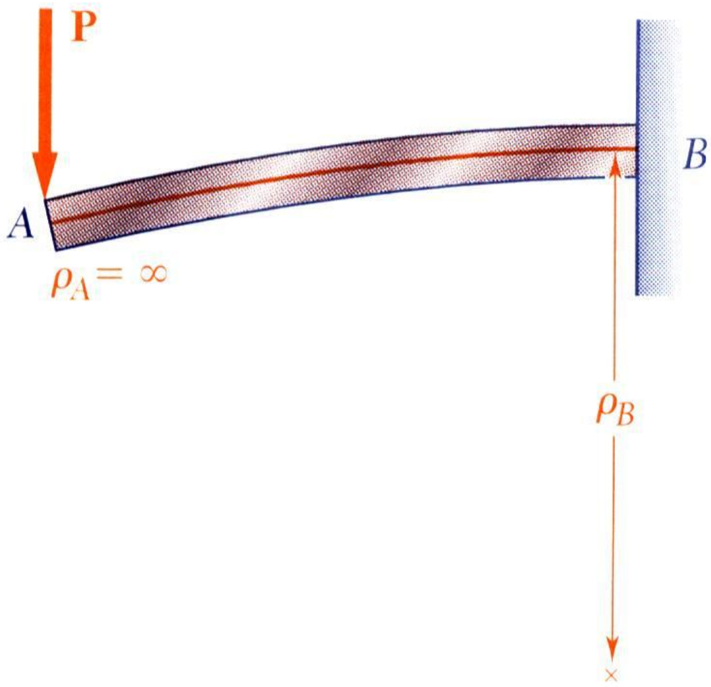
# Chapter # 9

Deflection of Beams

# Deflection of Beams

# Deformation of a Beam Under Transverse Loading




- Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Cantilever beam subjected to concentrated load at the free end,

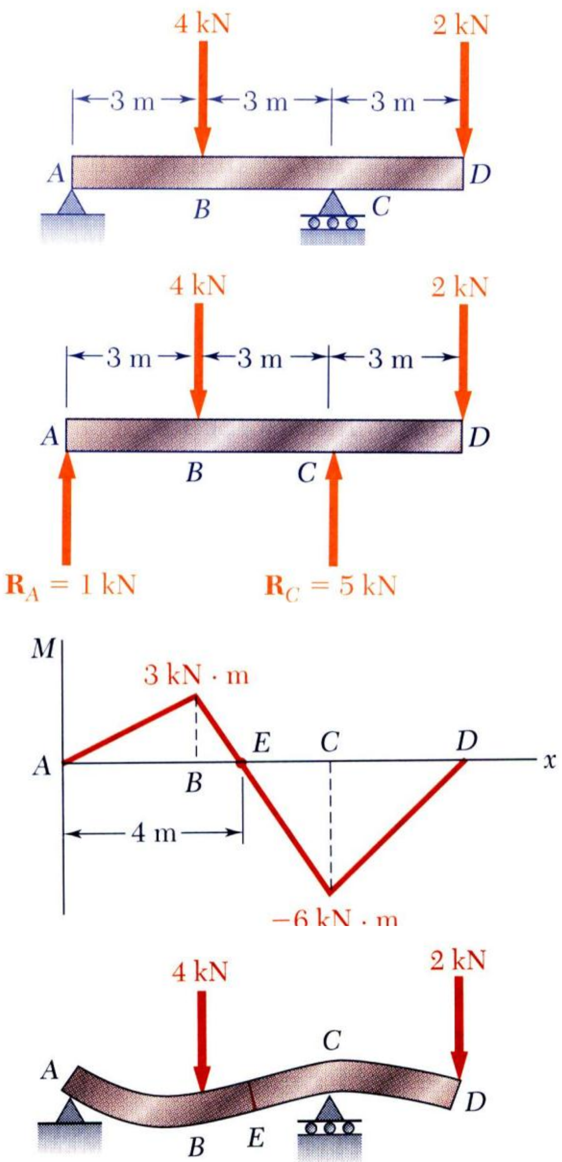
$$\frac{1}{\rho} = -\frac{Px}{EI}$$

- Curvature varies linearly with $x$

- At the free end $A$, $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$

- At the support $B$, $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

# Deformation of a Beam Under Transverse Loading

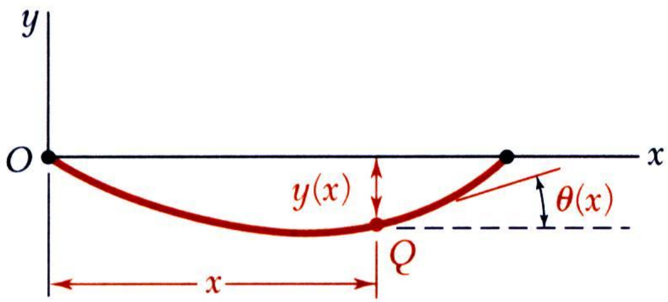



- Overhanging beam
- Reactions at *A* and *C*
- Bending moment diagram
- Curvature is zero at points where the bending moment is zero, i.e., at each end and at *E*.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Beam is concave upwards where the bending moment is positive and concave downwards where it is negative.
- Maximum curvature occurs where the moment magnitude is a maximum.
- An equation for the beam shape or *elastic curve* is required to determine maximum deflection and slope.

# Equation of the Elastic Curve




- From elementary calculus, simplified for beam parameters,

$$\frac{1}{\rho}=\frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}\approx\frac{d^2y}{dx^2}$$

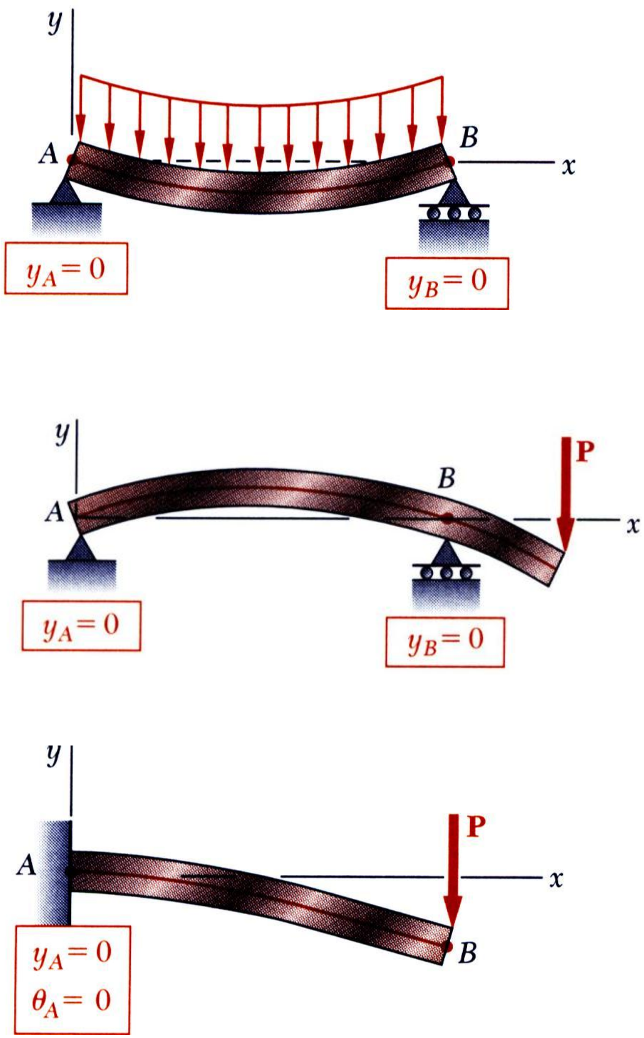
- Substituting and integrating,

$$EI\,\frac{1}{\rho}=EI\,\frac{d^2y}{dx^2}=M(x)$$

$$EI\,\theta\approx EI\,\frac{dy}{dx}=\int_0^x M(x)dx+C_1$$

$$EI\,y=\int_0^x dx\int_0^x M(x)dx+C_1x+C_2$$

# Equation of the Elastic Curve



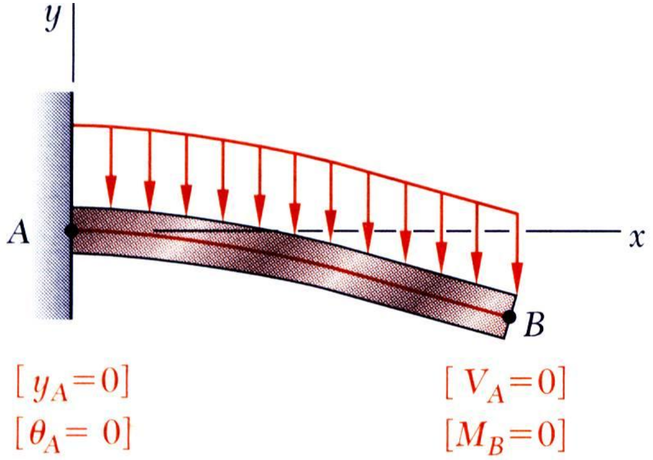

- Constants are determined from boundary conditions

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

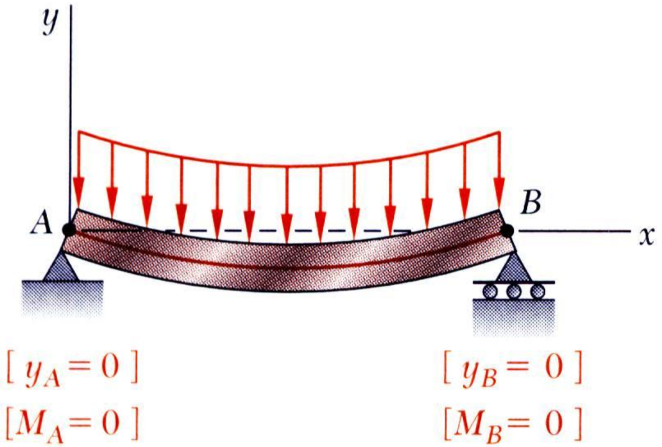
- Three cases for statically determinant beams,
  - Simply supported beam
    $y_A = 0, \quad y_B = 0$
  - Overhanging beam
    $y_A = 0, \quad y_B = 0$
  - Cantilever beam
    $y_A = 0, \quad \theta_A = 0$

- More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

# Direct Determination of the Elastic Curve From the Load Distribution




(a) Cantilever beam




(b) Simply supported beam

- For a beam subjected to a distributed load,

$$\frac{dM}{dx} = V(x) \qquad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x)$$

- Equation for beam displacement becomes

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = -w(x)$$

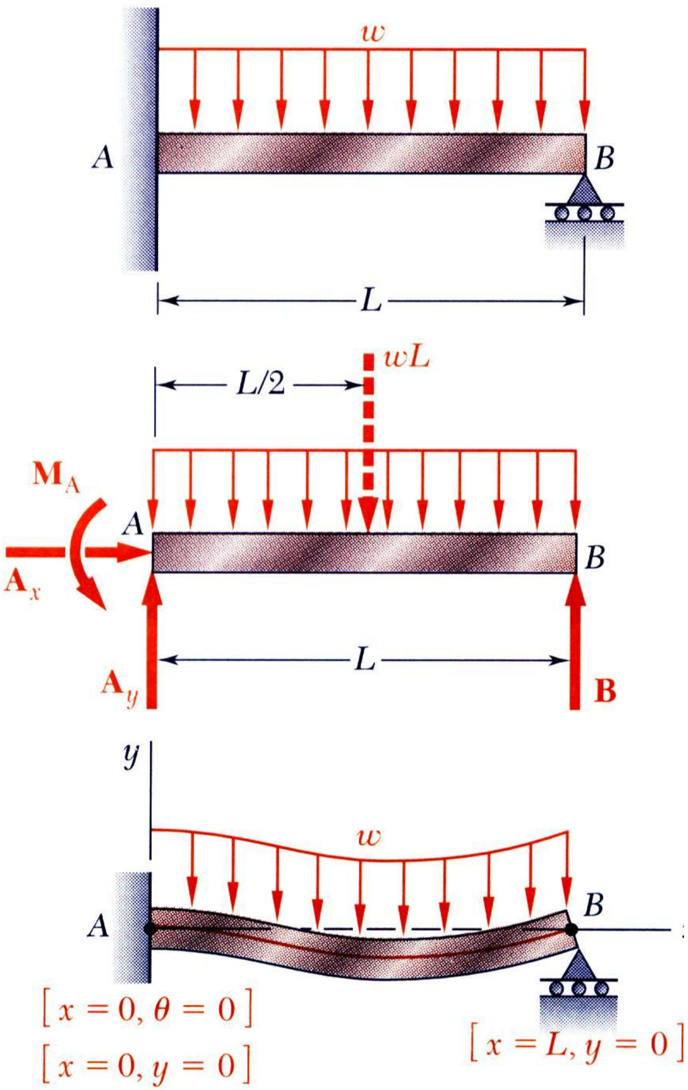
- Integrating four times yields

$$EI\,y(x) = -\int dx \int dx \int dx \int w(x)\,dx$$

$$+\tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from boundary conditions.

# Statically Indeterminate Beams




- Consider beam with fixed support at *A* and roller support at *B*.
- From free-body diagram, note that there are four unknown reaction components.
- Conditions for static equilibrium yield

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

The beam is statically indeterminate.

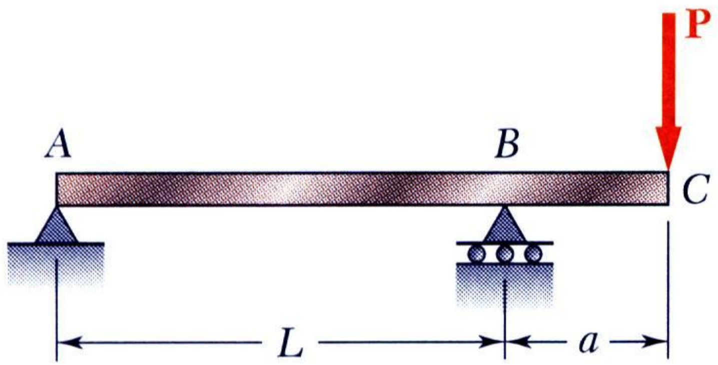
- Also have the beam deflection equation,

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

which introduces two unknowns but provides three additional equations from the boundary conditions:

$$\text{At } x = 0,\ \theta = 0\ y = 0 \qquad \text{At } x = L,\ y = 0$$

# Sample Problem 9.1




$W14 \times 68$ $\quad I = 723\,\text{in}^4$ $\quad E = 29 \times 10^6\,\text{psi}$

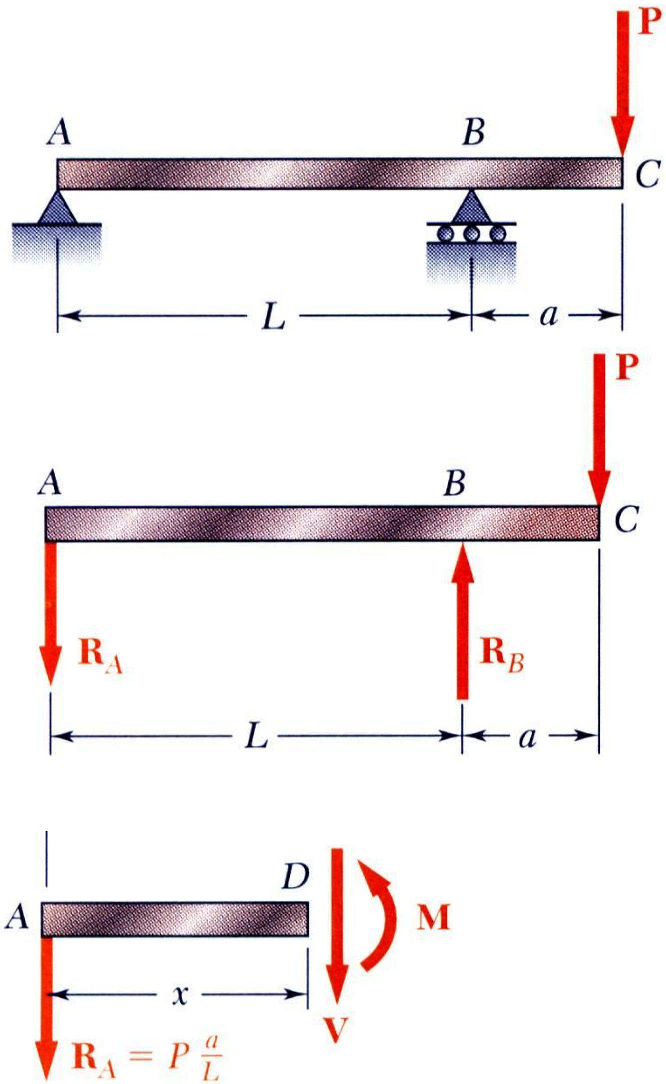
$P = 50\,\text{kips}$ $\quad L = 15\,\text{ft}$ $\quad a = 4\,\text{ft}$

For portion *AB* of the overhanging beam,
(*a*) derive the equation for the elastic curve,
(*b*) determine the maximum deflection,
(*c*) evaluate $y_{max}$.

SOLUTION:

- Develop an expression for M(x) and derive differential equation for elastic curve.
- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.
- Locate point of zero slope or point of maximum deflection.
- Evaluate corresponding maximum deflection.

# Sample Problem 9.1




SOLUTION:

- Develop an expression for M(x) and derive differential equation for elastic curve.

  - Reactions:

$$R_A = \frac{Pa}{L}\downarrow \quad R_B = P\left(1+\frac{a}{L}\right)\uparrow$$

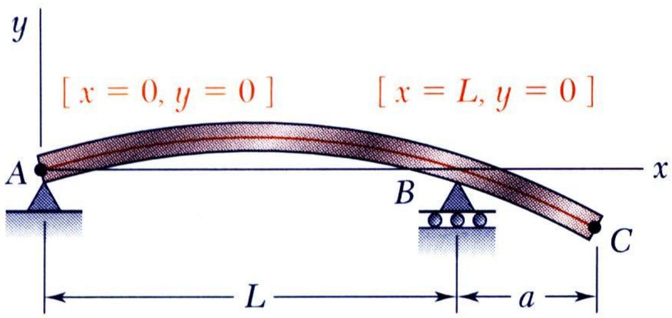
  - From the free-body diagram for section *AD*,

$$M = -P\frac{a}{L}x \quad (0 < x < L)$$

  - The differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

## Sample Problem 9.1




$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2$$

at $x = 0,\ y = 0: \quad C_2 = 0$

at $x = L,\ y = 0: \quad 0 = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \quad C_1 = \frac{1}{6}PaL$

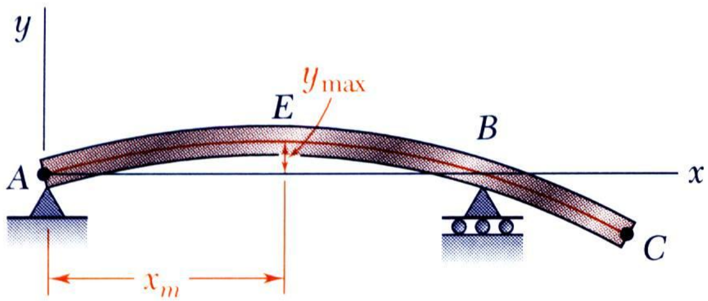
Substituting,

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right]$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx$$

$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

# Sample Problem 9.1




$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

- Locate point of zero slope or point of maximum deflection.

$$\frac{dy}{dx} = 0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

- Evaluate corresponding maximum deflection.

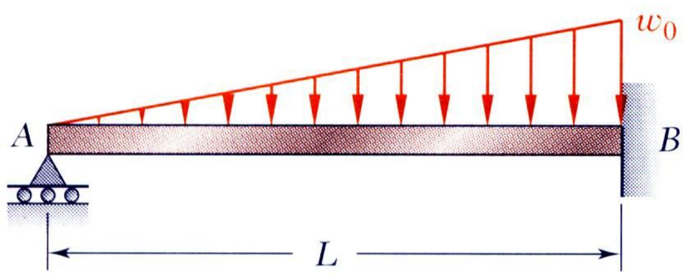
$$y_{max} = \frac{PaL^2}{6EI}\left[0.577 - (0.577)^3\right]$$

$$y_{max} = 0.0642\frac{PaL^2}{6EI}$$

$$y_{max} = 0.0642\frac{(50\,\text{kips})(48\,\text{in})(180\,\text{in})^2}{6(29\times10^6\,\text{psi})(723\,\text{in}^4)}$$

$$y_{max} = 0.238\,\text{in}$$

# Sample Problem 9.3

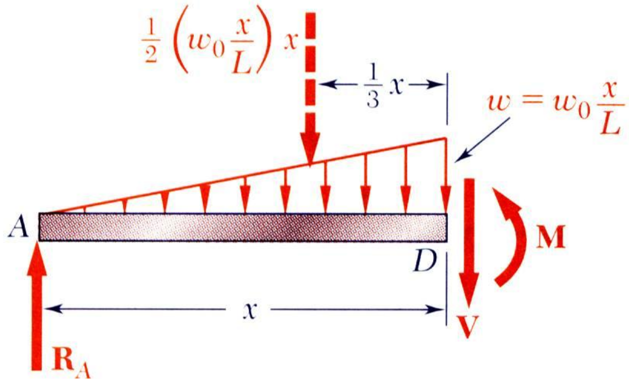



For the uniform beam, determine the reaction at *A*, derive the equation for the elastic curve, and determine the slope at *A*. (Note that the beam is statically indeterminate to the first degree)

SOLUTION:

- Develop the differential equation for the elastic curve (will be functionally dependent on the reaction at *A*).
- Integrate twice and apply boundary conditions to solve for reaction at *A* and to obtain the elastic curve.
- Evaluate the slope at *A*.

# Sample Problem 9.3




- Consider moment acting at section *D*,

$$\sum M_D = 0$$

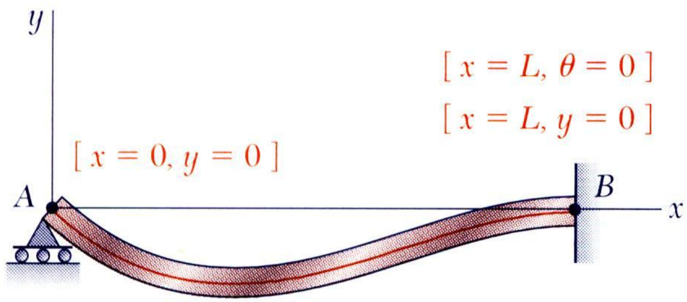
$$R_A x - \frac{1}{2}\left(\frac{w_0 x^2}{L}\right)\frac{x}{3} - M = 0$$

$$M = R_A x - \frac{w_0 x^3}{6L}$$

- The differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

# Sample Problem 9.3




$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

- Integrate twice

$$EI\frac{dy}{dx} = EI\theta = \frac{1}{2}R_A x^2 - \frac{w_0 x^4}{24L} + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2$$

- Apply boundary conditions:

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ \theta = 0:\quad \frac{1}{2}R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0$

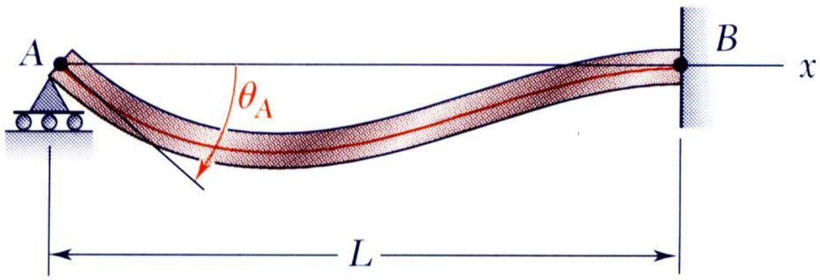
at $x = L,\ y = 0:\quad \frac{1}{6}R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + C_2 = 0$

- Solve for reaction at *A*

$$\frac{1}{3}R_A L^3 - \frac{1}{30}w_0 L^4 = 0$$

$$R_A = \frac{1}{10}w_0 L \uparrow$$

# Sample Problem 9.3




- Substitute for $C_1$, $C_2$, and $R_A$ in the elastic curve equation,

$$EI\,y = \frac{1}{6}\left(\frac{1}{10}w_0 L\right)x^3 - \frac{w_0 x^5}{120L} - \left(\frac{1}{120}w_0 L^3\right)x$$

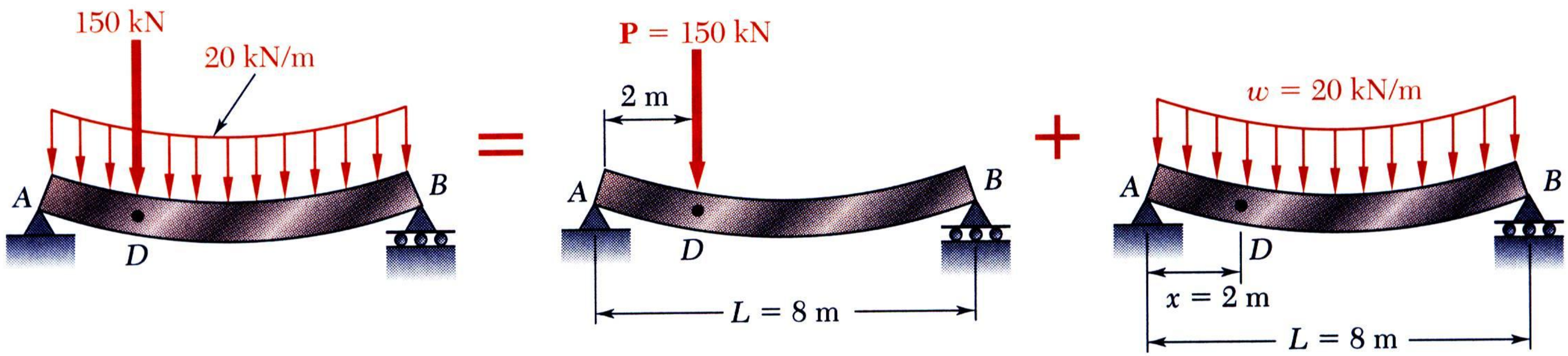
$$y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)$$

- Differentiate once to find the slope,

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EIL}\left(-5x^4 + 6L^2x^2 - L^4\right)$$

at $x = 0$, $$\theta_A = \frac{w_0 L^3}{120EI}$$

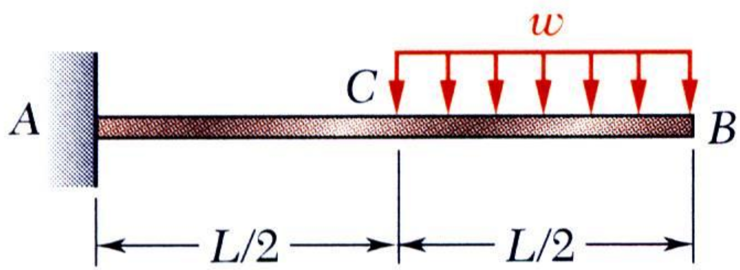
# Method of Superposition




Principle of Superposition:

- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

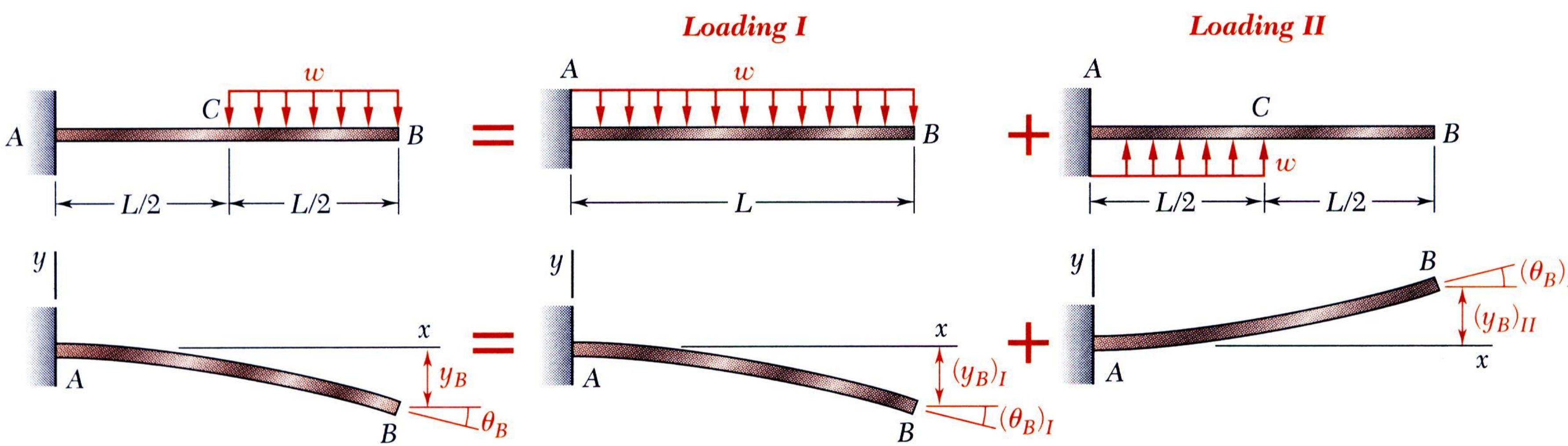
# Sample Problem 9.7



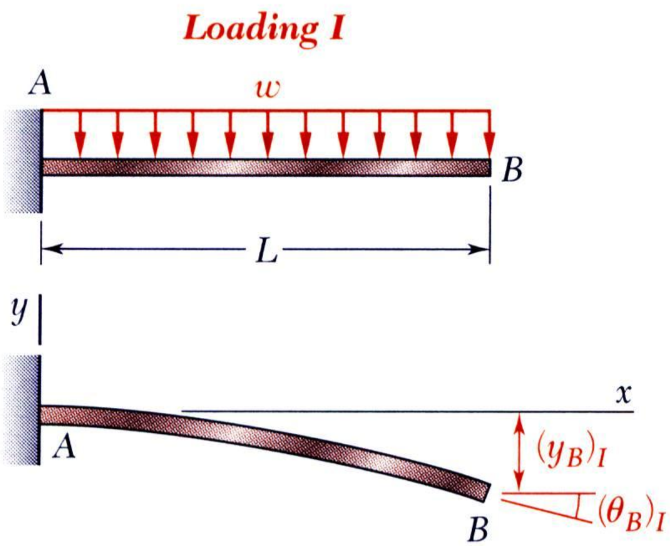

For the beam and loading shown, determine the slope and deflection at point *B*.

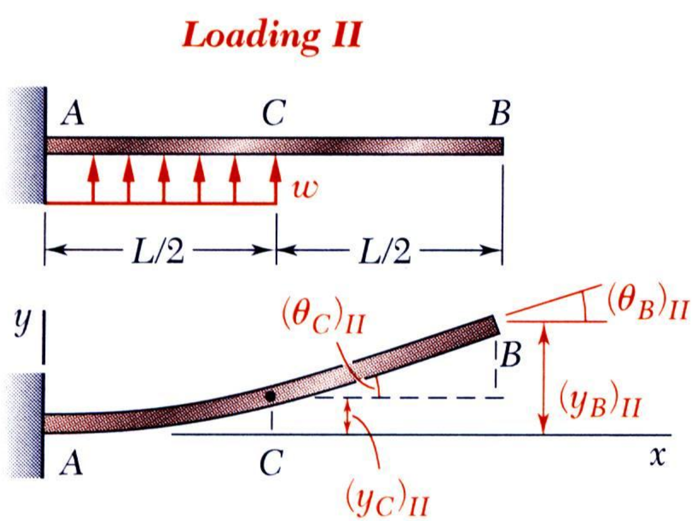
SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.



# Sample Problem 9.7




*Loading I*

$$(\theta_B)_I = -\frac{wL^3}{6EI} \qquad (y_B)_I = -\frac{wL^4}{8EI}$$

*Loading II*

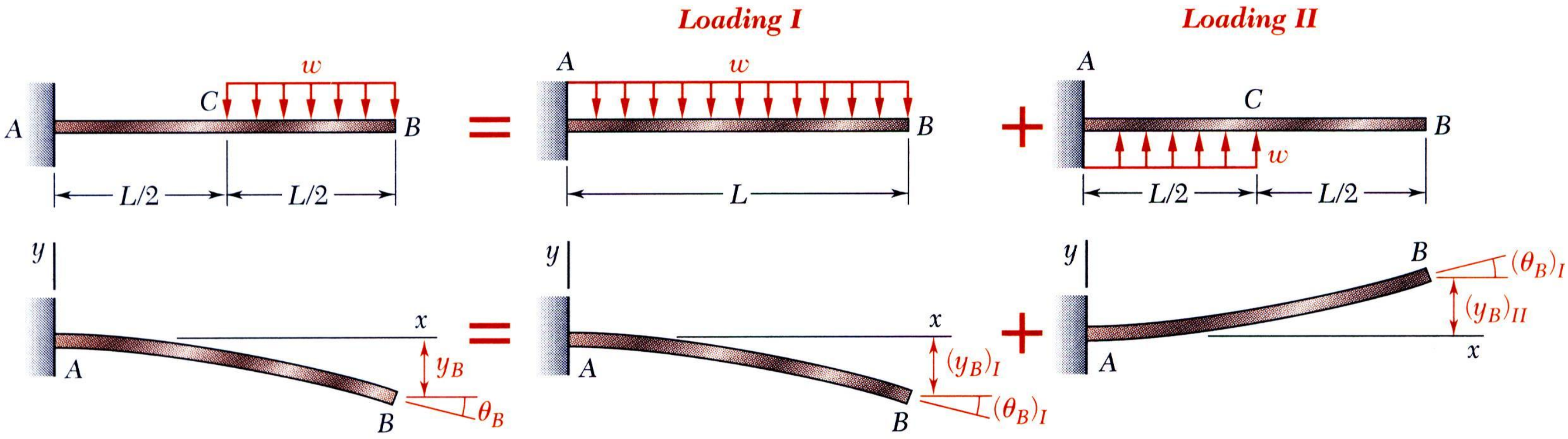
$$(\theta_C)_{II} = \frac{wL^3}{48EI} \qquad (y_C)_{II} = \frac{wL^4}{128EI}$$

In beam segment CB, the bending moment is zero and the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = \frac{wL^3}{48EI}$$

$$(y_B)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$

# Sample Problem 9.7




Combine the two solutions,

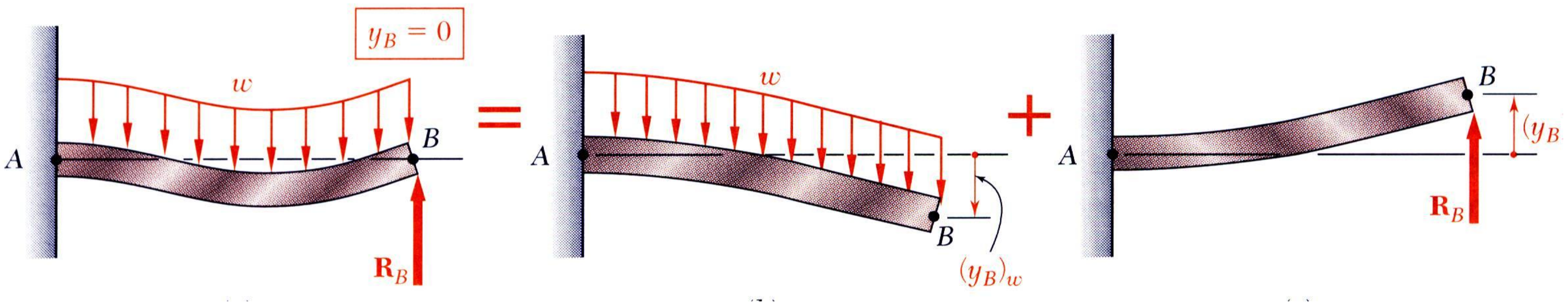
$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI}$$

$$\theta_B = \frac{7wL^3}{48EI}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI}$$

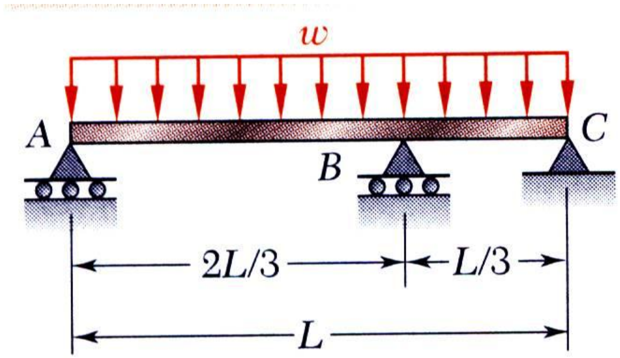
$$y_B = \frac{41wL^4}{384EI}$$

# Application of Superposition to Statically Indeterminate Beams




- Method of superposition may be applied to determine the reactions at the supports of statically indeterminate beams.
- Designate one of the reactions as redundant and eliminate or modify the support.
- Determine the beam deformation without the redundant support.
- Treat the redundant reaction as an unknown load which, together with the other loads, must produce deformations compatible with the original supports.

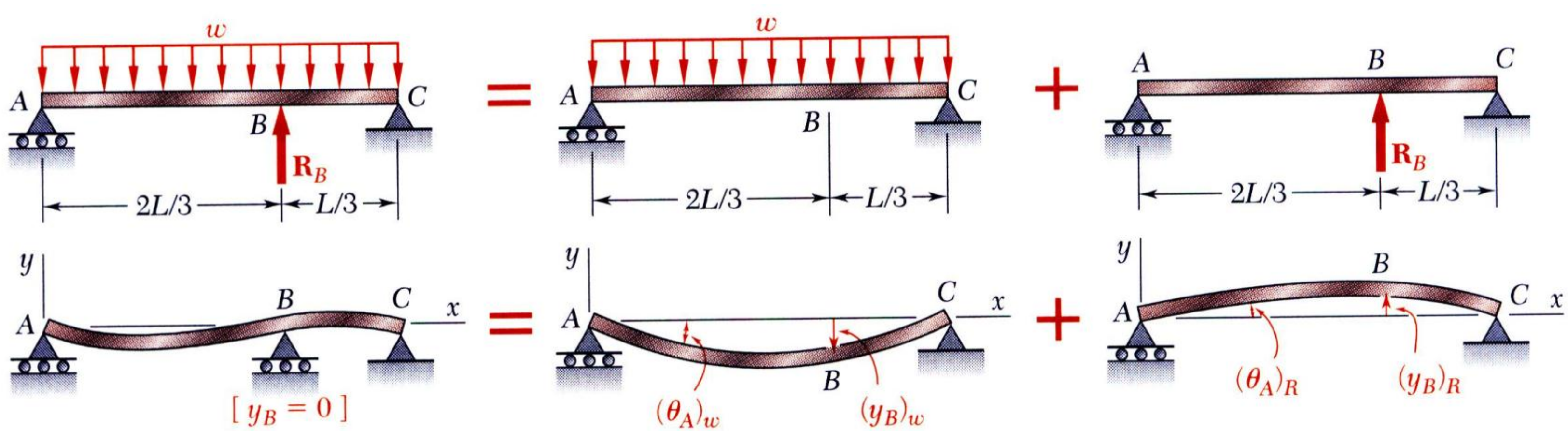
# Sample Problem 9.8



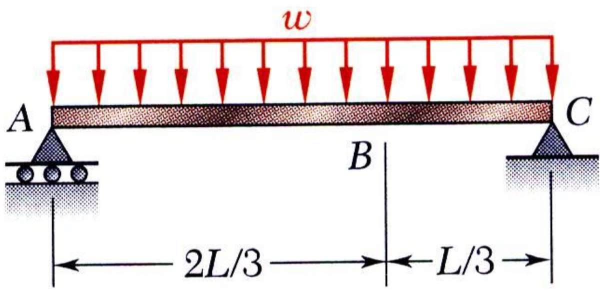

For the uniform beam and loading shown, determine the reaction at each support and the slope at end *A*.

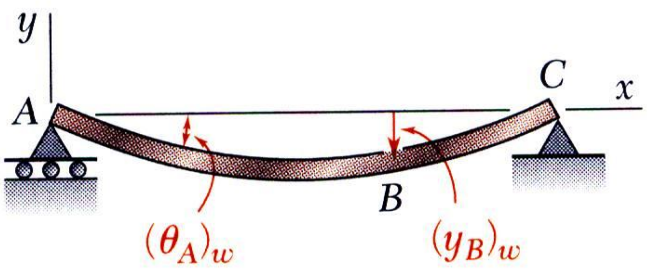
SOLUTION:

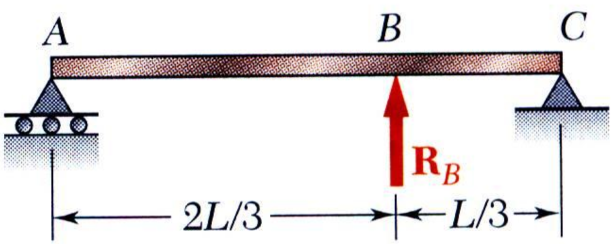
- Release the “redundant” support at B, and find deformation.
- Apply reaction at *B* as an unknown load to force zero displacement at *B*.

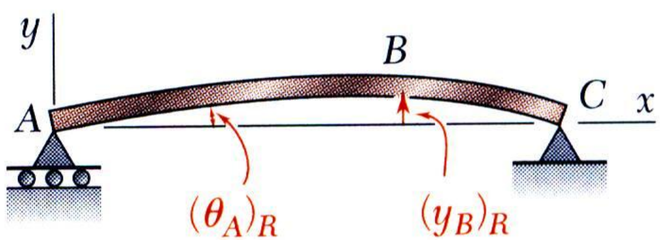


# Sample Problem 9.8




- Distributed Loading:

$$(y_B)_w = -\frac{w}{24EI}\left[\left(\frac{2}{3}L\right)^4 - 2L\left(\frac{2}{3}L\right)^3 + L^3\left(\frac{2}{3}L\right)\right]$$

$$= -0.01132\frac{wL^4}{EI}$$

- Redundant Reaction Loading:

$$(y_B)_R = \frac{R_B}{3EIL}\left(\frac{2}{3}L\right)^2\left(\frac{L}{3}\right)^2 = 0.01646\frac{R_B L^3}{EI}$$

- For compatibility with original supports, $y_B = 0$

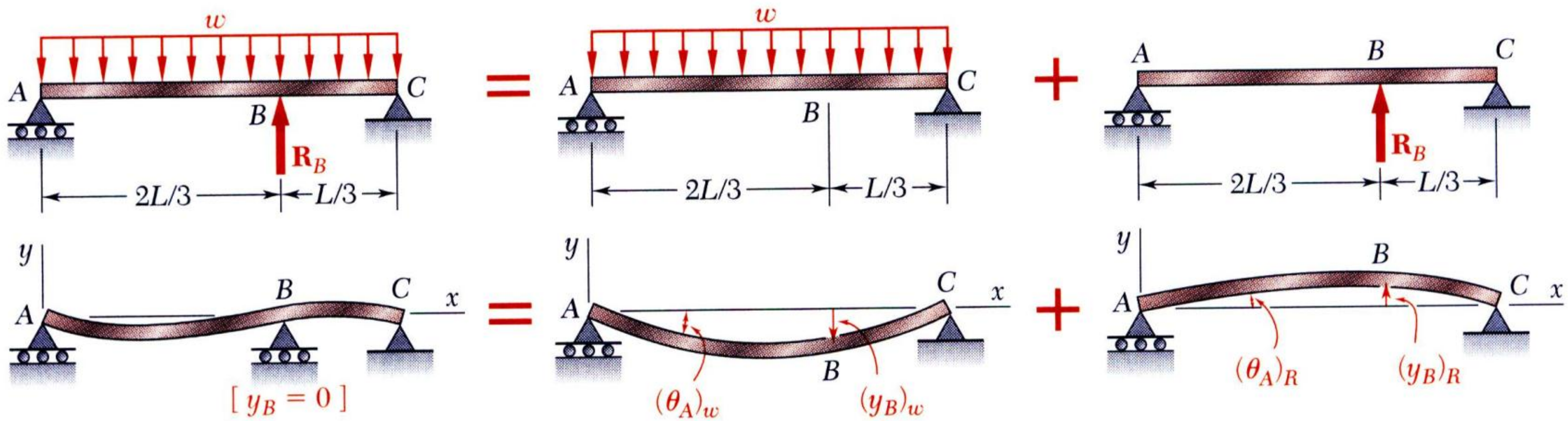
$$0 = (y_B)_w + (y_B)_R = -0.01132\frac{wL^4}{EI} + 0.01646\frac{R_B L^3}{EI}$$

$$R_B = 0.688wL \uparrow$$

- From statics,

$$R_A = 0.271wL \uparrow \qquad R_C = 0.0413wL \uparrow$$

# Sample Problem 9.8




Slope at end *A*,

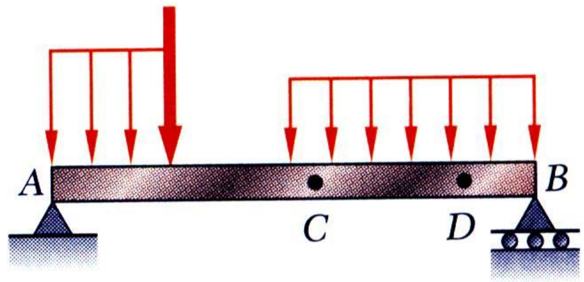
$$(\theta_A)_w = -\frac{wL^3}{24EI} = -0.04167\frac{wL^3}{EI}$$

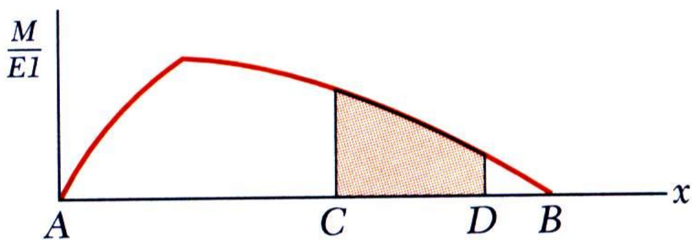
$$(\theta_A)_R = \frac{0.0688wL}{6EIL}\left(\frac{L}{3}\right)\left[L^2 - \left(\frac{L}{3}\right)^2\right] = 0.03398\frac{wL^3}{EI}$$

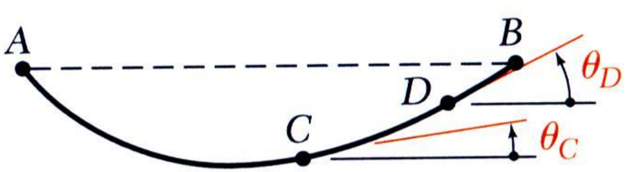
$$\theta_A = (\theta_A)_w + (\theta_A)_R = -0.04167\frac{wL^3}{EI} + 0.03398\frac{wL^3}{EI}$$

$$\theta_A = -0.00769\frac{wL^3}{EI}$$

# Moment-Area Theorems



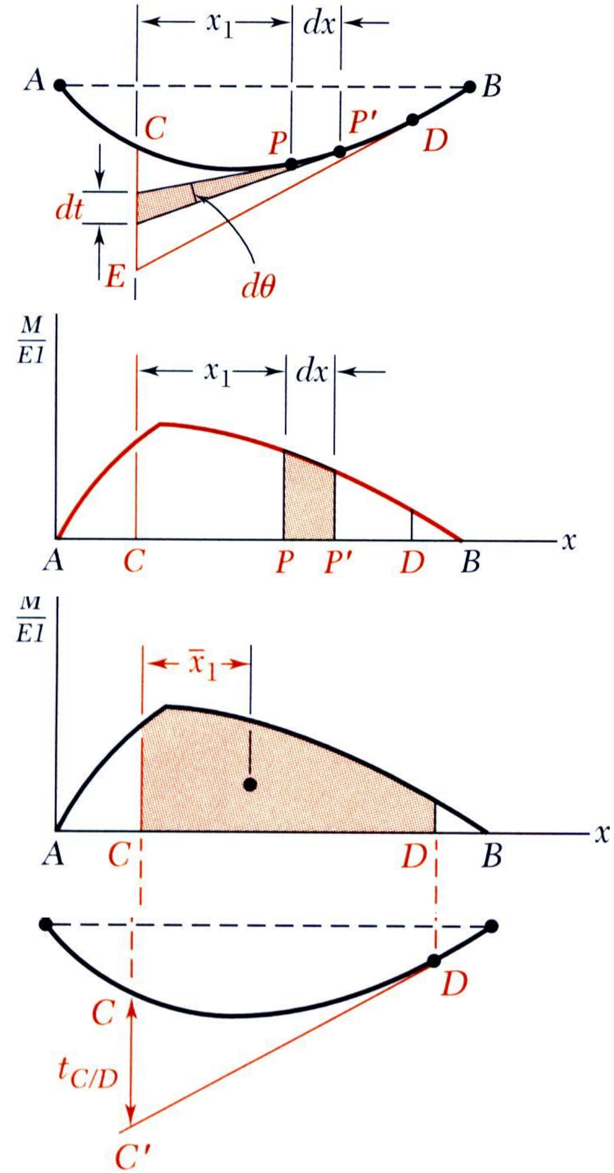





- Geometric properties of the elastic curve can be used to determine deflection and slope.
- Consider a beam subjected to arbitrary loading,
- *First Moment-Area Theorem*:

  area under *(M/EI)* diagram between *C* and *D*.

# Moment-Area Theorems

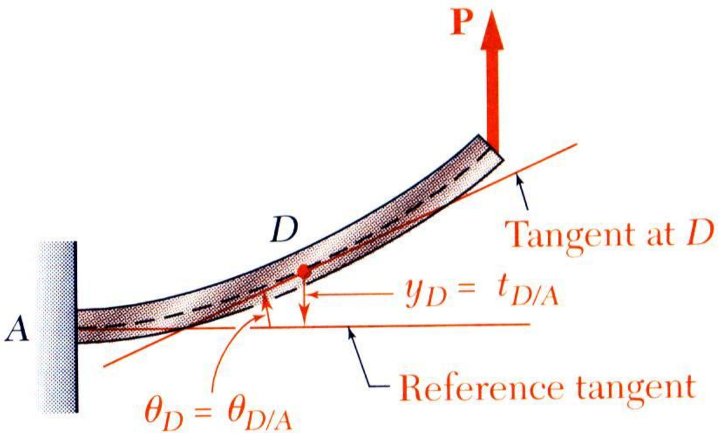



- Tangents to the elastic curve at *P* and *P’* intercept a segment of length *dt* on the vertical through *C*.

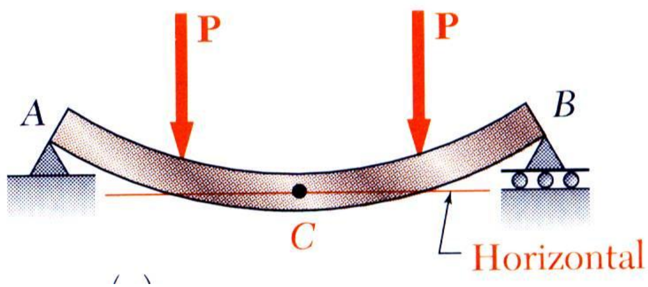
= tangential deviation of *C* with respect to *D*

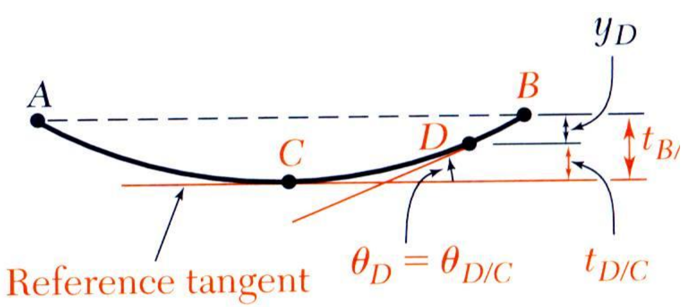
- *Second Moment-Area Theorem*: The tangential deviation of *C* with respect to *D* is equal to the first moment with respect to a vertical axis through *C* of the area under the (*M/EI*) diagram between *C* and *D*.

# Application to Cantilever Beams and Beams With Symmetric Loadings



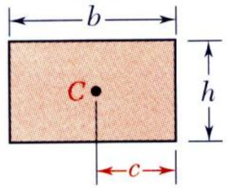
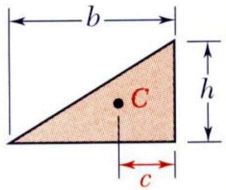
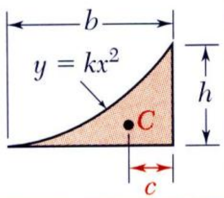
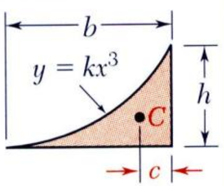
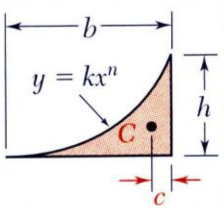

- Cantilever beam - Select tangent at *A* as the reference.




- Simply supported, symmetrically loaded beam - select tangent at *C* as the reference.

# Bending Moment Diagrams by Parts

| Shape | | Area | $c$ |
|---|---|---|---|
| Rectangle |  | $bh$ | $\frac{b}{2}$ |
| Triangle |  | $\frac{bh}{2}$ | $\frac{b}{3}$ |
| Parabolic spandrel |  | $\frac{bh}{3}$ | $\frac{b}{4}$ |
| Cubic spandrel |  | $\frac{bh}{4}$ | $\frac{b}{5}$ |
| General spandrel |  | $\frac{bh}{n+1}$ | $\frac{b}{n+2}$ |

- Determination of the change of slope and the tangential deviation is simplified if the effect of each load is evaluated separately.

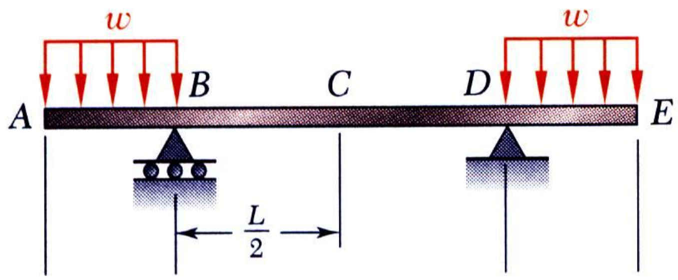
- Construct a separate (*M/EI*) diagram for each load.

  - The change of slope, $\theta_{D/C}$, is obtained by adding the areas under the diagrams.

  - The tangential deviation, $t_{D/C}$ is obtained by adding the first moments of the areas with respect to a vertical axis through D.

- Bending moment diagram constructed from individual loads is said to be *drawn by parts*.

# Sample Problem 9.11

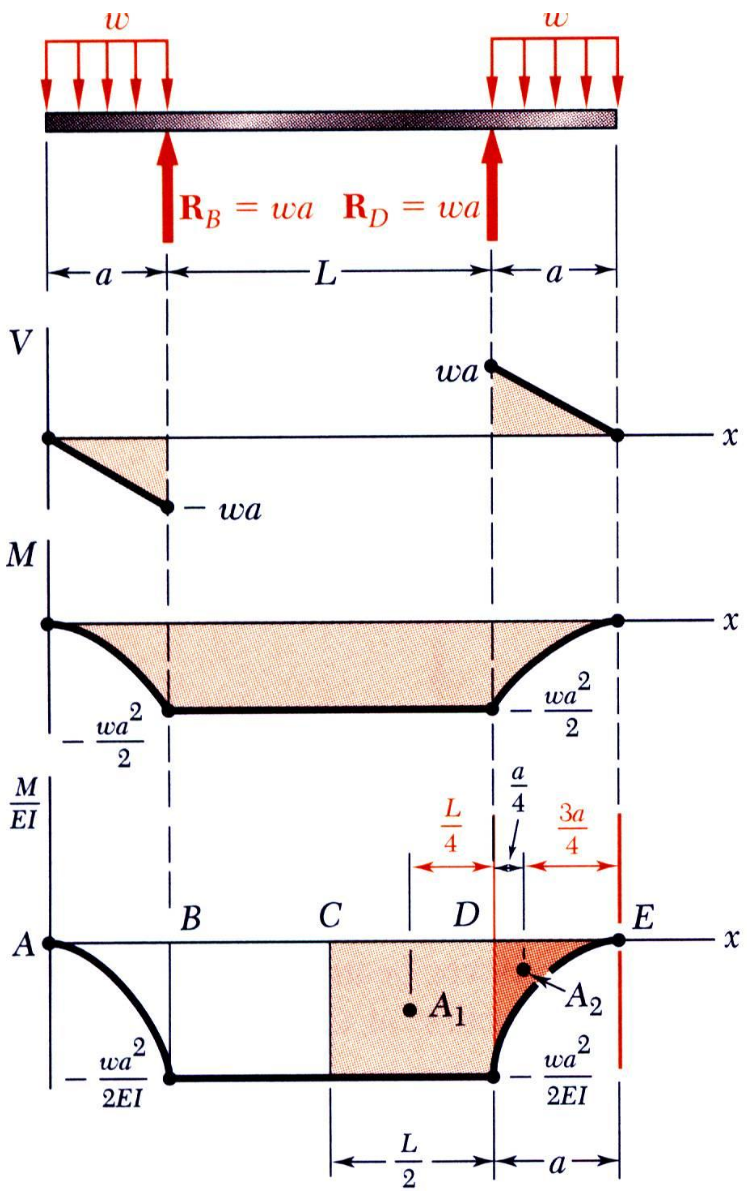



For the prismatic beam shown, determine the slope and deflection at *E*.

SOLUTION:

- Determine the reactions at supports.
- Construct shear, bending moment and (*M/EI*) diagrams.
- Taking the tangent at *C* as the reference, evaluate the slope and tangential deviations at *E*.

# Sample Problem 9.11




SOLUTION:

- Determine the reactions at supports.

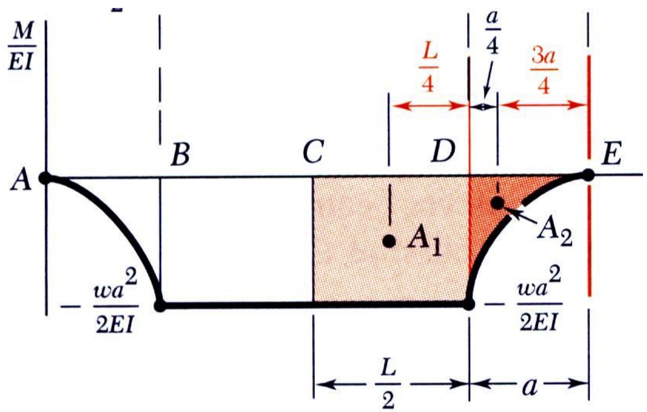
$$R_B = R_D = wa$$

- Construct shear, bending moment and ($M/EI$) diagrams.

$$A_1 = -\frac{wa^2}{2EI}\left(\frac{L}{2}\right) = -\frac{wa^2 L}{4EI}$$

$$A_2 = -\frac{1}{3}\left(\frac{wa^2}{2EI}\right)(a) = -\frac{wa^3}{6EI}$$

# Sample Problem 9.11

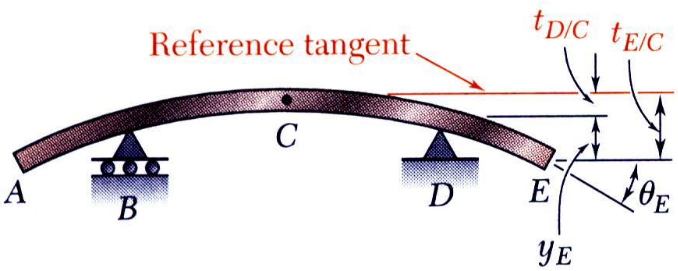



- Slope at E:

$$\theta_E = \theta_C + \theta_{E/C} = \theta_{E/C}$$

$$= A_1 + A_2 = -\frac{wa^2 L}{4EI} - \frac{wa^3}{6EI}$$

$$\theta_E = -\frac{wa^2}{12EI}(3L + 2a)$$




- Deflection at E:

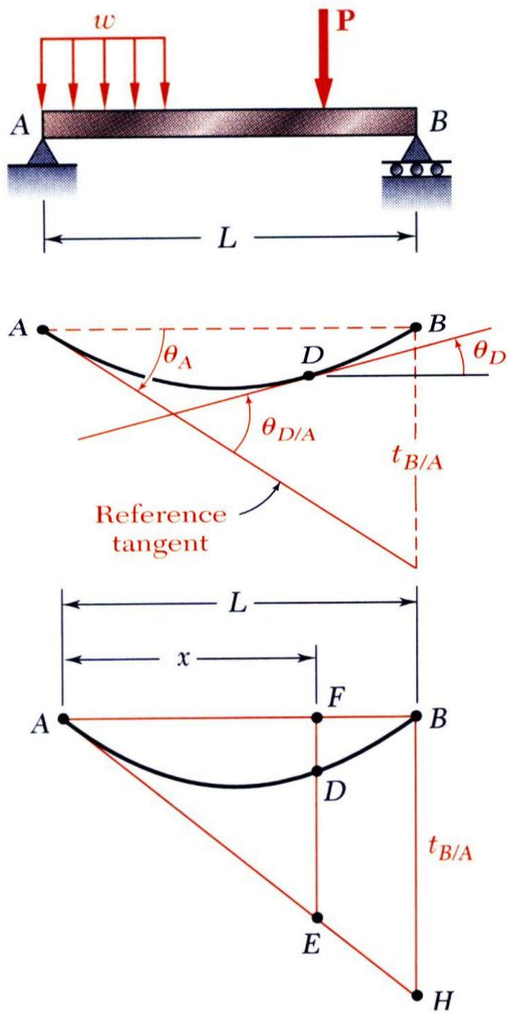
$$y_E = t_{E/C} - t_{D/C}$$

$$= \left[A_1\left(a + \frac{L}{4}\right) + A_2\left(\frac{3a}{4}\right)\right] - \left[A_1\left(\frac{L}{4}\right)\right]$$

$$= \left[-\frac{wa^3 L}{4EI} - \frac{wa^2 L^2}{16EI} - \frac{wa^4}{8EI}\right] - \left[-\frac{wa^2 L^2}{16EI}\right]$$

$$y_E = -\frac{wa^3}{8EI}(2L + a)$$

# Application of Moment-Area Theorems to Beams With Unsymmetric Loadings

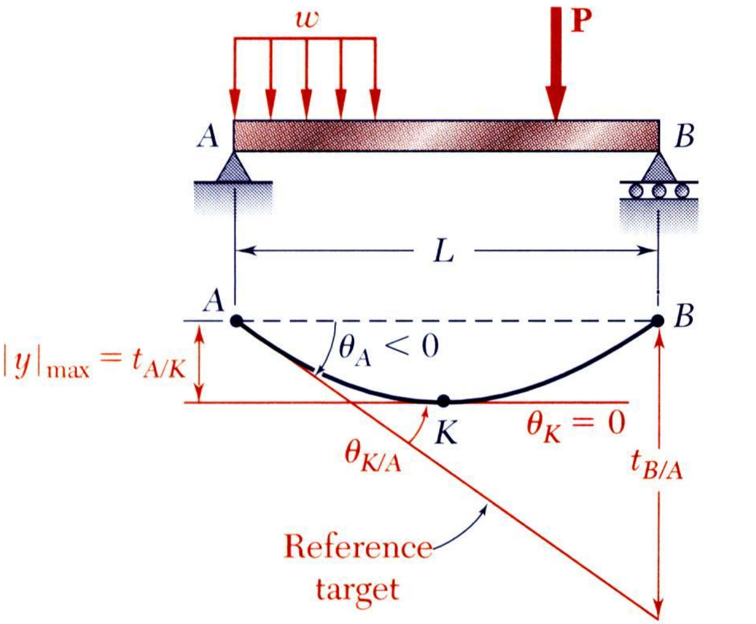


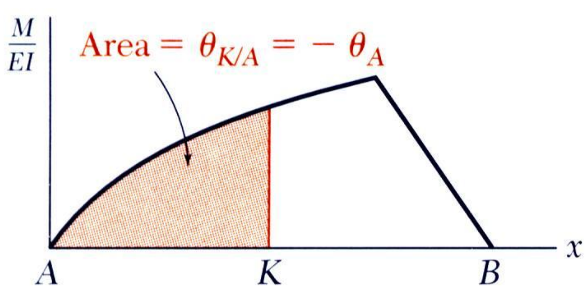

- Define reference tangent at support *A*. Evaluate $\theta_A$ by determining the tangential deviation at *B* with respect to *A*.
- The slope at other points is found with respect to reference tangent.

$$\theta_D = \theta_A + \theta_{D/A}$$

- The deflection at *D* is found from the tangential deviation at *D*.

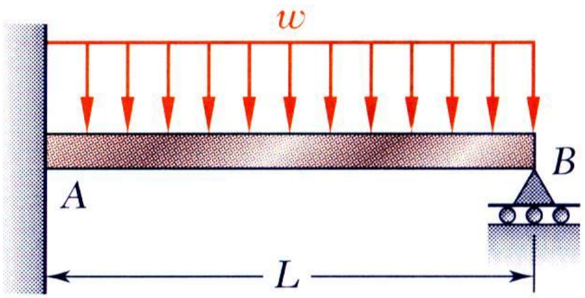
# Maximum Deflection




- Maximum deflection occurs at point $K$ where the tangent is horizontal.
- Point $K$ may be determined by measuring an area under the ($M/EI$) diagram equal to $-\theta_A$ .
- Obtain $y_{max}$ by computing the first moment with respect to the vertical axis through $A$ of the area between $A$ and $K$.

# Use of Moment-Area Theorems With Statically Indeterminate Beams




- Reactions at supports of statically indeterminate beams are found by designating a redundant constraint and treating it as an unknown load which satisfies a displacement compatibility requirement.
- The (*M/EI*) diagram is drawn by parts. The resulting tangential deviations are superposed and related by the compatibility requirement.
- With reactions determined, the slope and deflection are found from the moment-area method.