Chapter # 4 (Pure Bending)

Pure Bending

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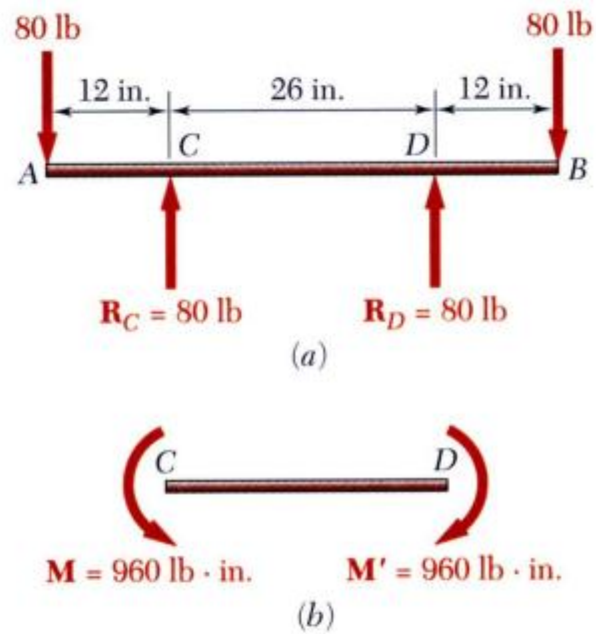
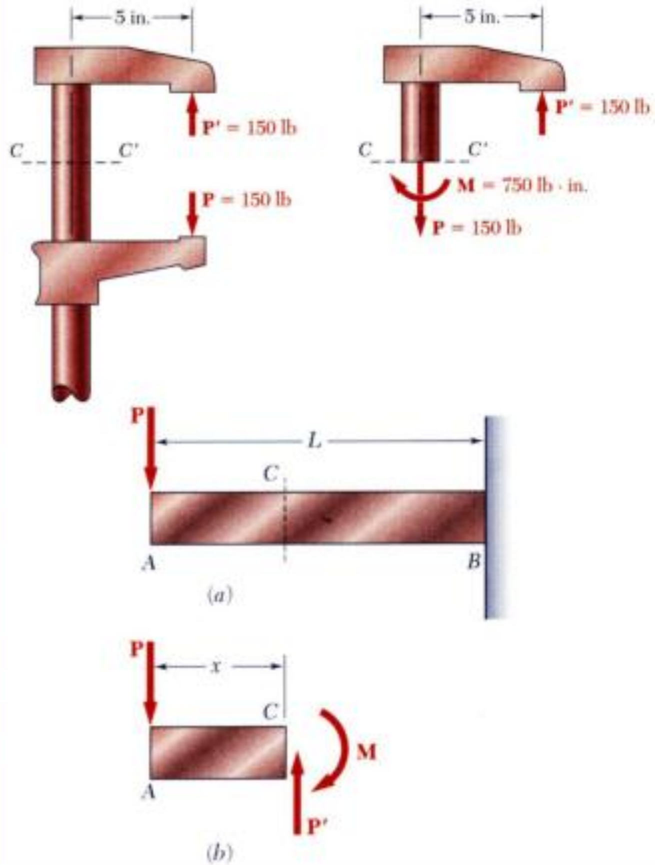


Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

Other Loading Types

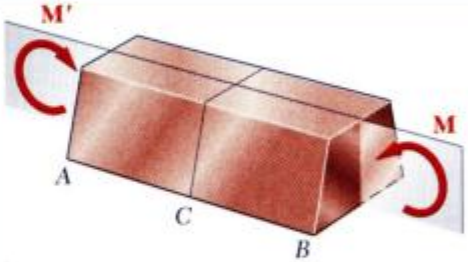


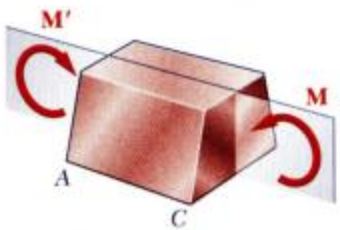
- *Eccentric Loading*: Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple

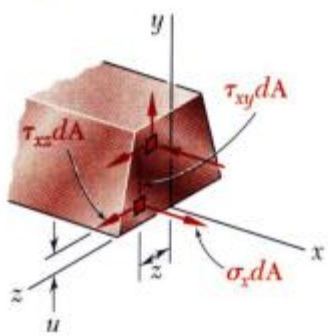
- *Transverse Loading*: Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple

- *Principle of Superposition*: The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.

Symmetric Member in Pure Bending



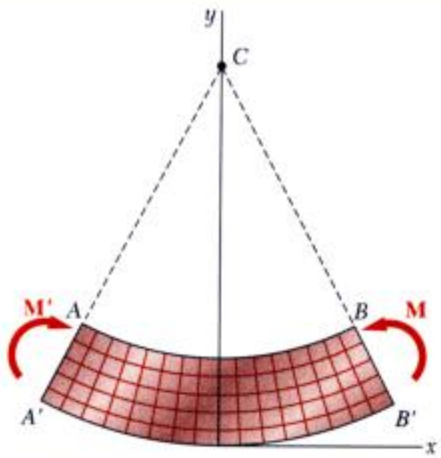




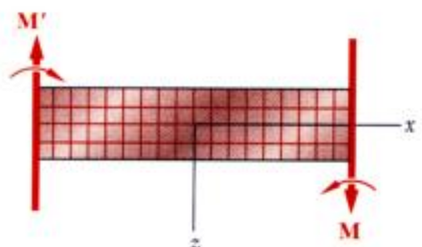
- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

$$F_x = \int \sigma_x \, dA = 0$$
$$M_y = \int z\sigma_x \, dA = 0$$
$$M_z = \int -y\sigma_x \, dA = M$$

Bending Deformations



(a) Longitudinal, vertical section (plane of symmetry)

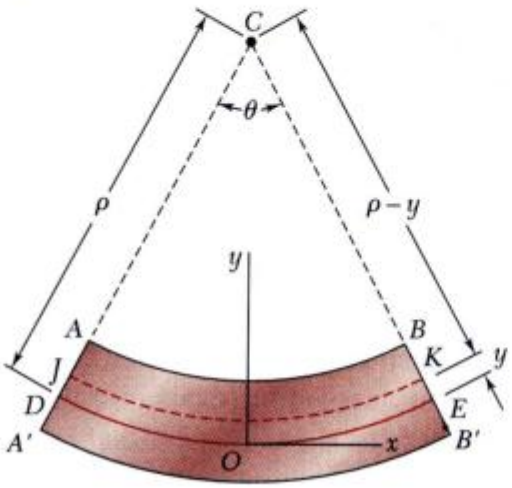


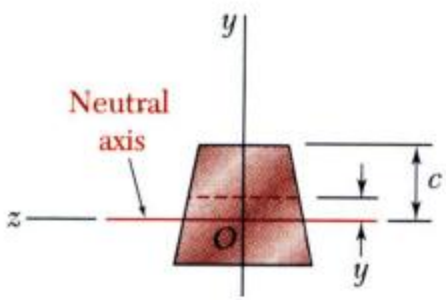
(b) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending





Consider a beam segment of length L.

After deformation, the length of the neutral surface remains L. At other sections,

$$L' = (\rho - y)\theta$$

$$\delta = L - L' = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Stress Due to Bending

- For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

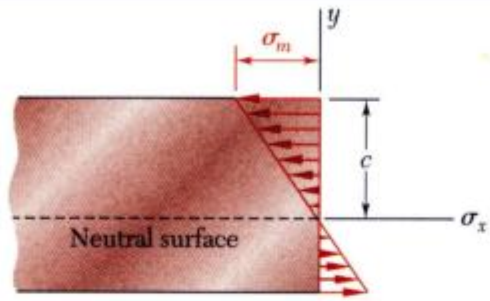
$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.



- For static equilibrium,

$$M = \int -y\sigma_x \, dA = \int -y\left(-\frac{y}{c}\sigma_m\right) dA$$

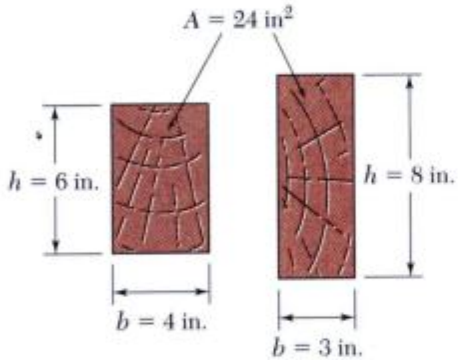
$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

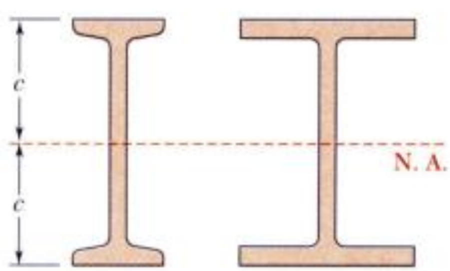
$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Substituting $\sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{My}{I}$$

Beam Section Properties





(a) S-beam (b) W-beam

- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

I = section moment of inertia

$$S = \frac{I}{c} = \text{section modulus}$$

A beam section with a larger section modulus will have a lower maximum stress

- Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^3 = \frac{1}{6}Ah$$

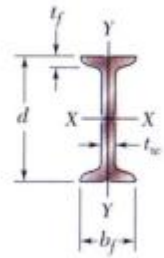
Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

- Structural steel beams are designed to have a large section modulus.

Properties of American Standard Shapes

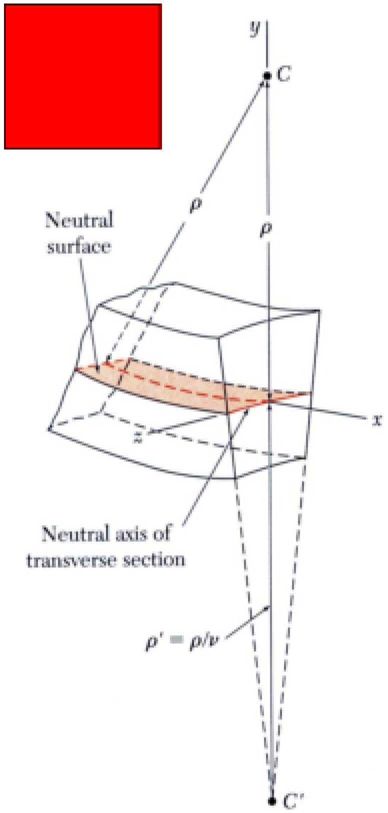
Appendix C. **Properties of Rolled-Steel Shapes**
(SI Units)

S Shapes
(American Standard Shapes)



Designation†	Area A, mm²	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm⁴	Axis X-X S_x 10^3 mm³	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm⁴	Axis Y-Y S_y 10^3 mm³	Axis Y-Y r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0
158	20100	622	200	27.7	15.7	1230	3950	247	32.5	321	39.9
149	19000	610	184	22.1	18.9	995	3260	229	20.2	215	32.3
134	17100	610	181	22.1	15.9	938	3080	234	19.0	206	33.0
119	15200	610	178	22.1	12.7	878	2880	240	17.9	198	34.0
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9
128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5
81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1
64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1

Deformations in a Transverse Cross Section



- Deformation due to bending moment M is quantified by the curvature of the neutral surface

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$
$$= \frac{M}{EI}$$

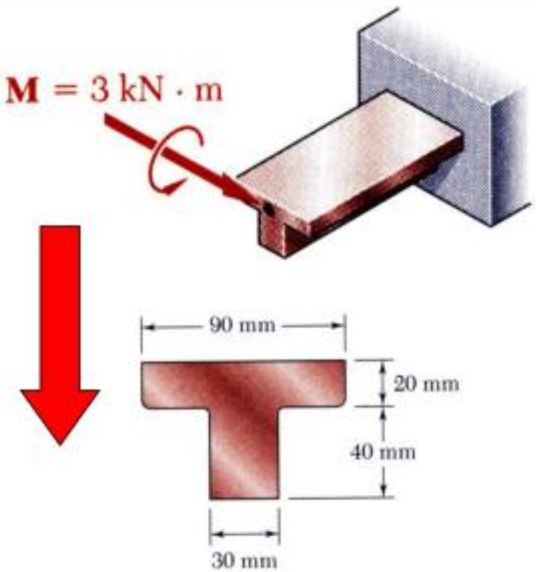
- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$

- Expansion above the neutral surface and contraction below it cause an in-plane curvature,

$$\frac{1}{\rho'} = \frac{\nu}{\rho} = \text{anticlastic curvature}$$

Sample Problem 4.2



A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing $E = 165$ GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \qquad I_{x'} = \sum\left(\bar{I} + Ad^2\right)$$

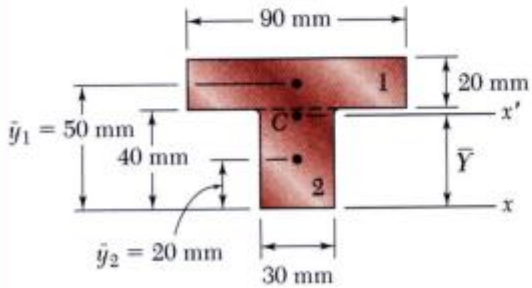
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

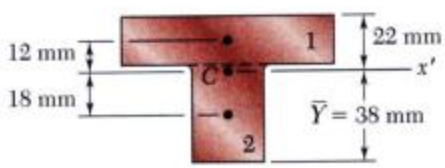
$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

Sample Problem 4.2





SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$20\times 90 = 1800$	50	90×10^3
2	$40\times 30 = 1200$	20	24×10^3
	$\sum A = 3000$		$\sum \bar{y}A = 114\times 10^3$

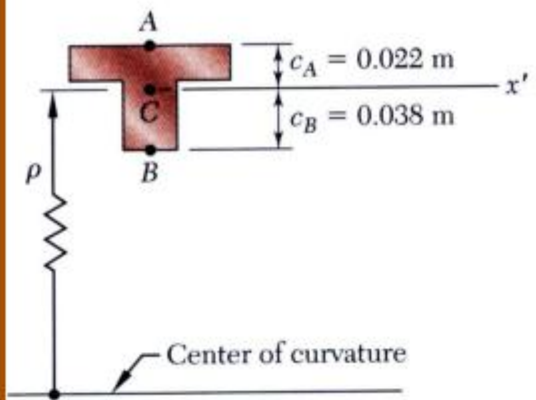
$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114\times 10^3}{3000} = 38\,\text{mm}$$

$$I_{x'} = \sum\left(\bar{I} + Ad^2\right) = \sum\left(\tfrac{1}{12}bh^3 + Ad^2\right)$$

$$= \left(\tfrac{1}{12}90\times 20^3 + 1800\times 12^2\right) + \left(\tfrac{1}{12}30\times 40^3 + 1200\times 18^2\right)$$

$$I = 868\times 10^3\,\text{mm} = 868\times 10^{-9}\,\text{m}^4$$

Sample Problem 4.2



- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{M\,c_A}{I} = \frac{3\,\text{kN}\cdot\text{m}\times 0.022\text{m}}{868\times 10^{-9}\,\text{mm}^4} \qquad \boxed{\sigma_A = +76.0\,\text{MPa}}$$

$$\sigma_B = -\frac{M\,c_B}{I} = -\frac{3\,\text{kN}\cdot\text{m}\times 0.038\text{m}}{868\times 10^{-9}\,\text{mm}^4} \qquad \boxed{\sigma_B = -131.3\,\text{MPa}}$$

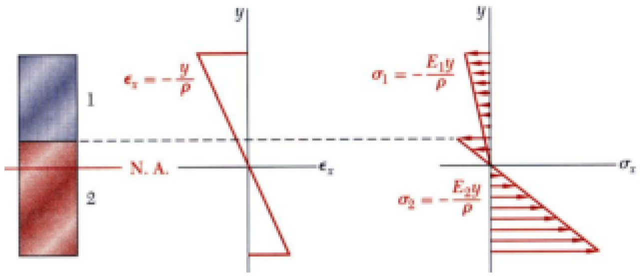
- Calculate the curvature

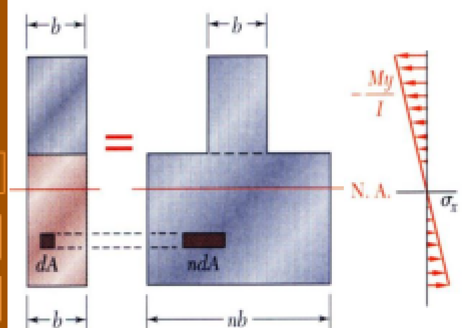
$$\frac{1}{\rho} = \frac{M}{EI}$$

$$= \frac{3\,\text{kN}\cdot\text{m}}{(165\,\text{GPa})(868\times 10^{-9}\,\text{m}^4)}$$

$$\boxed{\frac{1}{\rho} = 20.95\times 10^{-3}\,\text{m}^{-1} \qquad \rho = 47.7\,\text{m}}$$

Bending of Members Made of Several Materials





$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \qquad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with E_1 and E_2.
- Normal strain varies linearly.

$$\varepsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1\varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2\varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

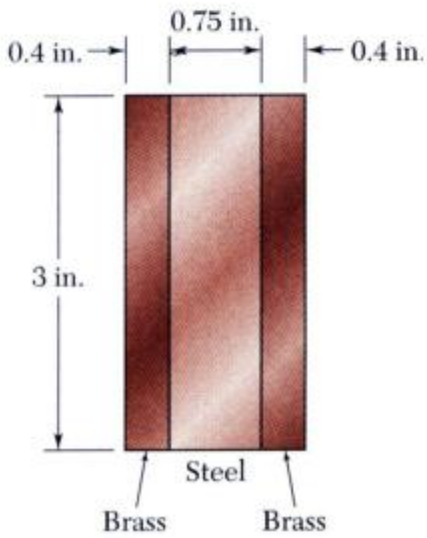
- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

Example 4.03

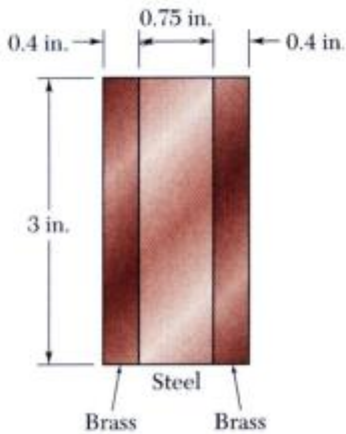


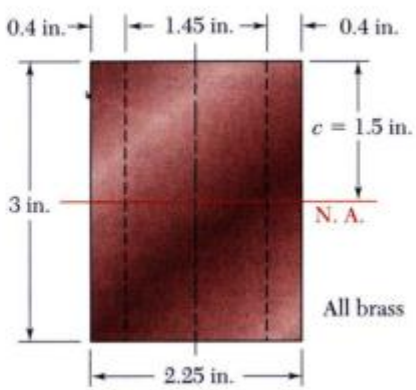
Bar is made from bonded pieces of steel (E_s = 29x10^6 psi) and brass (E_b = 15x10^6 psi). Determine the maximum stress in the steel and brass when a moment of 40 kip*in is applied.

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass
- Evaluate the cross sectional properties of the transformed section
- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.
- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.

Example 4.03





SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

$$n = \frac{E_s}{E_b} = \frac{29\times10^6\,\text{psi}}{15\times10^6\,\text{psi}} = 1.933$$

$$b_T = 0.4\,\text{in} + 1.933\times0.75\,\text{in} + 0.4\,\text{in} = 2.25\,\text{in}$$

- Evaluate the transformed cross sectional properties

$$I = \tfrac{1}{12} b_T h^3 = \tfrac{1}{12}(2.25\,\text{in.})(3\,\text{in})^3$$
$$= 5.063\,\text{in}^4$$

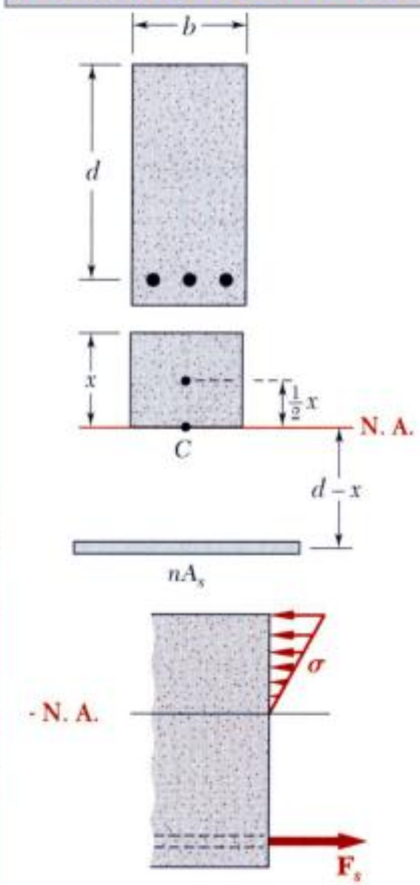
- Calculate the maximum stresses

$$\sigma_m = \frac{Mc}{I} = \frac{(40\,\text{kip}\cdot\text{in})(1.5\,\text{in})}{5.063\,\text{in}^4} = 11.85\,\text{ksi}$$

$$(\sigma_b)_{\max} = \sigma_m$$
$$(\sigma_s)_{\max} = n\sigma_m = 1.933\times11.85\,\text{ksi}$$

$$(\sigma_b)_{\max} = 11.85\,\text{ksi}$$
$$(\sigma_s)_{\max} = 22.9\,\text{ksi}$$

Reinforced Concrete Beams



- Concrete beams subjected to bending moments are reinforced by steel rods.
- The steel rods carry the entire tensile load below the neutral surface. The upper part of the concrete beam carries the compressive load.
- In the transformed section, the cross sectional area of the steel, A_s, is replaced by the equivalent area nA_s where $n = E_s/E_c$.
- To determine the location of the neutral axis,

$$(bx)\frac{x}{2} - n A_s(d - x) = 0$$

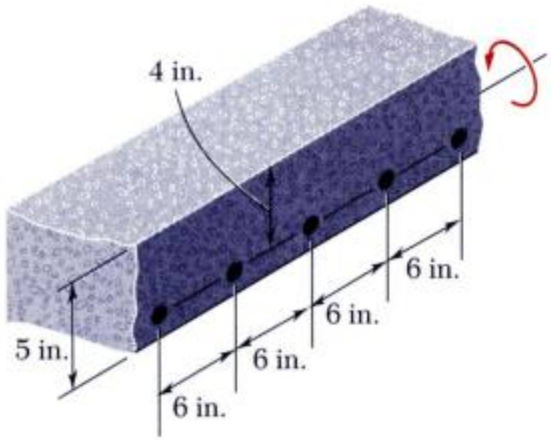
$$\tfrac{1}{2}bx^2 + n A_s x - n A_s d = 0$$

- The normal stress in the concrete and steel

$$\sigma_x = -\frac{My}{I}$$

$$\sigma_c = \sigma_x \qquad \sigma_s = n\sigma_x$$

Sample Problem 4.4

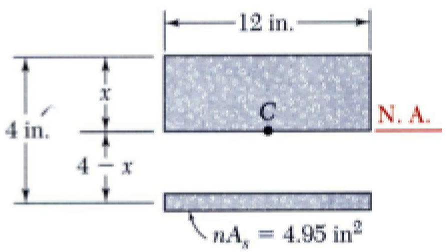


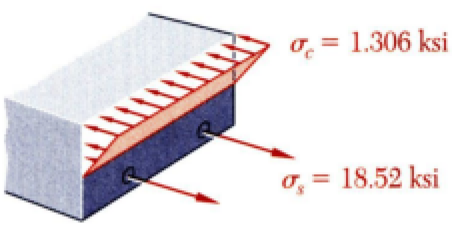
A concrete floor slab is reinforced with 5/8-in-diameter steel rods. The modulus of elasticity is 29x106psi for steel and 3.6x106psi for concrete. With an applied bending moment of 40 kip*in for 1-ft width of the slab, determine the maximum stress in the concrete and steel.

SOLUTION:

- Transform to a section made entirely of concrete.
- Evaluate geometric properties of transformed section.
- Calculate the maximum stresses in the concrete and steel.

Sample Problem 4.4





SOLUTION:

- Transform to a section made entirely of concrete.

$$n = \frac{E_s}{E_c} = \frac{29\times10^6\ \text{psi}}{3.6\times10^6\ \text{psi}} = 8.06$$

$$nA_s = 8.06\times 2\left[\frac{\pi}{4}\left(\frac{5}{8}\,\text{in}\right)^2\right] = 4.95\,\text{in}^2$$

- Evaluate the geometric properties of the transformed section.

$$12x\left(\frac{x}{2}\right) - 4.95(4-x) = 0 \qquad x = 1.450\,\text{in}$$

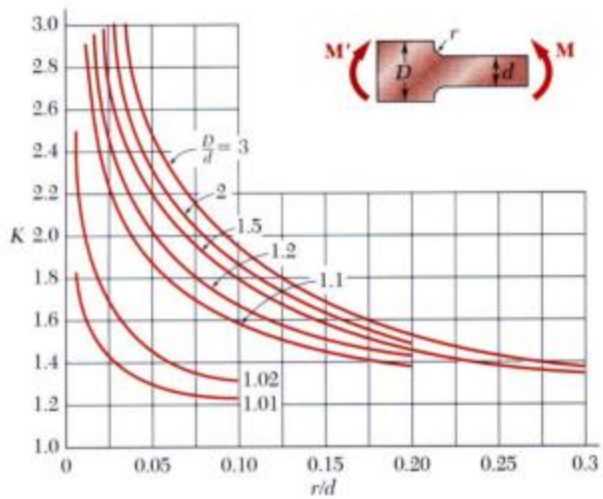
$$I = \tfrac{1}{3}(12\,\text{in})(1.45\,\text{in})^3 + \left(4.95\,\text{in}^2\right)(2.55\,\text{in})^2 = 44.4\,\text{in}^4$$

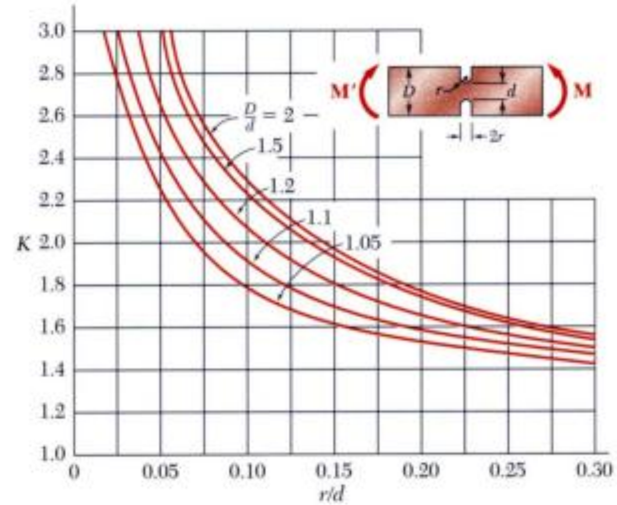
- Calculate the maximum stresses.

$$\sigma_c = \frac{Mc_1}{I} = \frac{40\,\text{kip}\cdot\text{in}\times1.45\,\text{in}}{44.4\,\text{in}^4} \qquad \boxed{\sigma_c = 1.306\,\text{ksi}}$$

$$\sigma_s = n\frac{Mc_2}{I} = 8.06\frac{40\,\text{kip}\cdot\text{in}\times2.55\,\text{in}}{44.4\,\text{in}^4} \qquad \boxed{\sigma_s = 18.52\,\text{ksi}}$$

Stress Concentrations



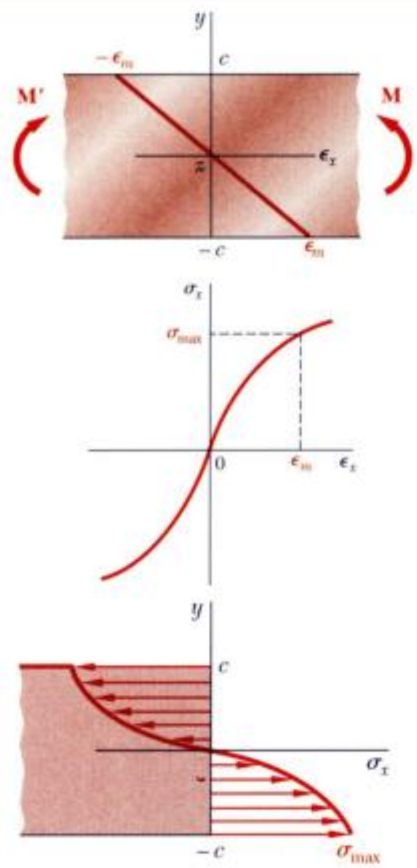


Stress concentrations may occur:

- in the vicinity of points where the loads are applied
- in the vicinity of abrupt changes in cross section

$$\sigma_m = K\frac{Mc}{I}$$

Plastic Deformations



- For any member subjected to pure bending

  $\varepsilon_x = -\frac{y}{c}\varepsilon_m$ strain varies linearly across the section

- <u>If</u> the member is made of a *linearly elastic material*, the neutral axis passes through the section centroid

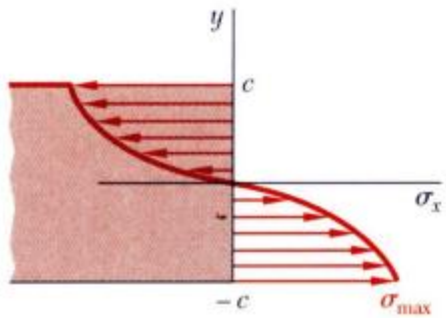
  and $\sigma_x = -\frac{My}{I}$

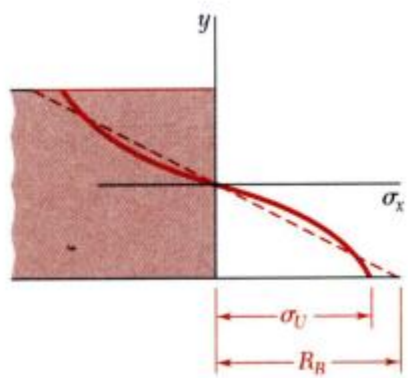
- For a material with a nonlinear stress-strain curve, the neutral axis location is found by satisfying

  $F_x = \int \sigma_x \, dA = 0 \qquad M = \int -y\sigma_x \, dA$

- For a member with vertical and horizontal planes of symmetry and a material with the same tensile and compressive stress-strain relationship, the neutral axis is located at the section centroid and the stress-strain relationship may be used to map the strain distribution from the stress distribution.

Plastic Deformations





- When the maximum stress is equal to the ultimate strength of the material, failure occurs and the corresponding moment M_U is referred to as the *ultimate bending moment*.
- The *modulus of rupture in bending*, R_B, is found from an experimentally determined value of M_U and a fictitious linear stress distribution.

$$R_B = \frac{M_U c}{I}$$

- R_B may be used to determine M_U of any member made of the same material and with the same cross sectional shape but different dimensions.