## 8.5 Conjugate-Beam Method

The conjugate-beam method was developed by H. Müller-Breslau in 1865. Essentially, it requires the same amount of computation as the moment-area theorems to determine a beam's slope or deflection; however, this method relies only on the principles of statics, and hence its application will be more familiar.

The basis for the method comes from the *similarity* of Eq. 4–1 and Eq. 4–2 to Eq. 8–2 and Eq. 8–4. To show this similarity, we can write these equations as follows:

$$\frac{dV}{dx} = w \qquad \Big| \qquad \frac{d^2M}{dx^2} = w$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \Big| \qquad \frac{d^2v}{dx^2} = \frac{M}{EI}$$

Or integrating,

$$V = \int w\,dx \qquad \Big| \qquad M = \int\left[\int w\,dx\right]dx$$

$$\updownarrow \qquad \updownarrow \qquad \Big| \qquad \updownarrow \qquad \updownarrow$$

$$\theta = \int\left(\frac{M}{EI}\right)dx \qquad \Big| \qquad v = \int\left[\int\left(\frac{M}{EI}\right)dx\right]dx$$

Here the *shear* $V$ compares with the *slope* $\theta$, the *moment* $M$ compares with the *displacement* $v$, and the *external load* $w$ compares with the $M/EI$ diagram. To make use of this comparison we will now consider a beam having the same length as the real beam, but referred to here as the "conjugate beam," Fig. 8–23. The conjugate beam is "loaded" with the $M/EI$ diagram derived from the load $w$ on the real beam. From the above comparisons, we can state two theorems related to the conjugate beam, namely,

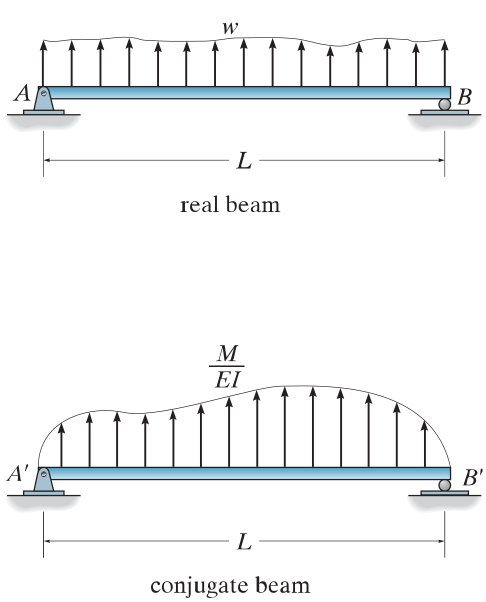


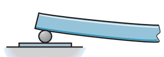
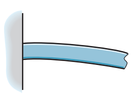
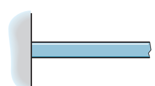
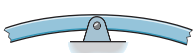
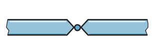
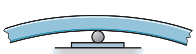
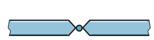
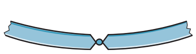
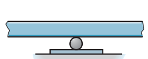
Fig. 8–23

**Theorem 1: The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.**

**Theorem 2: The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam.**

**Conjugate-Beam Supports.** When drawing the conjugate beam it is important that the shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beam at its supports, a consequence of Theorems 1 and 2. For example, as shown in Table 8–2, a pin or roller support at the end of the real beam provides *zero displacement,* but the beam has a nonzero slope. Consequently, from Theorems 1 and 2, the conjugate beam must be supported by a pin or roller, since this support has *zero moment* but has a shear or end reaction. When the real beam is fixed supported (3), both the slope and displacement at the support are zero. Here the conjugate beam has a free end, since at this end there is zero shear and zero moment. Corresponding real and conjugate-beam supports for other cases are listed in the table. Examples of real and conjugate beams are shown in Fig. 8–24. Note that, as a rule, neglecting axial force, statically determinate real beams have statically determinate conjugate beams; and statically indeterminate real beams, as in the last case in Fig. 8–24, become unstable conjugate beams. Although this occurs, the $M/EI$ loading will provide the necessary "equilibrium" to hold the conjugate beam stable.

**TABLE 8–2**

| | Real Beam | | Conjugate Beam | |
|---|---|---|---|---|
| 1) | $\theta$<br>$\Delta = 0$ | <br>pin | $V$<br>$M = 0$ | <br>pin |
| 2) | $\theta$<br>$\Delta = 0$ | <br>roller | $V$<br>$M = 0$ | <br>roller |
| 3) | $\theta = 0$<br>$\Delta = 0$ | <br>fixed | $V = 0$<br>$M = 0$ | <br>free |
| 4) | $\theta$<br>$\Delta$ | <br>free | $V$<br>$M$ | <br>fixed |
| 5) | $\theta$<br>$\Delta = 0$ | <br>internal pin | $V$<br>$M = 0$ | <br>hinge |
| 6) | $\theta$<br>$\Delta = 0$ | <br>internal roller | $V$<br>$M = 0$ | <br>hinge |
| 7) | $\theta$<br>$\Delta$ | <br>hinge | $V$<br>$M$ | <br>internal roller |

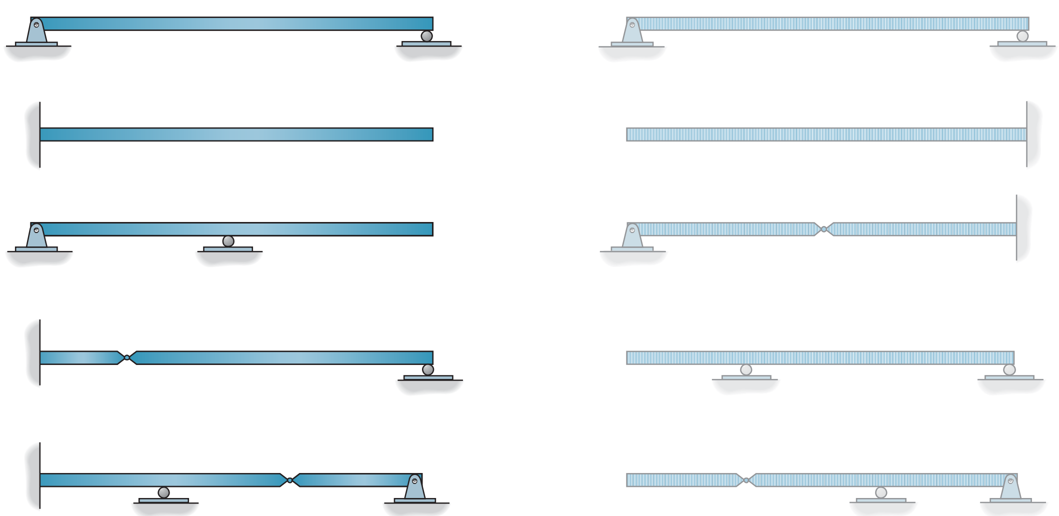



**Fig. 8–24**

## Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the conjugate-beam method.

### Conjugate Beam

- Draw the conjugate beam for the real beam. This beam has the same length as the real beam and has corresponding supports as listed in Table 8–2.
- In general, if the real support allows a *slope,* the conjugate support must develop a *shear*; and if the real support allows a *displacement,* the conjugate support must develop a *moment*.
- The conjugate beam is loaded with the real beam's $M/EI$ diagram. This loading is assumed to be *distributed* over the conjugate beam and is directed *upward* when $M/EI$ is *positive* and *downward* when $M/EI$ is *negative*. In other words, the loading always acts *away* from the beam.

### Equilibrium

- Using the equations of equilibrium, determine the reactions at the conjugate beam's supports.
- Section the conjugate beam at the point where the slope $\theta$ and displacement $\Delta$ of the real beam are to be determined. At the section show the unknown shear $V'$ and moment $M'$ acting in their positive sense.
- Determine the shear and moment using the equations of equilibrium. $V'$ and $M'$ equal $\theta$ and $\Delta$, respectively, for the real beam. In particular, if these values are *positive,* the *slope* is *counterclockwise* and the *displacement* is *upward*.

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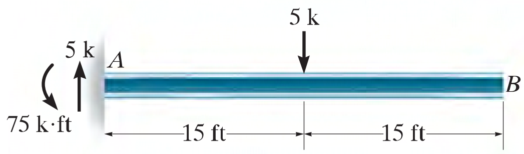
## EXAMPLE 8.13

Determine the slope and deflection at point $B$ of the steel beam shown in Fig. 8–25*a*. The reactions have been computed. $E = 29(10^3)$ ksi, $I = 800$ in$^4$.

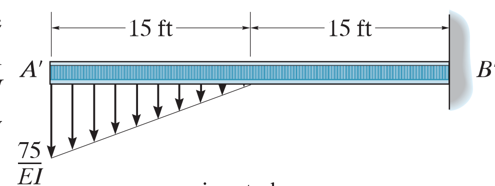



real beam
(a)

**Fig. 8–25**

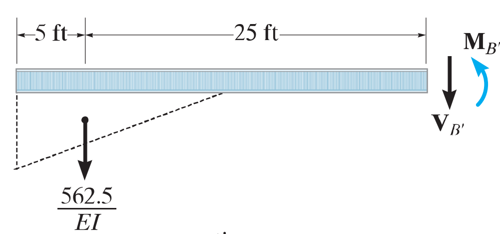
### SOLUTION

**Conjugate Beam.** The conjugate beam is shown in Fig. 8–25*b*. The supports at $A'$ and $B'$ correspond to supports $A$ and $B$ on the real beam, Table 8–2. It is important to understand why this is so. The $M/EI$ diagram is *negative,* so the distributed load acts *downward,* i.e., away from the beam.




conjugate beam
(b)

**Equilibrium.** Since $\theta_B$ and $\Delta_B$ are to be determined, we must compute $V_{B'}$ and $M_{B'}$ in the conjugate beam, Fig. 8–25*c*.




reactions
(c)

$$+\uparrow \Sigma F_y = 0; \qquad -\frac{562.5\ \text{k}\cdot\text{ft}^2}{EI} - V_{B'} = 0$$

$$\theta_B = V_{B'} = -\frac{562.5\ \text{k}\cdot\text{ft}^2}{EI}$$

$$= \frac{-562.5\ \text{k}\cdot\text{ft}^2}{29(10^3)\ \text{k/in}^2(144\ \text{in}^2/\text{ft}^2)800\ \text{in}^4(1\ \text{ft}^4/(12)^4\ \text{in}^4)}$$

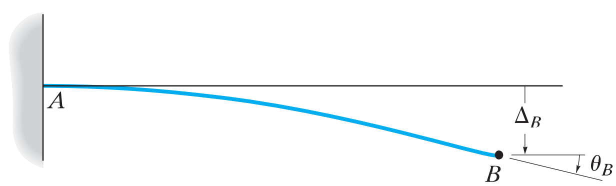
$$= -0.00349\ \text{rad} \qquad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad \frac{562.5\ \text{k}\cdot\text{ft}^2}{EI}(25\ \text{ft}) + M_{B'} = 0$$

$$\Delta_B = M_{B'} = -\frac{14\,062.5\ \text{k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-14\,062.5\ \text{k}\cdot\text{ft}^3}{29(10^3)(144)\ \text{k/ft}^2[800/(12)^4]\ \text{ft}^4}$$

$$= -0.0873\ \text{ft} = -1.05\ \text{in.} \qquad \textit{Ans.}$$




(d)

The negative signs indicate the slope of the beam is measured clockwise and the displacement is downward, Fig. 8–25*d*.

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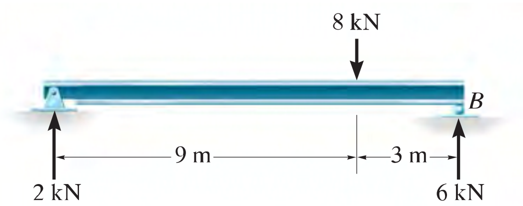
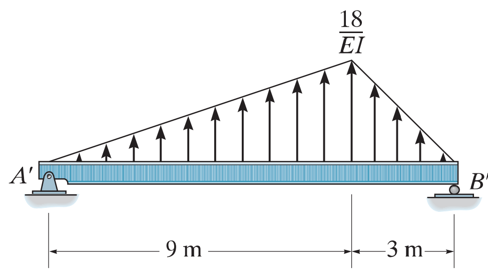
## EXAMPLE 8.14

Determine the maximum deflection of the steel beam shown in Fig. 8–26*a*. The reactions have been computed. $E = 200$ GPa, $I = 60(10^6)$ mm$^4$.




real beam

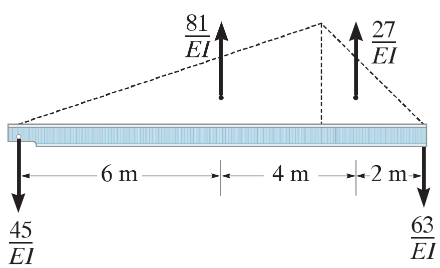
(a)




conjugate beam

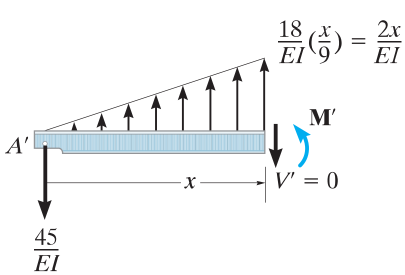
(b)

**Fig. 8–26**




external reactions

(c)




internal reactions

(d)

### SOLUTION

**Conjugate Beam.** The conjugate beam loaded with the *M/EI* diagram is shown in Fig. 8–26*b*. Since the *M/EI* diagram is positive, the distributed load acts upward (away from the beam).

**Equilibrium.** The external reactions on the conjugate beam are determined first and are indicated on the free-body diagram in Fig. 8–26*c*. *Maximum deflection* of the real beam occurs at the point where the *slope* of the beam is *zero*. This corresponds to the same point in the conjugate beam where the *shear* is *zero*. Assuming this point acts within the region $0 \le x \le 9$ m from $A'$, we can isolate the section shown in Fig. 8–26*d*. Note that the peak of the distributed loading was determined from proportional triangles, that is, $w/x = (18/EI)/9$. We require $V' = 0$ so that

$$+\uparrow \Sigma F_y = 0; \qquad -\frac{45}{EI} + \frac{1}{2}\left(\frac{2x}{EI}\right)x = 0$$

$$x = 6.71 \text{ m} \qquad (0 \le x \le 9 \text{ m}) \text{ OK}$$



Using this value for $x$, the maximum deflection in the real beam corresponds to the moment $M'$. Hence,

$$\circlearrowleft + \Sigma M = 0; \qquad \frac{45}{EI}(6.71) - \left[\frac{1}{2}\left(\frac{2(6.71)}{EI}\right)6.71\right]\frac{1}{3}(6.71) + M' = 0$$

$$\Delta_{max} = M' = -\frac{201.2 \text{ kN}\cdot\text{m}^3}{EI}$$

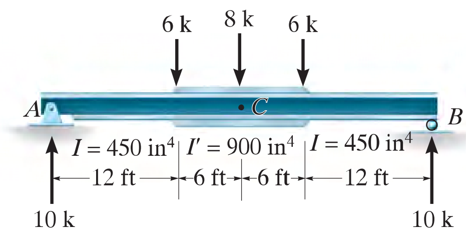
$$= \frac{-201.2 \text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2][60(10^6)\text{ mm}^4(1\text{ m}^4/(10^3)^4\text{ mm}^4)]}$$

$$= -0.0168 \text{ m} = -16.8 \text{ mm}$$ *Ans.*

The negative sign indicates the deflection is downward.

## EXAMPLE 8.15

The girder in Fig. 8–27*a* is made from a continuous beam and reinforced at its center with cover plates where its moment of inertia is larger. The 12-ft end segments have a moment of inertia of $I = 450\text{ in}^4$, and the center portion has a moment of inertia of $I' = 900\text{ in}^4$. Determine the deflection at the center $C$. Take $E = 29(10^3)$ ksi. The reactions have been calculated.



real beam

(a)

**Fig. 8–27**

### SOLUTION

**Conjugate Beam.** The moment diagram for the beam is determined first, Fig. 8–27*b*. Since $I' = 2I$, for simplicity, we can express the load on the conjugate beam in terms of the constant $EI$, as shown in Fig. 8–27*c*.

**Equilibrium.** The reactions on the conjugate beam can be calculated by the symmetry of the loading or using the equations of equilibrium. The results are shown in Fig. 8–27*d*. Since the deflection at $C$ is to be determined, we must compute the internal moment at $C'$. Using the method of sections, segment $A'C'$ is isolated and the resultants of the distributed loads and their locations are determined, Fig. 8–27*e*. Thus,

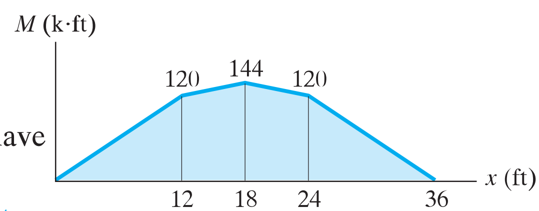
$$\circlearrowleft +\Sigma M_{C'} = 0; \quad \frac{1116}{EI}(18) - \frac{720}{EI}(10) - \frac{360}{EI}(3) - \frac{36}{EI}(2) + M_{C'} = 0$$

$$M_{C'} = -\frac{11\,736\text{ k}\cdot\text{ft}^3}{EI}$$

Substituting the numerical data for $EI$ and converting units, we have

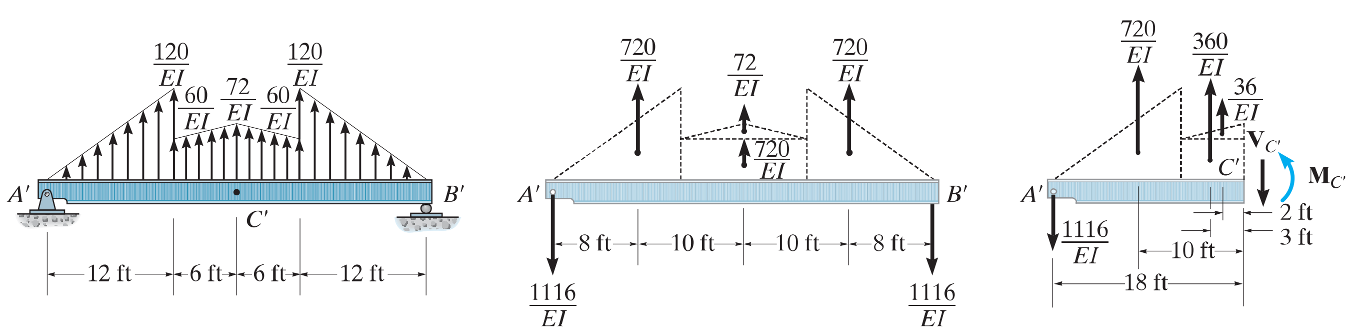
$$\Delta_C = M_{C'} = -\frac{11\,736\text{ k}\cdot\text{ft}^3(1728\text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(450\text{ in}^4)} = -1.55\text{ in.} \qquad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.



moment diagram

(b)



conjugate beam (c)

external reactions (d)

internal reactions (e)

## EXAMPLE 8.16

Determine the displacement of the pin at $B$ and the slope of each beam segment connected to the pin for the compound beam shown in Fig. 8–28*a*. $E = 29(10^3)$ ksi, $I = 30\text{ in}^4$.

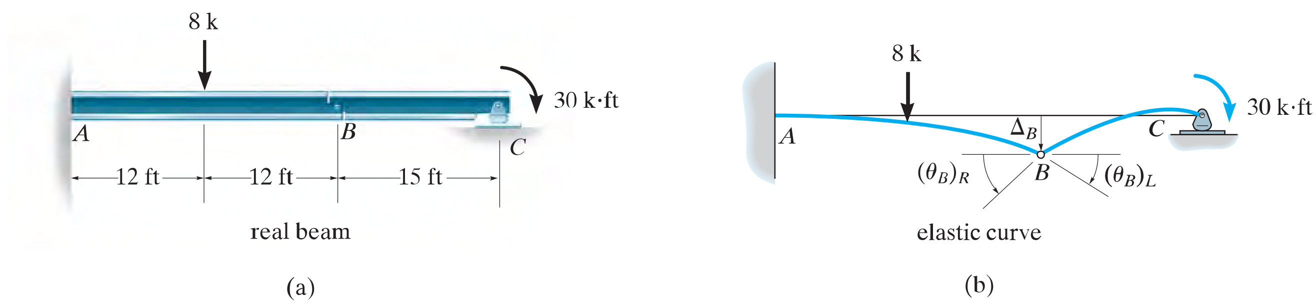
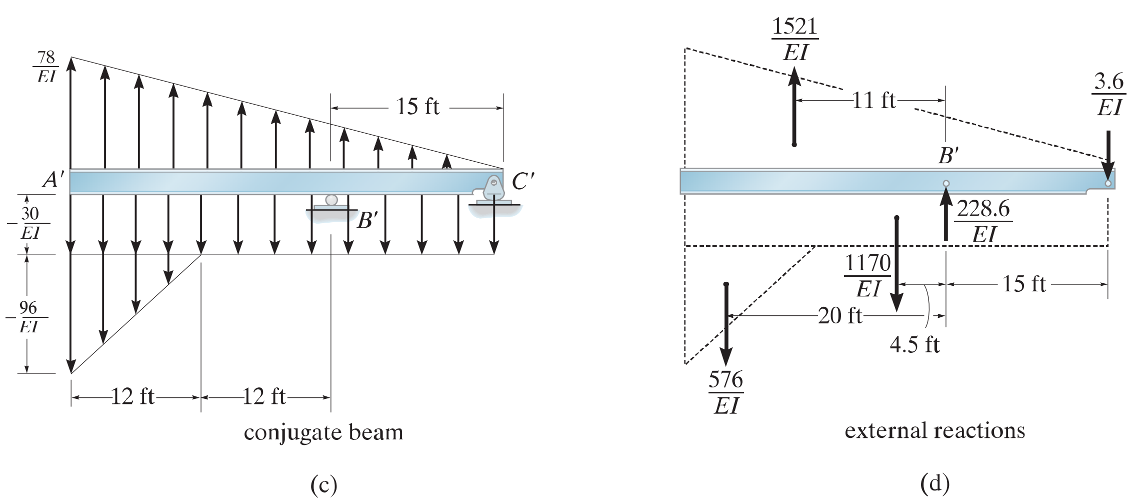



**Fig. 8–28**

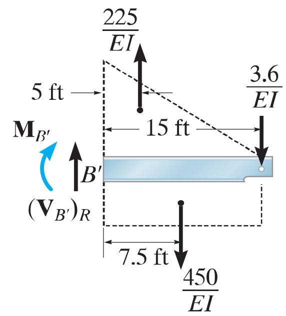
### SOLUTION

**Conjugate Beam.** The elastic curve for the beam is shown in Fig. 8–28*b* in order to identify the unknown displacement $\Delta_B$ and the slopes $(\theta_B)_L$ and $(\theta_B)_R$ to the left and right of the pin. Using Table 8–2, the conjugate beam is shown in Fig. 8–28*c*. For simplicity in calculation, the *M/EI* diagram has been drawn in *parts* using the principle of superposition as described in Sec. 4–5. In this regard, the real beam is thought of as cantilevered from the left support, $A$. The moment diagrams for the 8-k load, the reactive force $C_y = 2$ k, and the 30-k · ft loading are given. Notice that negative regions of this diagram develop a downward distributed load and positive regions have a distributed load that acts upward.





**Equilibrium.** The external reactions at $B'$ and $C'$ are calculated first and the results are indicated in Fig. 8–28*d*. In order to determine $(\theta_B)_R$, the conjugate beam is sectioned just to the *right* of $B'$ and the shear force $(V_{B'})_R$ is computed, Fig. 8–28*e*. Thus,




(e)

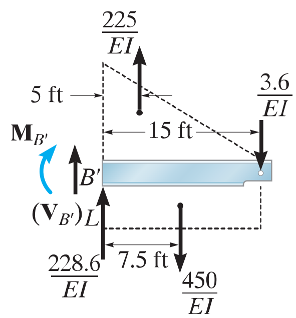
$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_R + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_R = (V_{B'})_R = \frac{228.6 \text{ k}\cdot\text{ft}^2}{EI}$$

$$= \frac{228.6 \text{ k}\cdot\text{ft}^2}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= 0.0378 \text{ rad} \qquad \textit{Ans.}$$

The internal moment at $B'$ yields the displacement of the pin. Thus,




(f)

$$\circlearrowleft\!+\Sigma M_{B'} = 0; \qquad -M_{B'} + \frac{225}{EI}(5) - \frac{450}{EI}(7.5) - \frac{3.6}{EI}(15) = 0$$

$$\Delta_B = M_{B'} = -\frac{2304 \text{ k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-2304 \text{ k}\cdot\text{ft}^3}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= -0.381 \text{ ft} = -4.58 \text{ in.} \qquad \textit{Ans.}$$

The slope $(\theta_B)_L$ can be found from a section of beam just to the *left* of $B'$, Fig. 8–28*f*. Thus,

$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_L + \frac{228.6}{EI} + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_L = (V_{B'})_L = 0 \qquad \textit{Ans.}$$

Obviously, $\Delta_B = M_{B'}$ for this segment is the *same* as previously calculated, since the moment arms are only slightly different in Figs. 8–28*e* and 8–28*f*.