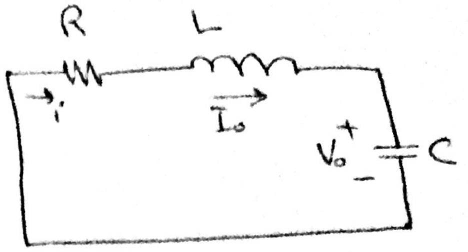
⇒ Natural Response of Series RLC Circuit :-

* Consider series RLC circuit as shown ⇒



The circuit is being excited by the energy initially stored in the capacitor and inductor.

V_o : initial voltage of capacitor

I_o : " inductor current.

By applying KVL :

$$iR + L\frac{di}{dt} + \underbrace{\frac{1}{C}\int_0^t i\,dt + V_o}_{\text{voltage through capacitor + initial voltage in it}} = 0$$

To eliminate the Integral, we differentiate w.r.t 't'. we get:

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = 0$$

$$\because \frac{1}{C}\int_0^t i\,dt \times \frac{d}{dt} = \frac{i}{C}$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

Here we see current w.r.t time 't'.

From previous practice on 1st order circuits we experience that solution is in exponential form.

So we assume: $i(t) = Ae^{st}$ —— (A)

where 'A' and 's' are constants to be determined

Now we write the characteristic eq. as:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \text{—— (B)}$$

we can solve it's quadratic eq as:

$$S_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \quad \text{—— (C)}$$

Q.E.:
eq. $5x^2+3x+3=0$
$\rightarrow ax^2+bx+c=0$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$S_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$S_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

we get two roots of above eq.

A more compact way of expressing those roots is:

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad \text{—— (D)}$$

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

- α is neper freq. expressed in np/s or damping factor.
- ω_0 " resonant " or undamped natural freq. in rad/s
- Neper freq: is measure of how circuit will die away.
- Resonant " : causing electric energy to oscillate between electric field of capacitor and magnetic field of inductor.

we can also write characteristic eq in (B) as:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

From eq. (C) we can say that there are two possible solutions for 'i' as:

$$i_1 = A_1 e^{s_1 t}, \quad i_2 = A_2 e^{s_2 t}$$

So, the overall natural response of series RLC circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

From eq (D) we infer that there are three types of solutions (3)

(1) If $\alpha > \omega_0$: Overdamped case (when the roots of characteristic eq. are unequal and real).

(2) $\alpha = \omega_0$: Critically damped case (when the roots are equal & real)

(3) $\alpha < \omega_0$: Underdamped case (when the roots are complex).

→ The Overdamped Case ($\alpha > \omega_0$):- We know from eq (C) and (E) that $C > \frac{4L}{R^2}$. When this happens both roots s_1 and s_2 are negative and real. So the response is.

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

→ The Critically Damped Case ($\alpha = \omega_0$):- When $\alpha = \omega_0$,

$C = \frac{4L}{R^2}$ and:

$$s_1 = s_2 = -\alpha = -\frac{R}{2L}$$

$$\boxed{i(t) = (A_1 + A_2 t)e^{-\alpha t}}$$

⇒ The Underdamped Case ($\alpha < \omega_0$):- when $\alpha < \omega_0$, $C < \frac{4L}{R^2}$

and
$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

we find $\boxed{i(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)}$

Example 8.3 :- In fig 8.8, $R = 40\,\Omega$, $L = 4H$, $C = \frac{1}{4}F$. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped or critically damped?

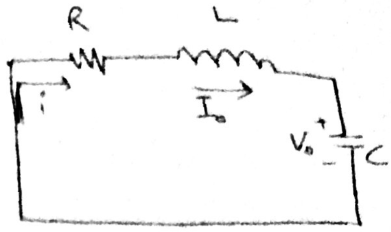


Fig: 8.8

Soln: we first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2\times4} = 5$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4\times\frac{1}{4}}} = 1$$

The roots are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -5 + \sqrt{(5)^2 - (1)^2}$$

$$s_1 = -5 + \sqrt{24}$$

$$s_1 = -0.101$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -5 - \sqrt{(5)^2 - (1)^2}$$

$$s_2 = -9.899$$

From above calculations we see that $\alpha > \omega_o$ i.e $5 > 1$.
So it means response is "Overdamped".
Also from roots s_1 and s_2 are real and negative, we can decide that the circuit response is "Overdamped."

P.P 8.3 :- If $R = 10\,\Omega$, $L = 5H$, $C = 2mF$. Find α, ω_o, s_1 and s_2. What type of natural response will the circuit have?

Soln: $\alpha = \frac{R}{2L} = \frac{10}{(2)(5)} = 1$, $\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5)(2mF)}} = 10$

The roots are.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \quad , \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$= -1 + \sqrt{(1)^2 - (10)^2} \quad , \quad s_2 = -1 - \sqrt{(1)^2 - (10)^2}$$

$$s_1 = -1 + j9.95 \qquad\qquad s_2 = -1 - j9.95.$$

From above calculations we see that $\alpha < \omega_o$ i.e. $1 < 10$.
So it means response is "Underdamped"

Example 8.4: Find i(t) in the circuit fig 8.10, Assume that circuit has reached steady state at $t = 0^-$.

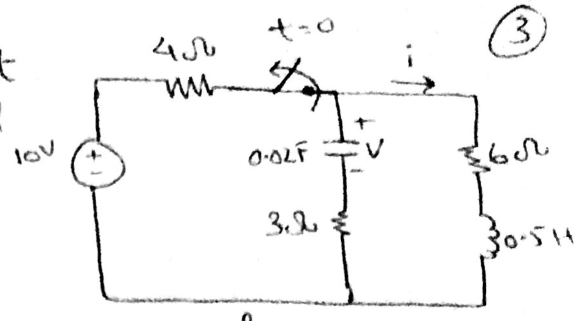


fig 8.10

Solution: For $t < 0$, switch is closed.
The capacitor behave as an open circuit,
and inductor " " " short circuit

So the equivalent circuit shown in fig 8.11 (a) for $t < 0$:

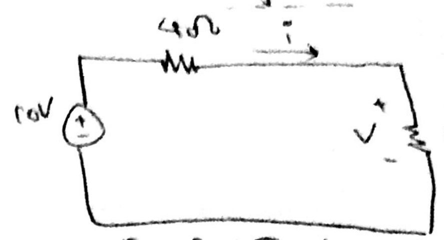


Fig 8.11 (a) $t < 0$

Thus at $t = 0$

$$i(0) = \frac{V}{R_{eq}} = \frac{10V}{4+6} = 1A$$

$$v(0) = \frac{R_2}{R_1 + R_2} V = \frac{6}{\cancel{(4+6)}} \times \cancel{10} = 6V$$

So now we know that $i(0)$ is the initial current through the inductor and $V(0)$ is the " voltage " capacitor.

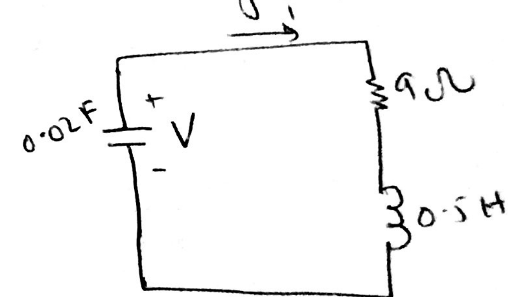


Fig 8.11 (b) $t > 0$

Now when $t > 0$, the switch is open and so the volt. source is disconnected.
Also the resistors 6Ω and 3Ω in fig. 8.10 are in series, so their equivalent resistance is $6\Omega + 3\Omega = 9\Omega$.

Now we calculate the roots s_1 and s_2. we first find α and ω_0 as:

$$\alpha = \frac{R}{2L} = \frac{9}{2(0.5)} = 9 \quad , \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.5)(0.02)}} = 10$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -9 + \sqrt{81 - 100}$$

$$s_1 = -9 + j4.359$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -9 - \sqrt{81 - 100}$$

$$s_2 = -\alpha - j4.359$$

From calculations we see that, $\alpha < \omega_0$ i.e $9 < 10$, it means response is "Underdamped". We use formulas for underdamped circuit response:

$$i(t) = A_1 e^{-\alpha t} \cos\omega_d t + B_2 e^{-\alpha t} \sin\omega_d t \quad \text{—— (A)}$$

Here $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$\omega_d = \sqrt{100 - 81}$

$\omega_d = 4.359$

Substitute values in eq (A): we get

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \quad \text{—— (B)}$$

We obtain A_1 and A_2 using initial conditions, At $t=0$

$i(0) = 1 = A_1$

The circuit eq of fig 8.11b is:

$$R\,i(0) + L\frac{di(0)}{dt} + V_0 = 0$$

$V(0) = V_0 = -6V$ because of sign convention.

or $$\frac{di(0)}{dt} = -\frac{1}{L}\left(R\,i(0) + V_0\right)$$

$$= -2(9(1) + (-6))$$

$$\left.\frac{di}{dt}\right|_{t=0} = -6\ A/s$$

Now taking derivative of $i(t)$ in eq (B):

$$\frac{d_i(t)}{dt} = e^{-9t}\frac{d}{dt}(A_1 \cos 4.359t + A_2 \sin 4.359t) + (A_1 \cos 4.359t + A_2 \sin 4.359t)\frac{d}{dt}e^{-9t}$$

$$= e^{-9t}[A_1(-\sin 4.359t)(4.359) + A_2 \cos 4.359t(4.359)] + (\quad " \quad)e^{-9t}\frac{d}{dt}(-9t)$$

$$\frac{di}{dt} = (e^{-9t})(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359) - 9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$

At $t=0$: we get above eq as:

$-6 = 4.359(-0 + A_2) - 9(A_1 + 0)$

$-6 = -9 + 4.359 B_2 \Rightarrow B_2 = 0.6882$

So the complete solution is

$$\boxed{i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t)\ A}$$