Chapter 3

Frequency Analysis of Discrete-time Signals

Contents

- Introduction
- Continuous-time Frequency Analysis

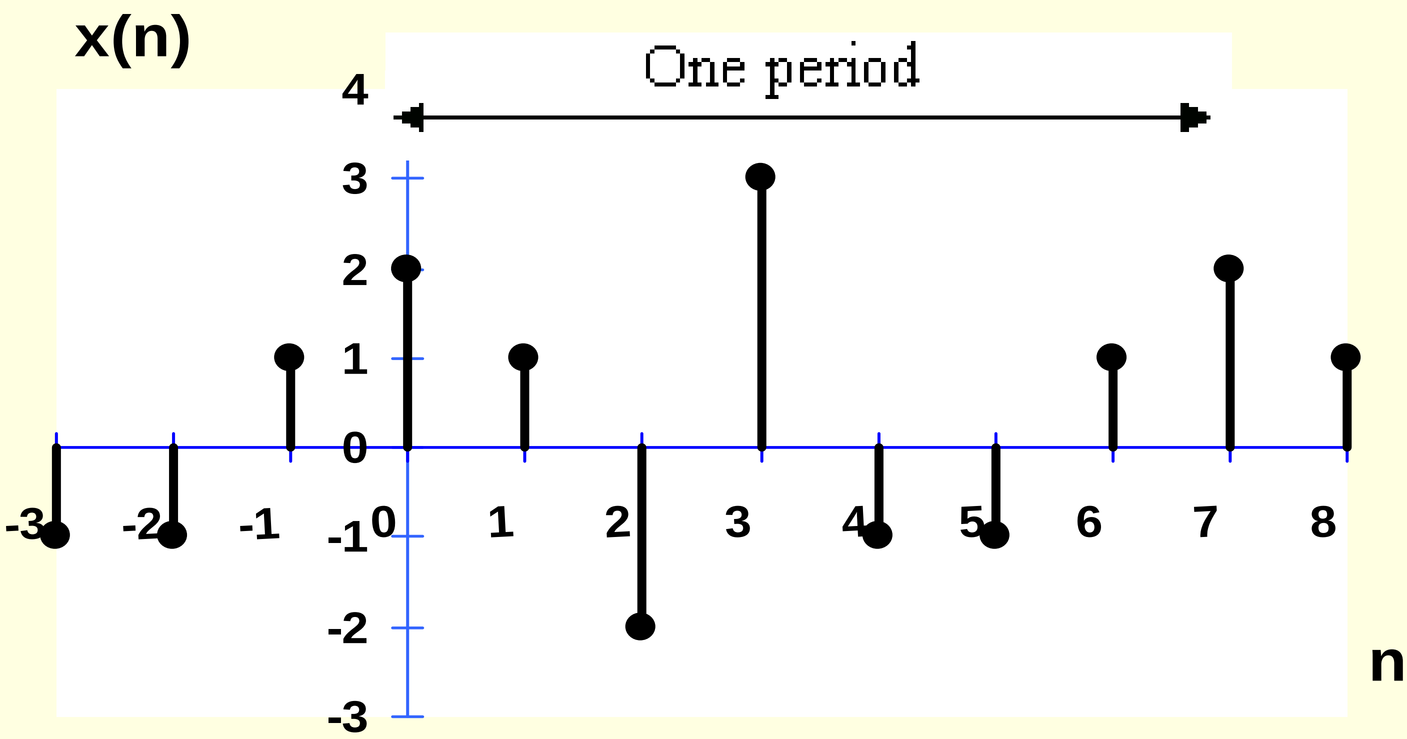
$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{\Omega \pi k F t} \rightarrow Synthesis$$

$$C_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi k F t} dt \rightarrow Analysis$$

Discrete-time Fourier Series

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \rightarrow Analysis$$

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} \rightarrow Synthesis$$


x(n)
One period
4
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3
4
5
6
7
8
n

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp(-j2\pi kn/N)$$

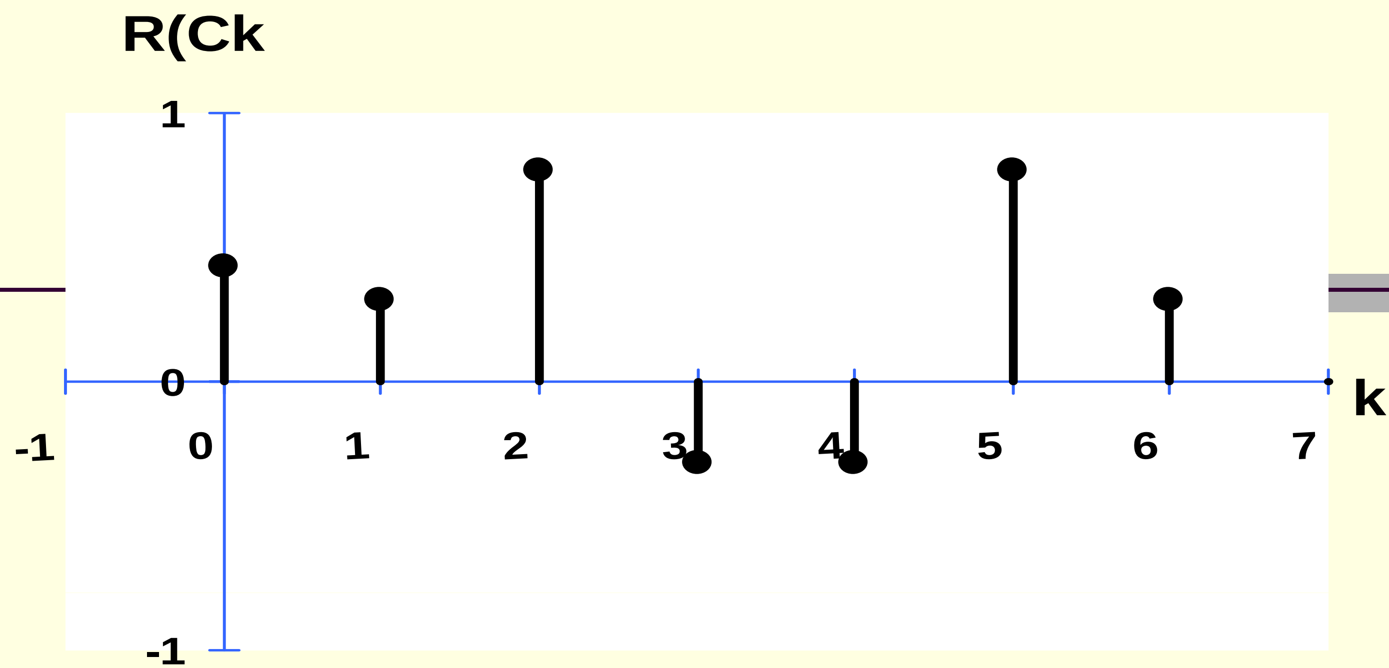
$$C_k = \frac{1}{7}\sum_{n=0}^{6} x[n]\exp(-j2\pi kn/7)$$

k	$\cos(2\pi kn/7)$							$-j\sin(2\pi kn/7)$						
	n							n						
	0	1	2	3	4	5	6	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1	0	0	0	0	0	0	0
	2	1	-2	3	-1	-1	1	0	0	0	0	0	0	0
				3/7=0.428										
1	1	.623	-.222	-.9	-.9	-.222	.623	0	-.974	.433	.781	.781	.433	-.974
	1	.623	-.444	-2.7	-.9	-.222	.623	0	-.974	-.867	2.34	-.781	.433	-.974
				2.11/7=.301							-.756/7=-.108			

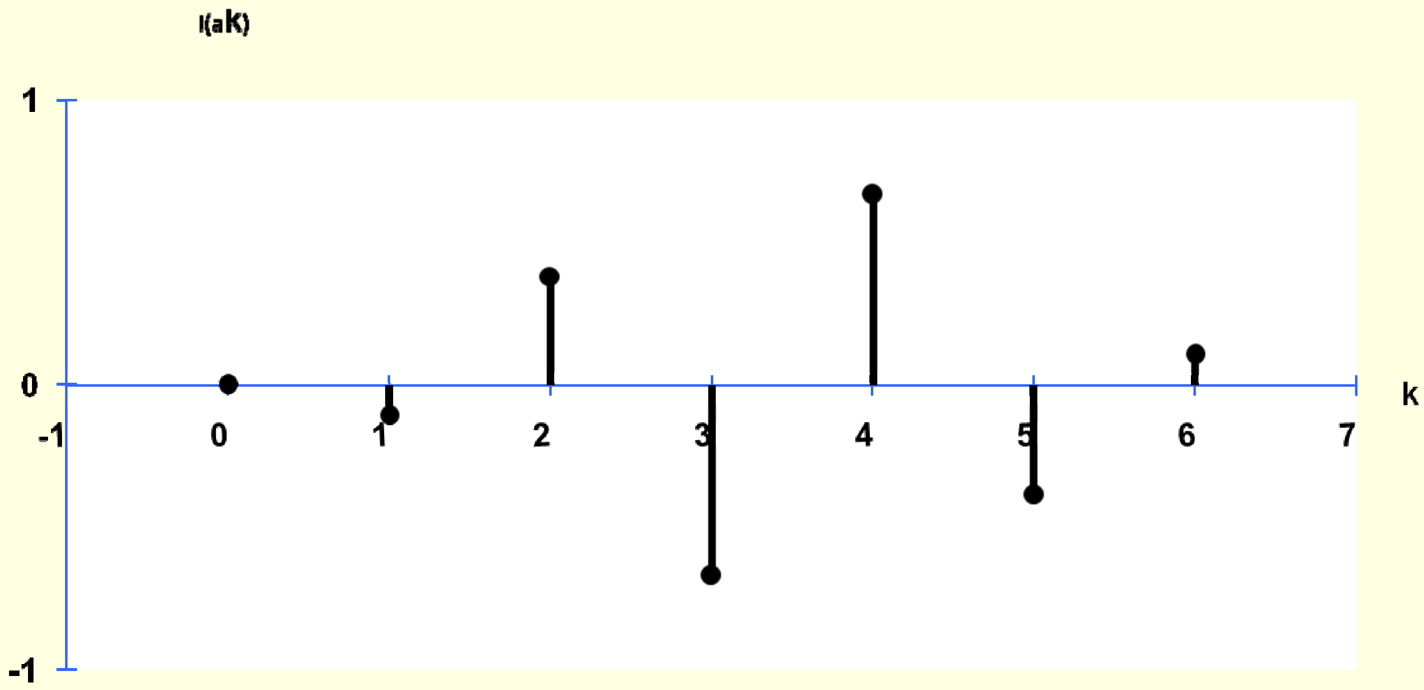
$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\exp(-j2\pi kn/N)$$

$$x(n) = \sum_{n=0}^{N-1} C_k \exp(j2\pi kn/N)$$

k	Spectral Coefficients C	
	Real part	Imaginary Part
0	0.4285715	0
1	0.3018007	-0.1086581
2	0.7864066	0.3847772
3	-0.3024935	-0.6687913
4	-0.3024928	0.6687927
5	0.7864058	-0.3847782
6	0.3018006	0.1086581



R(Ck
1
0
-1
-1
0
1
2
3
4
5
6
7
k



I(aK)
1
0
-1
-1
0
1
2
3
4
5
6
7
k

$$x[n]=\sum_{k=0}^{N-1} C_k\{\cos 2\pi kn/N\}+j\sum_{k=0}^{N-1} a_k\{\sin 2\pi kn/N)$$

$$x[n]=\sum_{k=0}^{N-1} C_k\{\cos 2\pi kn/7\}$$

$$x[0]=\sum_{k=0}^{N-1} 0.4285\{\cos 2\pi kn/7\}$$

$$x[0]=1.999\approx 2$$

Parseval’s Theorem:

- Periodic signals having infinite energy can be solved for PSD.
- Aperiodic signals having infinite power can be evaluated for ESD.

$$\frac{1}{N}\sum_{n=0}^{N-1}\{x(n)\}^2=\sum_{k=0}^{N-1}\left|C_k\right|^2$$

$$\frac{1}{N}\sum_{n=0}^{N-1}\{x[n]\}^2$$

$$=\frac{1}{7}(9+2.914+0+2.914+9+0.086+4+0.086)$$

$$=\frac{28}{7}\Longrightarrow 4$$

Power Spectral Density (PSD)

$$P_x = \frac{1}{T}\int \left| x(t) \right|^2 dt = \sum_{k=-\infty}^{\infty} \left| C_k \right|^2$$

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} \left| x(n) \right|^2$$

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} \left| x(n) \right|^2 = \sum_{k=0}^{N-1} \left| C_k \right|^2$$

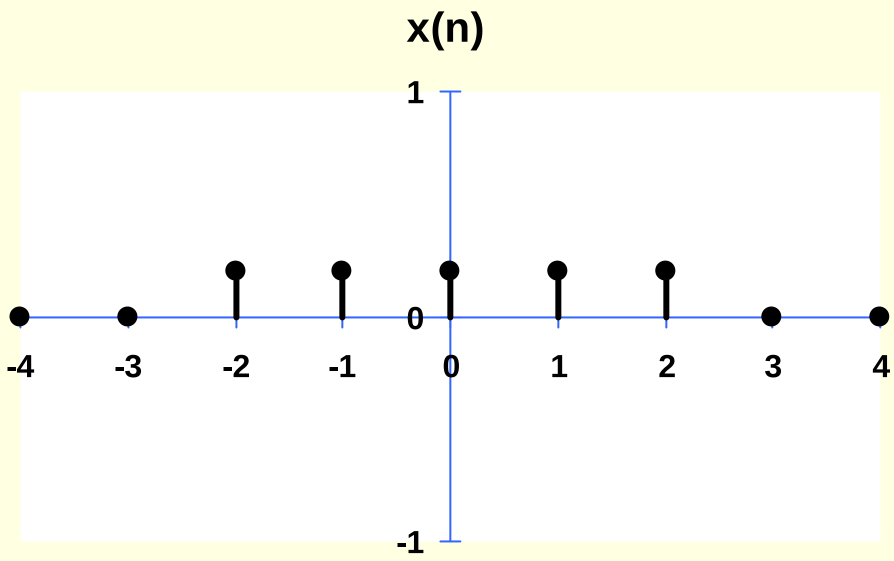
Properties of Discrete-Time Fourier Series

Discrete-time Fourier Transform

$$X(\omega)=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}\rightarrow Analysis$$

$$x(n)=\frac{1}{2\pi}\int_{2\pi}X(\omega)e^{j\omega n}d\omega\rightarrow Synthesis$$

Example:



$$X(\omega)=\{0.2\delta(n+2)+0.2\delta(n+1)+0.2\delta(n)+0.2\delta(n-1)$$
$$+0.2\delta(n-2)\}e^{-j\omega n}$$

$$X(\omega)=0.2\delta(n+2)e^{-j\omega n}+0.2\delta(n+1)e^{-j\omega n}+0.2\delta(n)e^{-j\omega n}$$
$$+0.2\delta(n-1)e^{-j\omega n}+0.2\delta(n-2)e^{-j\omega n}$$

$$X(\omega)=0.2\{e^{2j\omega}+e^{j\omega}+1+e^{-j\omega}+e^{-2j\omega}\}$$

$$X(\omega)=0.2\left\{1+2\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)+2\left(\frac{e^{2j\omega}+e^{-2j\omega}}{2}\right)\right\}$$

$$X(\omega)=0.2\{1+2\cos\omega+2\cos 2\omega\}$$

- Discrete-time periodic signal after the discrete-time Fourier transform application, the spectrum is also discrete and periodic.
- Fourier transform of aperiodic discrete-time signal is continuous and
- periodic.
- Fourier transform of periodic discrete-time signal is continuous and periodic.

Properties of DTFT

$$if \quad x(n) \leftrightarrow X(\omega)$$

1. Linearity:

$$a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_2 X_2(\omega) + a_2 X_2(\omega)$$

2. Time Shifting:

$$x(n-k) \leftrightarrow e^{-j\omega k} X(\omega)$$

3. Time Reversal:

$$x(-n) \leftrightarrow X(-\omega)$$

4. Convolution:

$$x_1(n) * x_2(n) = X_1(\omega) X_2(\omega)$$

5. Frequency Shifting:

$$e^{j\omega_0 n} x(n) \leftrightarrow X(\omega - \omega_0)$$

6. Differentiation in frequency domain:

$$nx(n) \leftrightarrow j \frac{dX(\omega)}{d\omega}$$

Energy Spectral Density

$$S_{xx}(\Omega) = |X(\Omega)|^2$$

Since

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$S_{xx} = |X(\omega)|^2$$

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$$

$$X^*(\omega) = X(-\omega)$$

$$|X(-\omega)| = |X(\omega)| \rightarrow \textit{Even Symmetry}$$

$$X(-\omega) = X(\omega) \quad \rightarrow \quad Odd\ Symmetry$$

$$S_{xx}(\omega)$$

Example:

Determine & sketch the ESD of the following signal.

$$x(n) = a^n U(n) \qquad , \qquad -1 < a < 1$$

Solution:

$$Taking FT of x(n)$$

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

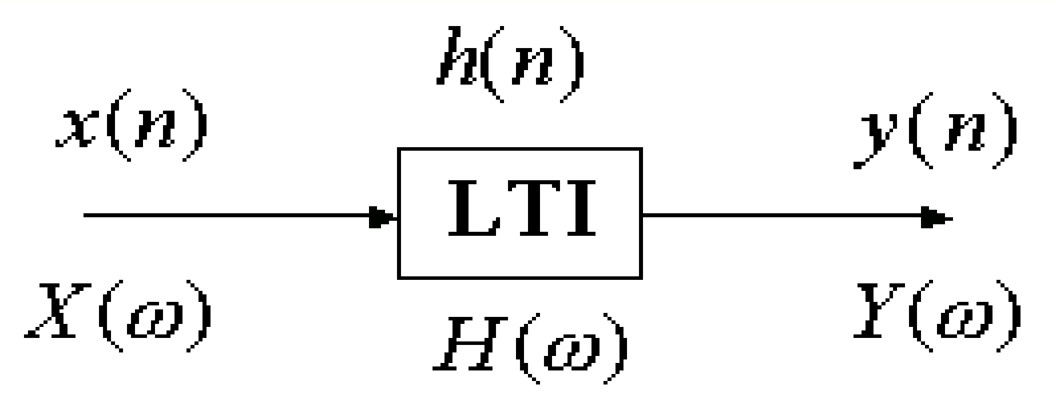
$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

$$S_{xx} = \left| X(\omega) \right|^2 = X(\omega) X^*(\omega)$$

$$S_{xx} = \frac{1}{\left(1 - ae^{-j\omega}\right)\left(1 - ae^{j\omega}\right)}$$

$$S_{xx} = \frac{1}{1 - 2a\cos\omega + a^2}$$

Response to a complex exponential input



Consider the convolution of x(n) with h(n) where x(n) is as follows

$$x(n) = Ae^{j\omega n}$$

$$y(n) = \sum_{-\infty}^{\infty} h(n)x(n-k)$$

$$= \sum_{-\infty}^{\infty} Ah(n)e^{j\omega(n-k)}$$

$$= \sum_{-\infty}^{\infty} Ah(n)e^{j\omega n}e^{-j\omega k}$$

$$= A\sum_{-\infty}^{\infty} h(n)e^{-j\omega k}e^{j\omega n}$$

$$where \quad \sum_{-\infty}^{\infty} h(n)e^{-j\omega k} = H(\omega)$$

$$y(n) = AH(\omega)e^{j\omega n}$$

The frequency response is multiplied by the applied input when the input is complex or sinusoidal.

$$H(\omega) = EigenValue$$

$$e^{j\omega n} = EigenFunction$$

Example:

Determine the output sequence of the system with impulse response.

$$x(n) = Ae^{j\pi n/2} \qquad -\infty < n < \infty$$

And input is complex exponential sequence.

$$h(n) = \left(\frac{1}{2}\right)^n U(n)$$

Solution:

Since we know from the previous slide

$$y(n) = AH(\omega)e^{-j\omega n} \rightarrow (1)$$

Taking Fourier Transform $h(n)$

$$H(\omega) = \sum_{n=0}^{\infty} h(n)e^{-j\omega n}$$

$$H(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} U(n)e^{-j\omega n}$$

$$H(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^{n}$$

$$H(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$H(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\pi/2}}$$

Put value of $H(\omega)$ *in* (1), *we get* ,

$$y(n) = \frac{Ae^{-j\pi n/2}}{1 - \frac{1}{2} e^{-j\pi/2}}$$

$$y(n) = \frac{Ae^{-j\pi n/2}}{1 + \frac{1}{2} j} \qquad [\,Euler\,]$$

Polar Form:

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(1)^2 + \left(\frac{1}{2}\right)^2}$$

$$r = \frac{\sqrt{5}}{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = 26.56$$

Replacing the rectangular form by polar form

$$y(n) = A\frac{1}{re^{j\theta}}e^{j\pi n/2}$$

$$y(n) = A\left\{\frac{1}{\frac{\sqrt{5}}{2}e^{j(26.56)}}\right\}e^{j\pi n/2}$$

$$y(n) = A\frac{2}{\sqrt{5}}e^{j(\pi n/2 - 26.56)}$$

The output is actually phase shifted of the input.