Lecture # 04

Discrete Structure

Biconditional

- The biconditional of p and q is “p if, and only if, q”
- Denoted by $p \leftrightarrow q$
- " $\leftrightarrow$" is the biconditional operator
- Biconditional statement is true if both p and q have the same truth values and false if p and q have opposite truth values
- “If and only if” is abbreviated as iff

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional (Cont.)

Examples:

1. "1+1 = 3 if and only if earth is flat"

   True

2. "Sky is blue iff 1 = 0"

   False

3. "Milk is white iff birds lay eggs"

   True

4. "33 is divisible by 4 if and only if horse has four legs"

   False

5. "$x > 5$ iff $x^2 > 25$"

   False

Biconditional (Cont.)

- In "**p if and only if q**", "**p if q**" means $\mathbf{q \rightarrow p}$ while "**p only if q**" means $\mathbf{p \rightarrow q}$
- So $\mathbf{p \leftrightarrow q}$ is logically equivalent to $\mathbf{(p \rightarrow q) \wedge (q \rightarrow p)}$

$$\mathbf{p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)}$$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Biconditional (Cont.)

Rephrasing propositions in the form “p if and only if q”

1. If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.

   You buy an ice cream cone if and only if it is hot outside.

2. For you to win the contest it is necessary and sufficient that you have the only winning ticket.

   You win the contest if and only if you hold the only winning ticket.

3. If you read the news paper every day, you will be informed and conversely.

   You will be informed if and only if you read the news paper every day.

4. This number is divisible by 6 precisely when it is divisible by both 2 and 3.

   This number is divisible by 6 if and only if it is divisible by both 2 and 3.

Hierarchy of Operations for Logical Connectives

1. ~(negation)
2. $\wedge$ (conjunction), $\vee$ (disjunction)
3. $\rightarrow$ (conditional), $\leftrightarrow$(biconditional)

- **p ∧ ~r ↔ q ∨ r**

p	q	r	~r	p∧~r	q∨r	p ∧ ~r ↔q∨r
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

Logical Equivalence Involving Biconditional

~p ↔ q and **p ↔ ~q** are logically equivalent

p	q	~p	~q	~p↔q	p↔~q
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

Logical Equivalence Involving Biconditional (Cont.)

- **~(p ⊕ q)** and **p ↔ q** are logically equivalent
- ⊕ is “Exclusive or” operator
- ⊕ has false truth value when both statements have same truth value.

p	q	p⊕q	~(p⊕q)	p↔q
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Laws Of Logic

1. Commutative Law: $p \leftrightarrow q \equiv q \leftrightarrow p$
2. Implication Laws: $p \to q \equiv \sim p \vee q$
   $\equiv \sim(p \wedge \sim q)$
3. Exportation Law: $(p \wedge q) \to r \equiv p \to (q \to r)$
4. Equivalence: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
5. Reductio ad absurdum $p \to q \equiv (p \wedge \sim q) \to c$

Applying Laws of Logic

Rewrite the statement forms without using the symbols $\rightarrow$ or $\leftrightarrow$

1. $p\wedge \sim q \rightarrow r$ **2. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$**

SOLUTION

1. $p\wedge \sim q \rightarrow r$

$\equiv\ (p \wedge \sim q) \rightarrow r$ Order of operations

$\equiv\ \equiv \sim(p \wedge \sim q) \vee r$ Implication law

2. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$

$\equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$ Implication law

$\equiv [(\sim p \vee r) \rightarrow (\sim q \vee r)] \wedge [(\sim q \vee r) \rightarrow (\sim p \vee r)]$ Equivalence of biconditional

$\equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$ Implication law

Applying Laws of Logic (Cont.)

Rewrite **$\sim p \vee q \rightarrow r \vee \sim q$** to a logically equivalent form that uses only **$\sim$** and $\wedge$

Solution:

Statement	Reason
$\sim p \vee q \rightarrow r \vee \sim q$	Given statement form
$\equiv (\sim p \vee q) \rightarrow (r \vee \sim q)$	Order of operations
$\equiv \sim[(\sim p \vee q) \wedge \sim (r \vee \sim q)]$	Implication law
$\equiv \sim[\sim(p \wedge \sim q) \wedge (\sim r \wedge q)]$	De Morgan’s law

Applying Laws of Logic (Cont.)

Show that **$\sim(p \rightarrow q) \rightarrow p$** is a tautology.

Solution:

Statement	**REASON**
$\sim(p \rightarrow q) \rightarrow p$	Given statement form
$\equiv \sim[\sim(p \wedge \sim q)] \rightarrow p$	Implication law
$\equiv (p \wedge \sim q) \rightarrow p$	Double negation law
$\equiv \sim(p \wedge \sim q) \vee p$	Implication law
$\equiv (\sim p \vee q) \vee p$	De Morgan's law
$\equiv (q \vee \sim p) \vee p$	Commutative law
$\equiv q \vee (\sim p \vee p)$	Associative law
$\equiv q \vee t$	Negation law
$\equiv t$	Universal bound law