Lecture # 03

Discrete Structure

Laws of Logic

Commutative Laws:	$p \wedge q \equiv q \wedge p$
	$p \vee q \equiv q \vee p$
Associative Laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws:	$p \wedge t \equiv p$
	$p \vee c \equiv p$
Negation laws:	$p \vee \sim p \equiv t$
	$p \wedge \sim p \equiv c$
Double negation law:	$\sim (\sim p) \equiv p$

Laws of Logic (Cont.)

Idempotent laws:	$p \wedge p \equiv p$
	$p \vee p \equiv p$
DeMorgan's laws:	$\sim (p \wedge q) \equiv \sim p \vee \sim q$
	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Universal bound laws:	$p \vee t \equiv t$
	$p \wedge c \equiv c$
Absorption laws:	$p \vee (p \wedge q) \equiv p$
	$p \wedge (p \vee q) \equiv p$
Negations of t and c:	$\sim t \equiv c$
	$\sim c \equiv t$

Applying Laws of Logic

Using law of logic, simplify $p \vee [\sim(\sim p \wedge q)]$

Solution:

$p \vee [\sim(\sim p \wedge q)]$	
$\equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan's Law
$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law
$\equiv [p \vee p] \vee (\sim q)$	Associative Law
$\equiv p \vee (\sim q)$	Indempotent Law

Which is the simplified statement form

Applying Laws of Logic (Cont.)

Using Laws of Logic, verify the logical equivalence

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

SOLUTION

Consider $\sim(\sim p \wedge q) \wedge (p \vee q)$

$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	DeMorgan's Law
$\equiv (p \vee \sim q) \wedge (p \vee q)$	Double Negative Law
$\equiv p \vee (\sim q \wedge q)$	Distributive Law
$\equiv p \vee c$	Negation Law
$\equiv p$	Identity Law

Hence the logical equivalence has been shown.

Simplifying a Statement

"You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains."

Let

p = You are hardworking

q = The sun shines

r = It rains

Statement form for above sentence: $(p \wedge q) \vee (p \wedge r)$

Using distributive law in reverse: $(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r)$

Putting $p \wedge (q \vee r)$ back into English, the sentence is rephrased as:

" You will get an A if you are hardworking and the sun shines or it rains."

Conditional Statements

- "If you earn an A in Math, then I'll buy you a computer."
- Let p = "You earn an A in Math."
- Q = "I will buy you a computer."
- if p is true, then q is true, or, more simply, if p, then q, or, p implies q, denoted by $p \to q$
- The arrow "$\to$ " is the conditional operator
- p is the hypothesis (antecedent) and q is the conclusion (consequent)

Truth Table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional Statements (Cont.)

Determine the truth value of each of the following conditional statements:

1. “If 1 = 1, then 3 = 3.”	TRUE
2. “If 1 = 1, then 2 = 3.”	FALSE
3. “If 1 = 0, then 3 = 3.”	TRUE
4. “If 1 = 2, then 2 = 3.”	TRUE
5. “If 1 = 1,then 1 = 2 and 2 = 3.”	FALSE
6. “If 1 = 3 or 1 = 2 then 3 = 3.”	TRUE

Translating English Sentences to Symbols

p = “you get an A on the final exam”

q = “you do every exercise in this book”

r = “you get an A in this class”

- To get an A in this class it is necessary for you to get an A on the final.

  $\mathbf{p \rightarrow r}$

- You do every exercise in this book; You get an A on the final, implies, you get an A in the class.

  $\mathbf{p \land q \rightarrow r}$

- Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

  $\mathbf{p \land q \rightarrow r}$

Translating Symbolic Propositions to English

p = “you have the flu”

q = “you miss the final exam”

r = “you pass the course”

- **$p \rightarrow q$**

  If you have flu, then you will miss the final exam.

- **$\sim q \rightarrow r$**

  If you don’t miss the final exam, you will pass the course.

- **$\sim p \wedge \sim q \rightarrow r$**

  If you neither have flu nor miss the final exam, then you will pass the course.

Hierarchy of Operations for Logical Connectives

1. ~(negation)
2. $\wedge$ (conjunction), $\vee$ (disjunction)
3. $\rightarrow$ (conditional)

Example

p $\vee$ ~ q $\rightarrow$ ~ p

(p $\vee$ (~ q)) $\rightarrow$ (~ p)

p	q	~q	~p	p $\vee$ ~q	**p $\vee$ ~ q $\rightarrow$ ~ p**
T	T	F	F	T	F
T	F	T	F	T	F
F	T	F	T	F	T
F	F	T	T	T	T

Hierarchy of Operations for Logical Connectives (Cont.)

$$(p \to q) \wedge (\sim p \to r)$$

p	q	r	$p \to q$	$\sim p$	$\sim p \to r$	$(p \to q) \wedge (\sim p \to r)$
T	T	T	T	F	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

Logical Equivalence Involving Implication

$$P \to q \equiv {\sim}q \to {\sim}p$$

p	q	~q	~p	p→q	~q → ~p
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

Implication Law

$$P \rightarrow q \equiv \sim p \vee q$$

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Negation of a Conditional Statement

- Implication Law:

  $p \rightarrow q \equiv \sim p \vee q$
- Taking Negation on both side:

  $\sim (p \rightarrow q) \equiv \sim (\sim p \vee q)$

  $\equiv \sim (\sim p) \wedge (\sim q)$ by De Morgan's law

  $\equiv p \wedge \sim q$ by the Double Negative law

- Negation of "**if p then q**" is logically equivalent to "**p and not q**"
- Negation of implication and implication are not equivalent

Negation of a Conditional Statement (Cont.)

Statements

1. If Ali lives in Pakistan then he lives in Lahore.
2. If my car is in the repair shop, then I cannot get to class.
3. If x is prime then x is odd **or** x is 2.
4. If n is divisible by 6, then n is divisible by 2 **and** n is divisible by 3.

Negations of statements

1. Ali lives in Pakistan and he does not live in Lahore.
2. My car is in the repair shop and I can get to class.
3. x is prime but x is not odd **and** x is not 2.
4. n is divisible by 6 but n is not divisible by 2 **or** by 3.

Inverse of a Conditional Statement

- The inverse of the conditional statement **$p \rightarrow q$** is **$\sim p \rightarrow \sim q$**
- A conditional and its inverse are not equivalent

p	q	p→q	~p	~q	~p →~q
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

Converse of a Conditional Statement

- The converse of the conditional statement **p → q** is **q → p**
- A conditional and its converse are not equivalent
- "→" is not a commutative operator

p	q	p→q	q→p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Contrapositive of a Conditional Statement

- Contrapositive of the conditional statement $\mathbf{p \rightarrow q}$ is $\mathbf{\sim q \rightarrow \sim p}$
- A conditional and its contrapositive are equivalent

$$\mathbf{p \rightarrow q \equiv \sim q \rightarrow \sim p}$$

p	q	p→q	~ q → ~ p
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Converse, Inverse, & Contrapositive

- Conditional statement: $\mathbf{p \to q}$
- Inverse of the conditional statement: $\mathbf{\sim p \to \sim q}$
- Converse of the conditional statement: $\mathbf{q \to p}$
- Contrapositive of the conditional statement: $\mathbf{\sim q \to \sim p}$
- $\mathbf{p \to q \equiv \sim q \to \sim p}$ **&** $\mathbf{\sim p \to \sim q \equiv q \to p}$

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim p \to \sim q$	$q \to p$	$\sim q \to \sim p$
T	T	T	F	F	T	T	T
T	F	F	F	T	T	T	F
F	T	T	T	F	F	F	T
F	F	T	T	T	T	T	T