Set Identities Using Membership Table (Cont.)

Prove that $A - B = A \cap B^c$

Solution:

A	B	$A - B$	B^c	$A \cap B^c$
1	1	0	0	0
1	0	1	1	1
0	1	0	0	0
0	0	0	1	0

Entries in the 3rd and last columns are same so $A - B = A \cap B^c$