{Electric Network Analysis}

→ In previous chapter we studied circuits which contained only one energy storage element i.e. Capacitor or Inductor
- RL circuit } First-Order Circuits. (1^{st} differential eq.)
- RC circuit

→ Now in this chapter we will study "Second Order Differential Equations" / "Second-Order Circuits".

⇒ Second-Order Circuit (RLC Circuit):- "is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements".

* The second-order circuit or the RLC circuit has three kinds of passive elements present in it. i.e resistor, inductor and capacitor.

- 2nd order differential:

e.g.

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$f''(x) = 6x \quad \checkmark$$

or

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

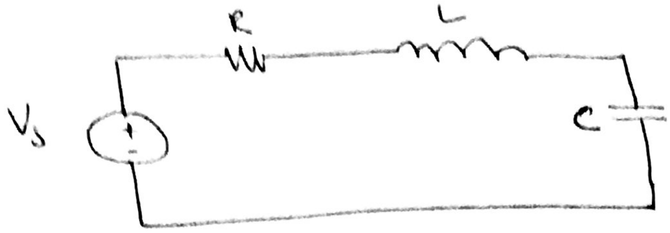
$$\frac{d^2y}{dx^2} = 6x \quad \checkmark$$

* These RLC circuits are occurs routinely in wide variety of applications: e.g oscillators, frequency filters.
* Modelling of practical situations e.g temperature controllers, automobile suspension system etc.

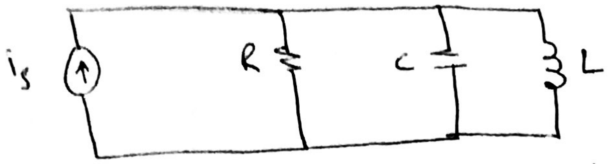
⇒ Types of RLC Circuits:-

Two basic types of RLC circuits are on basis of their combinations i.e.

- Parallel Connection / Parallel RLC Circuit
- Series " / Series " "



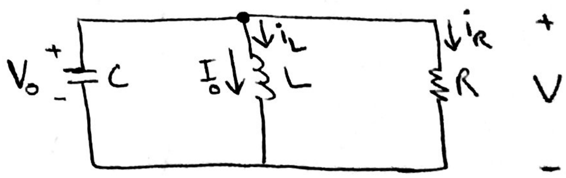
"Series RLC Circuit"



"Parallel RLC Circuit".

⇒ Natural Response of Parallel RLC Circuits

* Consider a parallel RLC circuit in figure ⟶



we apply KCL to it:

$$i_c + i_L + i_R = 0$$

$$C\frac{dv}{dt} + \underbrace{\frac{1}{L}\int_0^t v\,dt + I_o}_{\text{current through the inductor + initial current in inductor.}} + \frac{V}{R} = 0$$

$$\because \frac{1}{L}\int_0^t v\,dt \;\frac{d}{dt} = \frac{V}{L}$$

Taking 2nd order differential:

$$C\frac{d^2V}{dt^2} + \frac{V}{L} + \frac{1}{R}\frac{dV}{dt} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}v = 0$$

here we see voltage w.r.t time.

* From previous practice on first order circuits, we experience that solution is in exponential form.
So, we can assume

$$V(t) = Ae^{st} \quad \text{——— (A)}$$

where 'A' and 's' are constants to be determined.

Now we write the characteristic equation as below:

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0 \quad \text{——— (B)}$$

We solve this quadratic equation as,

$$S_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad \text{——— (C)}$$

So,

$$\left.\begin{aligned} S_1 &= -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \\ S_2 &= -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \end{aligned}\right\} \text{we get two roots.}$$

∴ A more compact way of expressing these roots is

$$\boxed{\begin{aligned} S_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ S_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}} \quad \text{——— (D)}$$

where $\alpha = \frac{1}{2RC}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

* 'α' is neper frequency expressed in np/s
* 'ω_0' is resonant " or underdamped natural freq. rad/s
* 'α' nepere freq. is ~~the~~ measure of how fast the transient responce of circuit will die away after stimulus is removed.
* resonant freq: The resonance of RLC causes the electrical energy to oscilate between electrical field of capacitor and magnitude of inductor. This is the underdamped natural freq. response

* We can also write characteristic eq. as:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

The two values of 's' in eq (C) indicate that there are two possible solutions for 'v', each of which is of the form of the assumed solution in (A) that is:

$$v_1 = A_1 e^{s_1 t} \quad \text{and} \quad v_2 = A_2 e^{s_2 t}$$

So the overall natural response of parallel RLC circuit is:

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

From eq (D) it can be concluded that there are three types of solutions:

(1) If $\alpha > \omega_0$: Over damped case (response). Both roots s_1 and s_2 will be unequal and real numbers.

(2) If $\alpha = \omega_0$: Critically damped response. Both roots s_1 and s_2 are equal and real. This is special case

(3) If $\alpha < \omega_0$: Underdamped response. Both roots s_1 and s_2 are complex numbers i.e. non-zero imaginary components.

→ <u>The Overdamped Case</u> ($\alpha > \omega_0$):- From (C) and (D) we know that $L > 4R^2C$. The natural response of eq. is:

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

→ <u>The Critically Damped Case</u> ($\alpha = \omega_0$):- For $\alpha = \omega_0$, $L = 4R^2C$
The roots are real and equal so that the response is:

$$V(t) = (A_1 + A_2 t)e^{-\alpha t}$$

→ <u>The Underdamped Case</u> ($\alpha < \omega_0$):- When $\alpha < \omega_0$, $L < 4R^2C$,
In this case the roots are complex and expressed as:

$$s_{1,2} = -\alpha \pm j\omega_d \qquad \text{where } \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

The response is:

$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

Example 8.5:- In parallel circuit of fig. 8.13, find $v(t)$ for $t>0$ assuming $V(0) = 5V$, $i(0) = 0$, $L = 1H$, and $C = 10mF$. Consider these cases: $R = 1.293\Omega$, $R = 5\Omega$ and $R = 6.25\Omega$

Solution: Case # 1: If $R = 1.923\Omega$

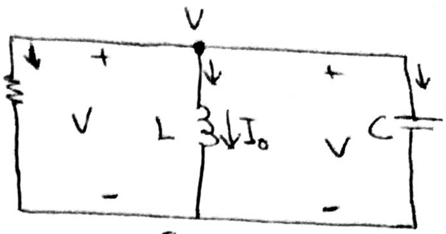


Fig. 8.13

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Here, we see that $\alpha > \omega_o$, so the circuit response is "Overdamped". The roots of characteristic eq. are:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

$$s_1 = -26 + \sqrt{(26)^2 - (10)^2} = -2$$

$$s_2 = -26 - \sqrt{(26)^2 - (10)^2} = -50$$

We can see that roots s_1 and s_2 are unequal and real. Now we find the corresponding response.

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{—— (a)}$$

To get A_1 and A_2, we first apply initial conditions:

$$V(0) = 5 = A_1 e^0 + A_2 e^0$$

$$V(0) = 5 = A_1 + A_2 \quad \text{—— (b)}$$

Circuit eq. of fig. 8.13 is:

$$C\frac{dv}{dt} + I_o + \frac{V_o}{R} = 0$$

or
$$\frac{dv}{dt} + \frac{I_o}{C} + \frac{V_o}{RC} = 0$$

$$\frac{dV(0)}{dt} = -\left(\frac{I_o}{C} + \frac{V_o}{RC}\right) = -\left(\frac{0}{C} + \frac{5}{1.923 \times 10 \times 10^{-3}}\right)$$

$$\frac{dV(0)}{dt} = -260$$

We differentiate both sides of (a) as below.

$$\frac{dv}{dt} = -2A_1e^{-2t} - 50A_2e^{-50t}$$

$$\frac{dv(0)}{dt} = -2A_1 - 50A_2$$

$$-260 = -2A_1 - 50A_2 \quad \text{——— (c)}$$

From eq (b) and eq (c) we find:

$A_1 = -0.2083$

and $A_2 = 5.208$

Substituting A_1 and A_2 in eq (a) we get.

$$\boxed{V(t) = -0.2083e^{-2t} + 5.208e^{-50t}}$$

Case #2: When $R = 5\Omega$

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times5\times10\times10^{-3}} = 10$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times10\times10^{-3}}} = 10$$

Since, $\alpha = \omega_0 = 10$, it means response is Critically Damped

The roots of characteristic eq. are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -10 + \sqrt{100-100} = -10$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -10 - \sqrt{100-100} = -10$$

The corresponding response of circuit is:

$$V(t) = (A_1 + A_2t)\,e^{-10t} \quad \text{——— (d)}$$

To find A_1 and A_2 apply initial condition.

$V(0) = 5 = A_1$

$$\frac{dv(0)}{dt} = -\left(\frac{I_0}{C} + \frac{V_0}{RC}\right) = -\left(0 + \frac{5}{5\times10\times10^{-3}}\right) = -100$$

Differential eq (d) we get:

$$\frac{dv}{dt} = \frac{d}{dt}A_1e^{-10t} + \frac{1}{dt}A_2te^{-10t}$$

$$\frac{dv}{dt} = (A_1 e^{-10t} \cdot (-10)) + A_2(t \cdot e^{-10t}(-10) + e^{-10t}(1))$$ (11)

$$= -10 A_1 e^{-10t} + A_2(-10t e^{-10t} + e^{-10t})$$

$$= -10 A_1 e^{-10t} - 10 A_2 t e^{-10t} + A_2 e^{-10t}$$

$$= e^{-10t}(-10 A_1 - 10 A_2 t + A_2)$$

At $t = 0$,

$$-100 = -10 A_1 + A_2$$

$$A_2 = -50$$

So, substituting values in (d) we get

$$\boxed{v(t) = (5 - 50t)e^{-10t}\ V}$$

Case #3 :- If $R = 6.25\,\Omega$.

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8.$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

As, $\alpha < \omega_0$ so response is "Underdamped".

The roots are:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

As we can see roots are complex. The corresponding response of circuit is:

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \quad \text{— (e)}$$

To obtain A_1 and A_2, we apply initial conditions:

$$V(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\left(\frac{I_0}{C} + \frac{V_0}{RC}\right) = -\left(0 + \frac{5}{6.25 \times 10 \times 10^{-3}}\right)$$

$$\frac{dv(0)}{dt} = -80$$

Differentiating eq ⓔ as:

$$\frac{dv}{dt} = \left(A_1 \cos 6t\, e^{-8t} + A_2 \sin 6t\, e^{-8t}\right)\frac{d}{dt}$$

$$\frac{dv}{dt} = \left(A_1 \left(-6\sin 6t\, e^{-8t} - 8\cos 6t\, e^{-8t}\right)\right) + \left(A_2 \left(6\cos 6t e^{-8t} - 8\sin 6t\, e^{-8t}\right)\right)$$

$$= -A_1 6\sin 6t e^{-8t} - 8A_1 \cos 6t e^{-8t} + A_2 6\cos 6t e^{-8t} - 8A_2 \sin 6t e^{-8t}$$

$$= \left(-A_1 6\sin 6t - 8A_2 \sin 6t - 8A_1 \cos 6t + A_2 6\cos 6t\right)e^{-8t}$$

At $t = 0$,

$$-80 = -8A_1 + 6A_2$$

$$-80 = -8(5) + 6(A_2)$$

$$A_2 = -6.667$$

Substituting in eq. ⓔ we get

$$\boxed{v(t) = \left(5\cos 6t - 6.667\sin 6t\right)e^{-8t}}$$