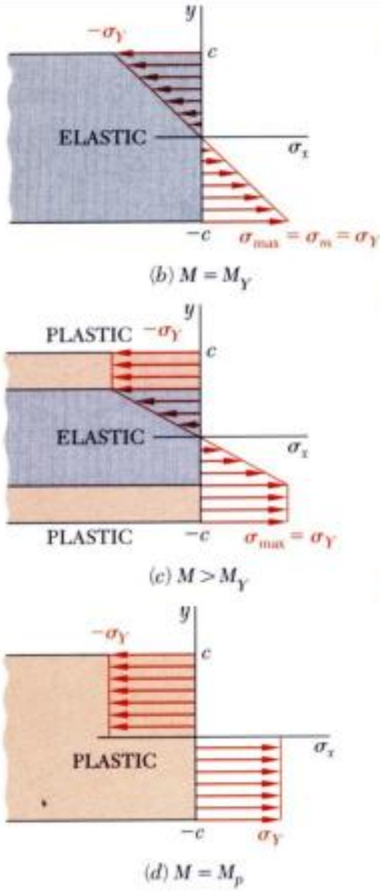
Members Made of an Elastoplastic Material



- Rectangular beam made of an elastoplastic material

$$\sigma_x \le \sigma_Y \quad \sigma_m = \frac{Mc}{I}$$

$$\sigma_m = \sigma_Y \quad M_Y = \frac{I}{c}\sigma_Y = \text{ maximum elastic moment}$$

- If the moment is increased beyond the maximum elastic moment, plastic zones develop around an elastic core.

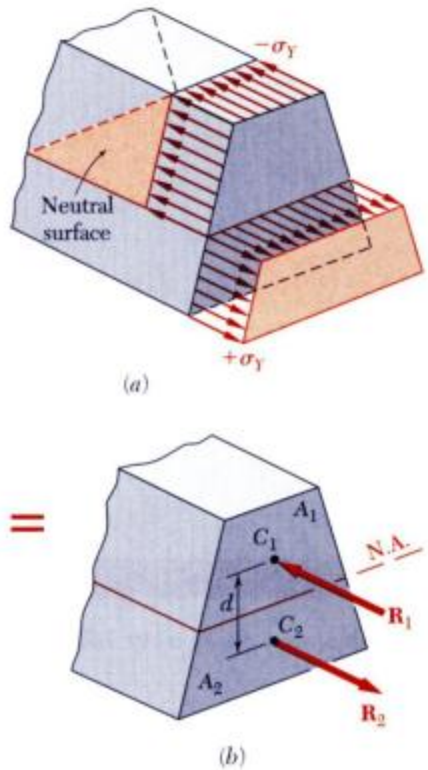
$$M = \tfrac{3}{2} M_Y \left(1 - \tfrac{1}{3}\frac{y_Y^2}{c^2}\right) \qquad y_Y = \text{elastic core half - thickness}$$

- In the limit as the moment is increased further, the elastic core thickness goes to zero, corresponding to a fully plastic deformation.

$$M_p = \tfrac{3}{2} M_Y = \text{plastic moment}$$

$$k = \frac{M_p}{M_Y} = \text{shape factor (depends only on cross section shape)}$$

Plastic Deformations of Members With a Single Plane of Symmetry



- Fully plastic deformation of a beam with only a vertical plane of symmetry.
- The neutral axis cannot be assumed to pass through the section centroid.
- Resultants R_1 and R_2 of the elementary compressive and tensile forces form a couple.

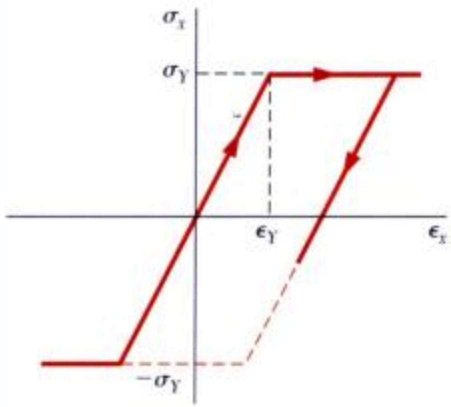
$$R_1 = R_2$$
$$A_1\sigma_Y = A_2\sigma_Y$$

The neutral axis divides the section into equal areas.

- The plastic moment for the member,

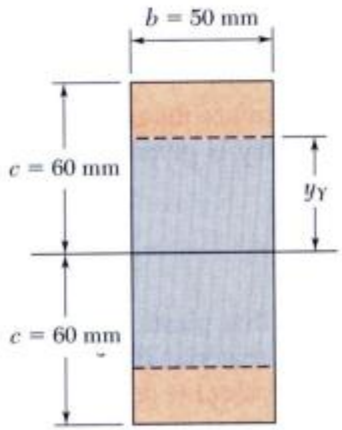
$$M_p = \left(\tfrac{1}{2} A\sigma_Y\right)d$$

Residual Stresses



- Plastic zones develop in a member made of an elastoplastic material if the bending moment is large enough.
- Since the linear relation between normal stress and strain applies at all points during the unloading phase, it may be handled by assuming the member to be fully elastic.
- Residual stresses are obtained by applying the principle of superposition to combine the stresses due to loading with a moment *M* (elastoplastic deformation) and unloading with a moment -*M* (elastic deformation).
- The final value of stress at a point will not, in general, be zero.

Example 4.05, 4.06

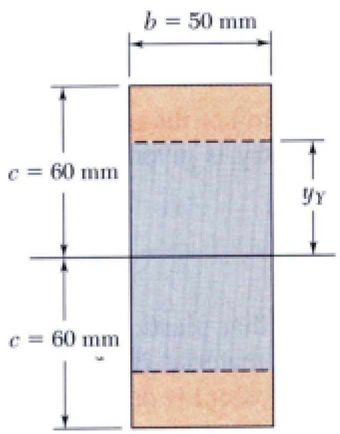


A member of uniform rectangular cross section is subjected to a bending moment M = 36.8 kN-m. The member is made of an elastoplastic material with a yield strength of 240 MPa and a modulus of elasticity of 200 GPa.

Determine (a) the thickness of the elastic core, (b) the radius of curvature of the neutral surface.

After the loading has been reduced back to zero, determine (c) the distribution of residual stresses, (d) radius of curvature.

Example 4.05, 4.06



- Maximum elastic moment:

$$\frac{I}{c} = \tfrac{2}{3}bc^2 = \tfrac{2}{3}\left(50\times10^{-3}m\right)\left(60\times10^{-3}m\right)^2$$

$$= 120\times10^{-6}\,\text{m}^3$$

$$M_Y = \frac{I}{c}\sigma_Y = \left(120\times10^{-6}\,\text{m}^3\right)(240\,\text{MPa})$$

$$= 28.8\,\text{kN}\cdot\text{m}$$

- Thickness of elastic core:

$$M = \tfrac{3}{2}M_Y\left(1-\tfrac{1}{3}\frac{y_Y^2}{c^2}\right)$$

$$36.8\,\text{kN}\cdot\text{m} = \tfrac{3}{2}(28.8\,\text{kN}\cdot\text{m})\left(1-\tfrac{1}{3}\frac{y_Y^2}{c^2}\right)$$

$$\frac{y_Y}{c} = \frac{y_Y}{60\,\text{mm}} = 0.666$$

$$\boxed{2y_Y = 80\,\text{mm}}$$

- Radius of curvature:

$$\varepsilon_Y = \frac{\sigma_Y}{E} = \frac{240\times10^6\,\text{Pa}}{200\times10^9\,\text{Pa}}$$

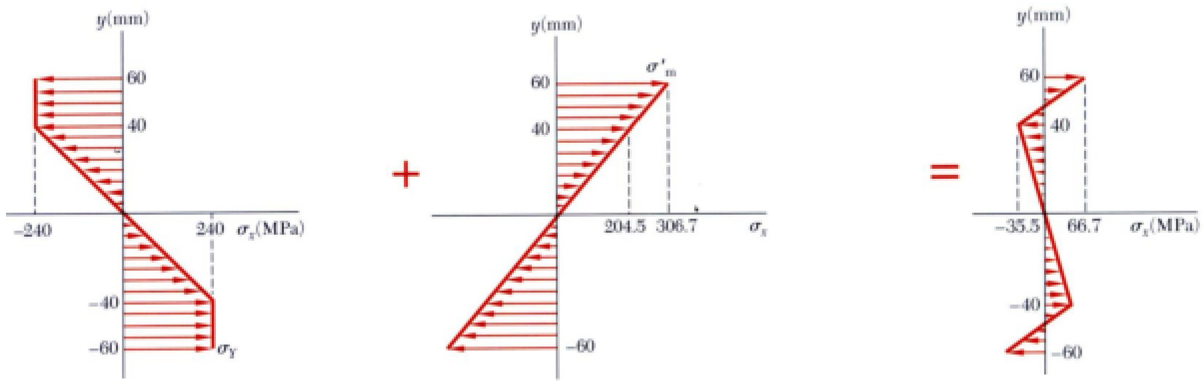
$$= 1.2\times10^{-3}$$

$$\varepsilon_Y = \frac{y_Y}{\rho}$$

$$\rho = \frac{y_Y}{\varepsilon_Y} = \frac{40\times10^{-3}\,\text{m}}{1.2\times10^{-3}}$$

$$\boxed{\rho = 33.3\,\text{m}}$$

Example 4.05, 4.06

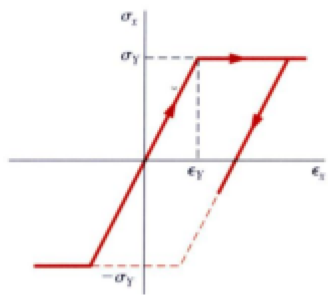


- $M = 36.8$ kN-m

$$y_Y = 40\,\text{mm}$$
$$\sigma_Y = 240\,\text{MPa}$$

- $M = -36.8$ kN-m

$$\sigma'_m = \frac{Mc}{I} = \frac{36.8\,\text{kN}\cdot\text{m}}{120\times10^6\,\text{m}^3}$$
$$= 306.7\,\text{MPa} < 2\sigma_Y$$



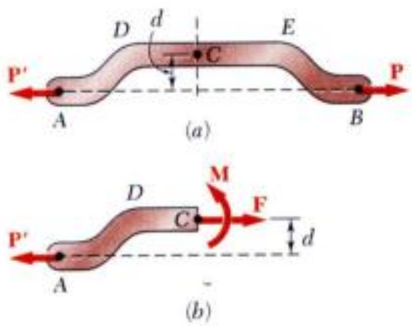
- $M = 0$

At the edge of the elastic core,

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{-35.5\times10^6\,\text{Pa}}{200\times10^9\,\text{Pa}}$$
$$= -177.5\times10^{-6}$$
$$\rho = -\frac{y_Y}{\varepsilon_x} = \frac{40\times10^{-3}\,\text{m}}{177.5\times10^{-6}}$$

$$\boxed{\rho = 225\,\text{m}}$$

Eccentric Axial Loading in a Plane of Symmetry



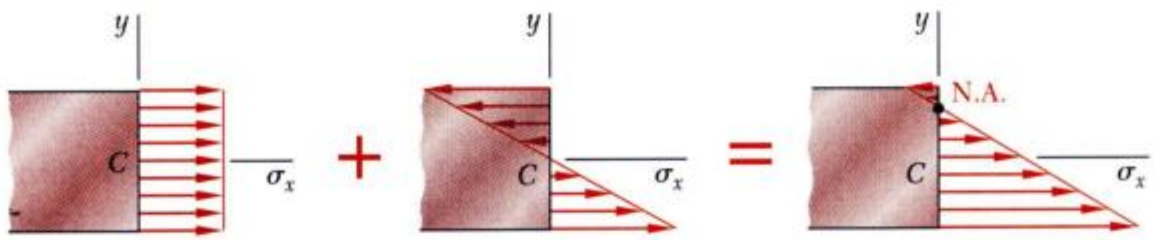
- Eccentric loading

$$F = P$$
$$M = Pd$$

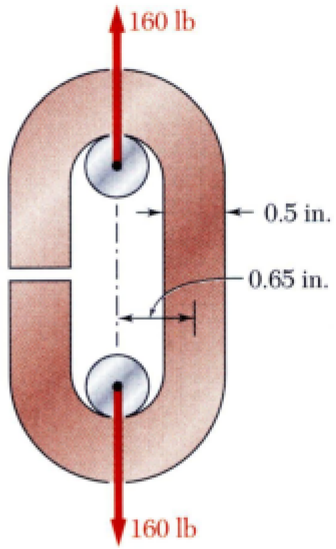
- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due a pure bending moment

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$
$$= \frac{P}{A} - \frac{My}{I}$$

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.



Example 4.07

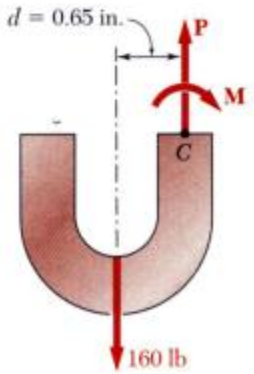


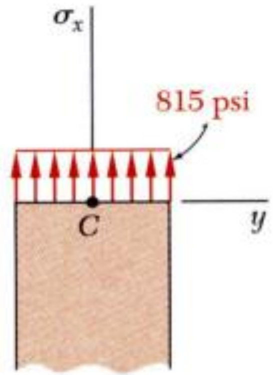
An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 160 lb load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

SOLUTION:

- Find the equivalent centric load and bending moment
- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.
- Find the neutral axis by determining the location where the normal stress is zero.

Example 4.07



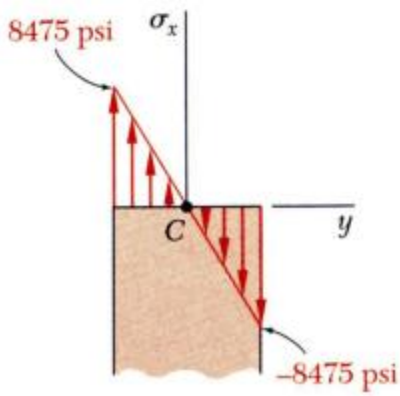


- Normal stress due to a centric load

$$A = \pi c^2 = \pi(0.25\text{in})^2$$
$$= 0.1963\text{in}^2$$
$$\sigma_0 = \frac{P}{A} = \frac{160\text{lb}}{0.1963\text{in}^2}$$
$$= 815\text{psi}$$

- Equivalent centric load and bending moment

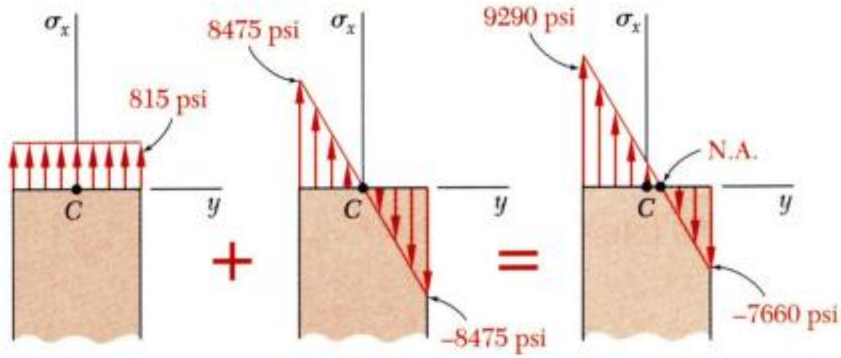
$$P = 160\text{lb}$$
$$M = Pd = (160\text{lb})(0.6\text{in})$$
$$= 104\text{lb}\cdot\text{in}$$



- Normal stress due to bending moment

$$I = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi(0.25)^4$$
$$= 3.068\times10^{-3}\text{in}^4$$
$$\sigma_m = \frac{Mc}{I} = \frac{(104\text{lb}\cdot\text{in})(0.25\text{in})}{.068\times10^{-3}\text{in}^4}$$
$$= 8475\text{psi}$$

Example 4.07



- Maximum tensile and compressive stresses

$$\sigma_t = \sigma_0 + \sigma_m = 815 + 8475$$

$$\boxed{\sigma_t = 9260\text{psi}}$$

$$\sigma_c = \sigma_0 - \sigma_m = 815 - 8475$$

$$\boxed{\sigma_c = -7660\text{psi}}$$

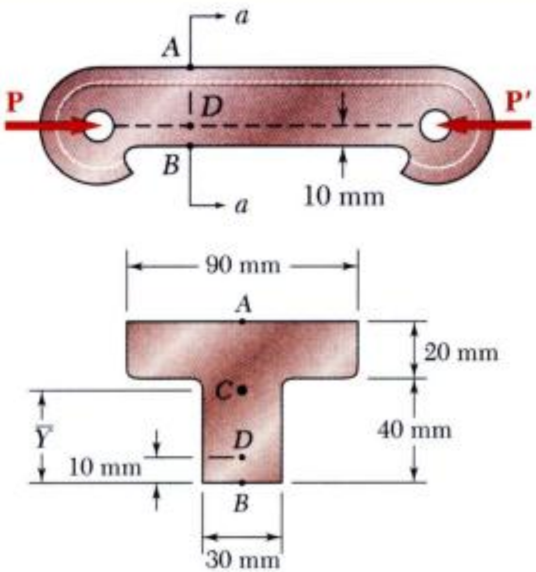
- Neutral axis location

$$0 = \frac{P}{A} - \frac{My_0}{I}$$

$$y_0 = \frac{P}{A}\frac{I}{M} = (815\text{psi})\frac{3.068\times10^{-3}\text{in}^4}{105\text{lb}\cdot\text{in}}$$

$$\boxed{y_0 = 0.0240\text{in}}$$

Sample Problem 4.8



Section *a–a*

From Sample Problem 2.4,

$A = 3\times10^{-3}\,\text{m}^2$

$\bar{Y} = 0.038\text{m}$

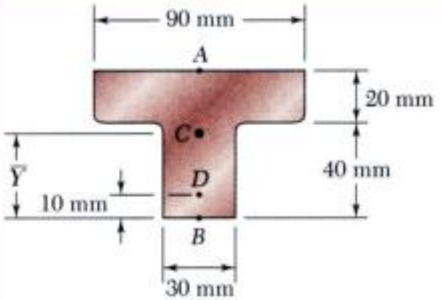
$I = 868\times10^{-9}\,\text{m}^4$

The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force *P* which can be applied to the link.

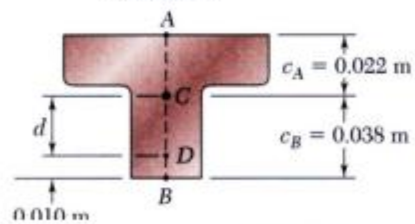
SOLUTION:

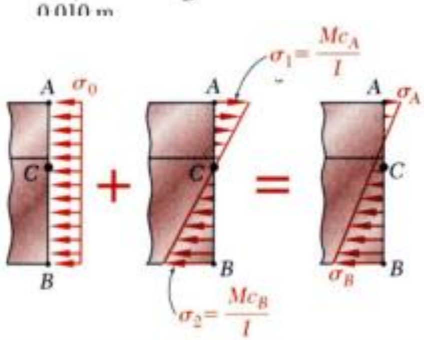
- Determine an equivalent centric load and bending moment.
- Superpose the stress due to a centric load and the stress due to bending.
- Evaluate the critical loads for the allowable tensile and compressive stresses.
- The largest allowable load is the smallest of the two critical loads.

Sample Problem 4.8



Section a–a





- Determine an equivalent centric and bending loads.

  $d = 0.038 - 0.010 = 0.028\text{m}$

  $P = \text{centric load}$

  $M = Pd = 0.028P = \text{bending moment}$

- Superpose stresses due to centric and bending loads

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} + \frac{(0.028P)(0.022)}{868\times10^{-9}} = +377P$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} - \frac{(0.028P)(0.022)}{868\times10^{-9}} = -1559P$$

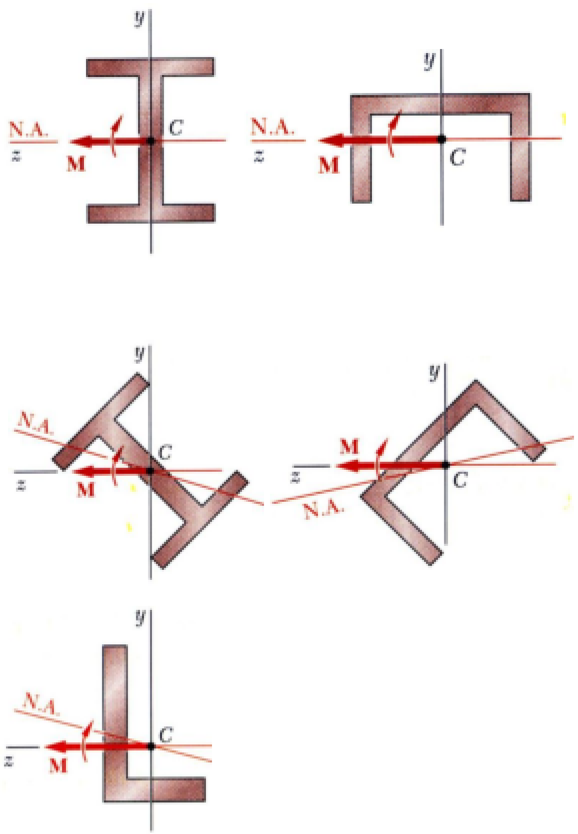
- Evaluate critical loads for allowable stresses.

  $\sigma_A = +377P = 30\text{MPa} \qquad P = 79.6\text{kN}$

  $\sigma_B = -1559P = -120\text{MPa} \qquad P = 79.6\text{kN}$

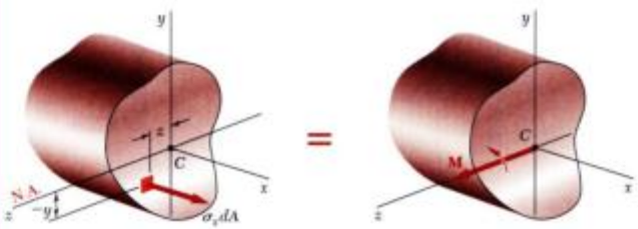
- The largest allowable load $\boxed{P = 77.0 \text{ kN}}$

Unsymmetric Bending



- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple
- Will now consider situations in which the bending couples do not act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.

Unsymmetric Bending



Wish to determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy

  $F_x = 0 = M_y \quad M_z = M = \textit{applied couple}$

- $0 = F_x = \int \sigma_x dA = \int \left(-\frac{y}{c}\sigma_m \right) dA$

  or $0 = \int y\, dA$

  neutral axis passes through centroid

- $M = M_z = -\int y \left(-\frac{y}{c}\sigma_m \right) dA$

  or $M = \frac{\sigma_m I}{c} \quad I = I_z = \textit{moment of inertia}$

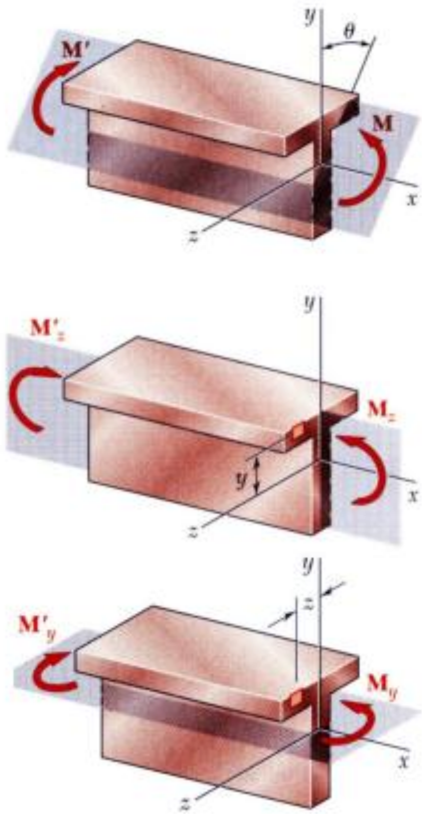
  defines stress distribution

- $0 = M_y = \int z\sigma_x dA = \int z \left(-\frac{y}{c}\sigma_m \right) dA$

  or $0 = \int yz\, dA = I_{yz} = \textit{product of inertia}$

  couple vector must be directed along a principal centroidal axis

Unsymmetric Bending



Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.

  $M_z = M\cos\theta \qquad M_y = M\sin\theta$

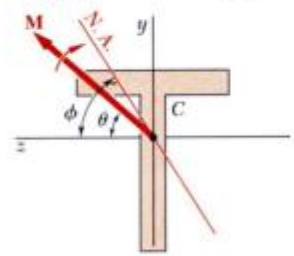
- Superpose the component stress distributions

  $$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

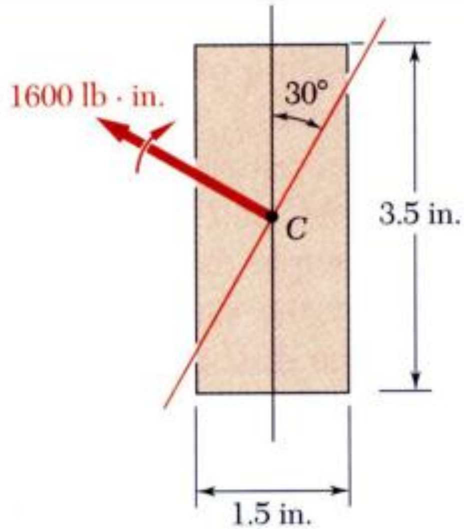
- Along the neutral axis,

  $$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y} = -\frac{(M\cos\theta)y}{I_z} + \frac{(M\sin\theta)y}{I_y}$$

  $$\tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$



Example 4.08



A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

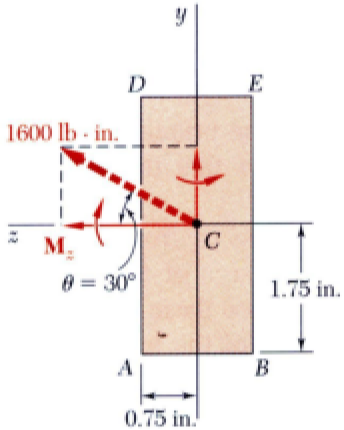
- Combine the stresses from the component stress distributions.

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

- Determine the angle of the neutral axis.

$$\tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$

Example 4.08



- Resolve the couple vector into components and calculate the corresponding maximum stresses.

$$M_z = (1600\,\text{lb}\cdot\text{in})\cos 30 = 1386\,\text{lb}\cdot\text{in}$$

$$M_y = (1600\,\text{lb}\cdot\text{in})\sin 30 = 800\,\text{lb}\cdot\text{in}$$

$$I_z = \tfrac{1}{12}(1.5\,\text{in})(3.5\,\text{in})^3 = 5.359\,\text{in}^4$$

$$I_y = \tfrac{1}{12}(3.5\,\text{in})(1.5\,\text{in})^3 = 0.9844\,\text{in}^4$$

The largest tensile stress due to M_z occurs along AB

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386\,\text{lb}\cdot\text{in})(1.75\,\text{in})}{5.359\,\text{in}^4} = 452.6\,\text{psi}$$

The largest tensile stress due to M_z occurs along AD

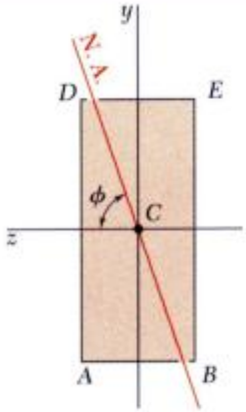
$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800\,\text{lb}\cdot\text{in})(0.75\,\text{in})}{0.9844\,\text{in}^4} = 609.5\,\text{psi}$$

- The largest tensile stress due to the combined loading occurs at A.

$$\sigma_{max} = \sigma_1 + \sigma_2 = 452.6 + 609.5$$

$$\boxed{\sigma_{max} = 1062\,\text{psi}}$$

Example 4.08

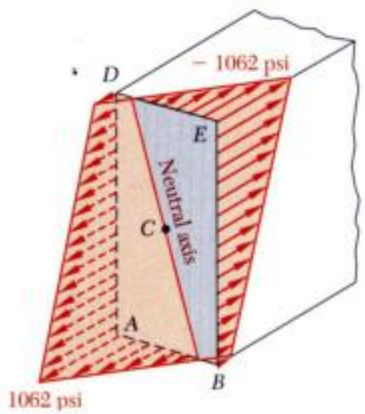


- Determine the angle of the neutral axis.

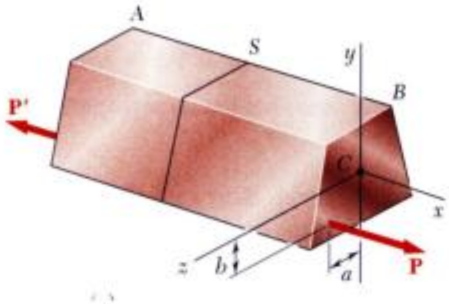
$$\tan\phi = \frac{I_z}{I_y}\tan\theta = \frac{5.359\text{in}^4}{0.9844\text{in}^4}\tan 30$$

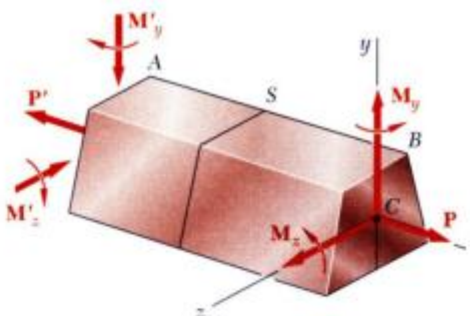
$$= 3.143$$

$$\phi = 72.4^o$$



General Case of Eccentric Axial Loading





- Consider a straight member subject to equal and opposite eccentric forces.
- The eccentric force is equivalent to the system of a centric force and two couples.

  $P = \text{centric force}$

  $M_y = Pa \qquad M_z = Pb$

- By the principle of superposition, the combined stress distribution is

  $$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- If the neutral axis lies on the section, it may be found from

  $$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$$