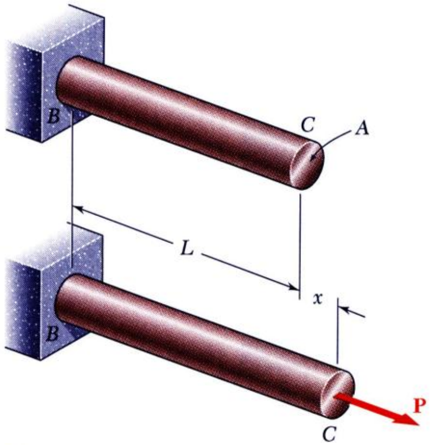
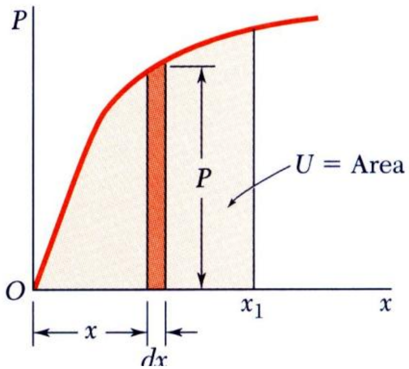
Chapter # 11

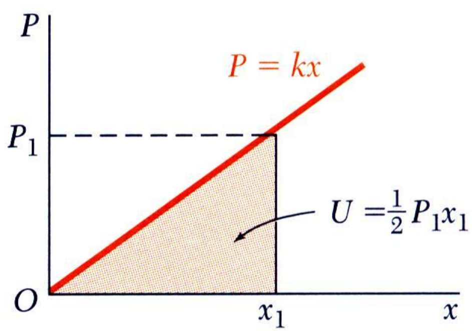
Energy Methods

Energy Methods

Strain Energy







- A uniform rod is subjected to a slowly increasing load
- The *elementary work* done by the load P as the rod elongates by a small *dx* is

$$dU = P\,dx = \textit{elementary work}$$

which is equal to the area of width *dx* under the load-deformation diagram.

- The *total work* done by the load for a deformation x_1,

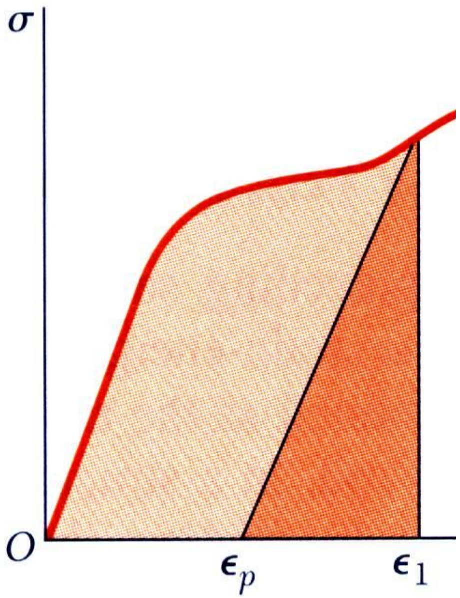
$$U = \int_0^{x_1} P\,dx = \textit{total work} = \textit{strain energy}$$

which results in an increase of *strain energy* in the rod.

- In the case of a linear elastic deformation,

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2} k x_1^2 = \tfrac{1}{2} P_1 x_1$$

Strain Energy Density



- To eliminate the effects of size, evaluate the strain-energy per unit volume,

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A}\,\frac{dx}{L}$$

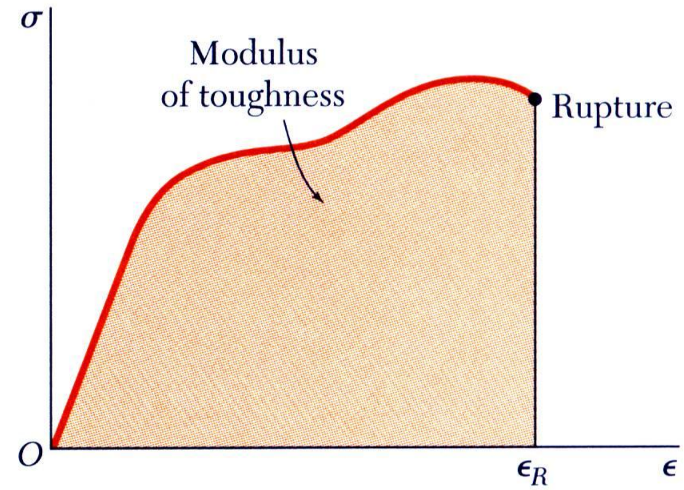
$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon = \text{strain energy density}$$

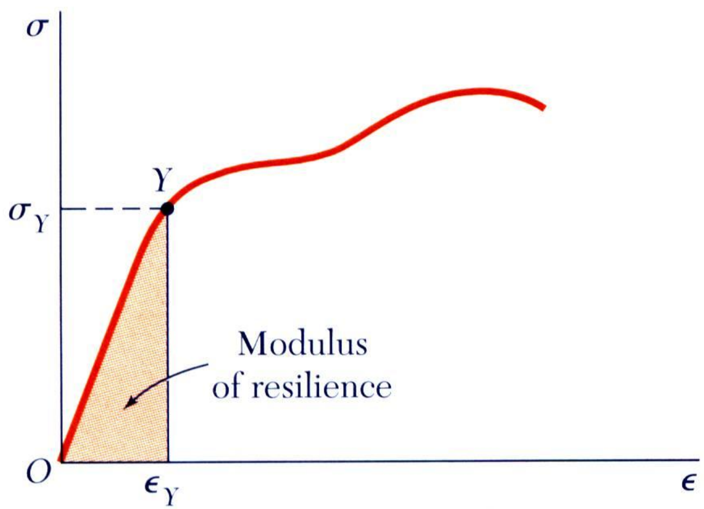
- The total strain energy density resulting from the deformation is equal to the area under the curve to ε_1.

- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.

- Remainder of the energy spent in deforming the material is dissipated as heat.

Strain-Energy Density





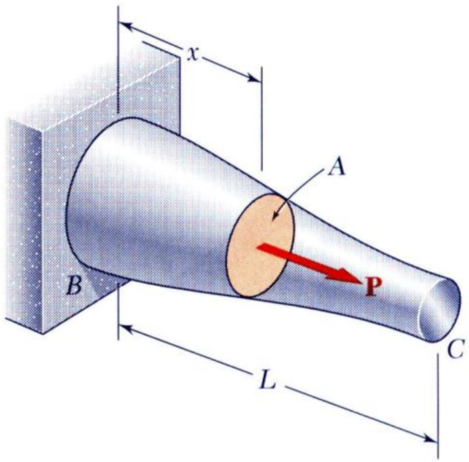
- The strain energy density resulting from setting $\varepsilon_1 = \varepsilon_R$ is the *modulus of toughness*.
- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.
- If the stress remains within the proportional limit,

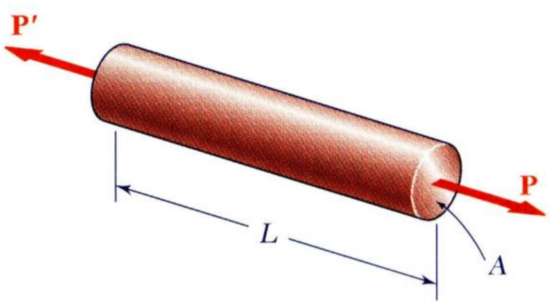
$$u = \int_0^{\varepsilon_1} E\varepsilon_1 \, d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

- The strain energy density resulting from setting $\sigma_1 = \sigma_Y$ is the *modulus of resilience*.

$$u_Y = \frac{\sigma_Y^2}{2E} = \text{modulus of resilience}$$

Elastic Strain Energy for Normal Stresses





- In an element with a nonuniform stress distribution,

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

- For values of $u < u_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} dV = \textit{elastic strain energy}$$

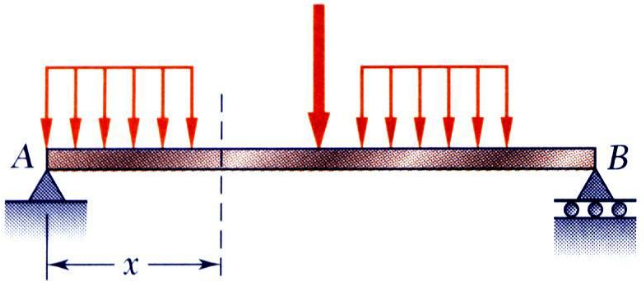
- Under axial loading, $\sigma_x = P/A \qquad dV = A\,dx$

$$U = \int_0^L \frac{P^2}{2AE} dx$$

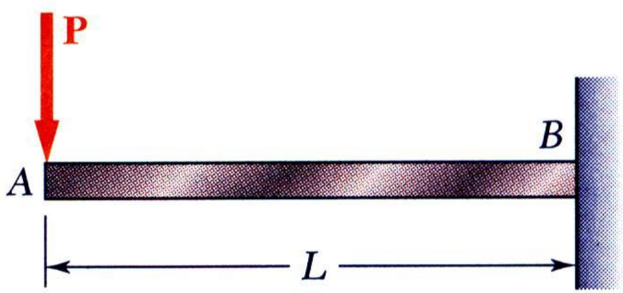
- For a rod of uniform cross-section,

$$U = \frac{P^2 L}{2AE}$$

Elastic Strain Energy for Normal Stresses



$$\sigma_x = \frac{My}{I}$$



- For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting $dV = dA\, dx$,

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\, dx = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx$$

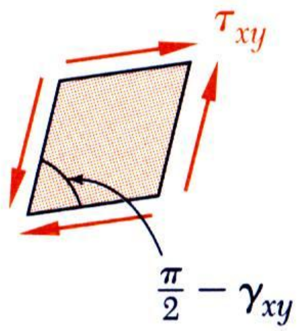
$$= \int_0^L \frac{M^2}{2EI} dx$$

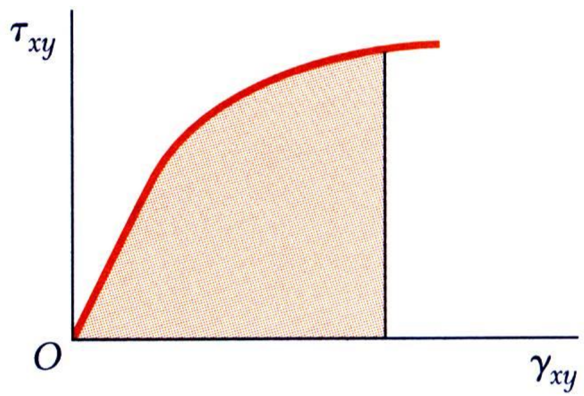
- For an end-loaded cantilever beam,

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

Strain Energy For Shearing Stresses





- For a material subjected to plane shearing stresses,

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}$$

- For values of τ_{xy} within the proportional limit,

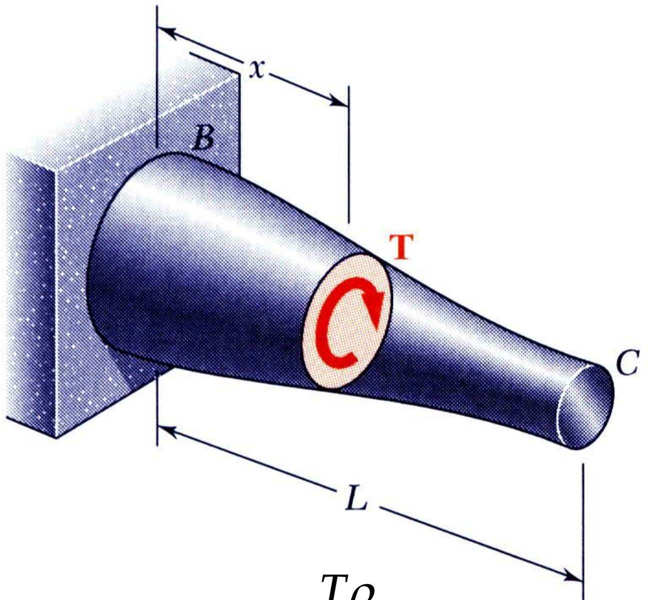
$$u = \tfrac{1}{2} G\gamma_{xy}^2 = \tfrac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

- The total strain energy is found from

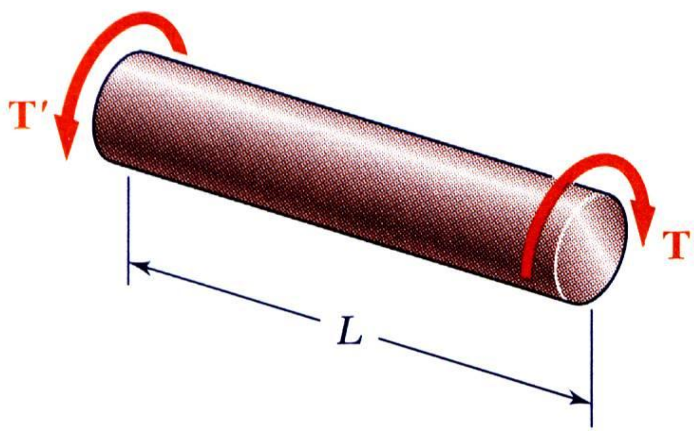
$$U = \int u\, dV$$

$$= \int \frac{\tau_{xy}^2}{2G}\, dV$$

Strain Energy For Shearing Stresses



$$\tau_{xy} = \frac{T\rho}{J}$$



- For a shaft subjected to a torsional load,

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GJ^2} dV$$

- Setting $dV = dA\,dx$,

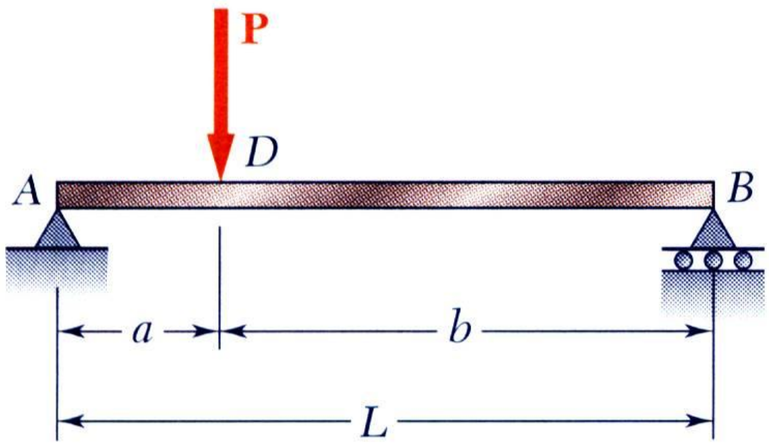
$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} dA\,dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

$$= \int_0^L \frac{T^2}{2GJ} dx$$

- In the case of a uniform shaft,

$$U = \frac{T^2 L}{2GJ}$$

Sample Problem 11.2



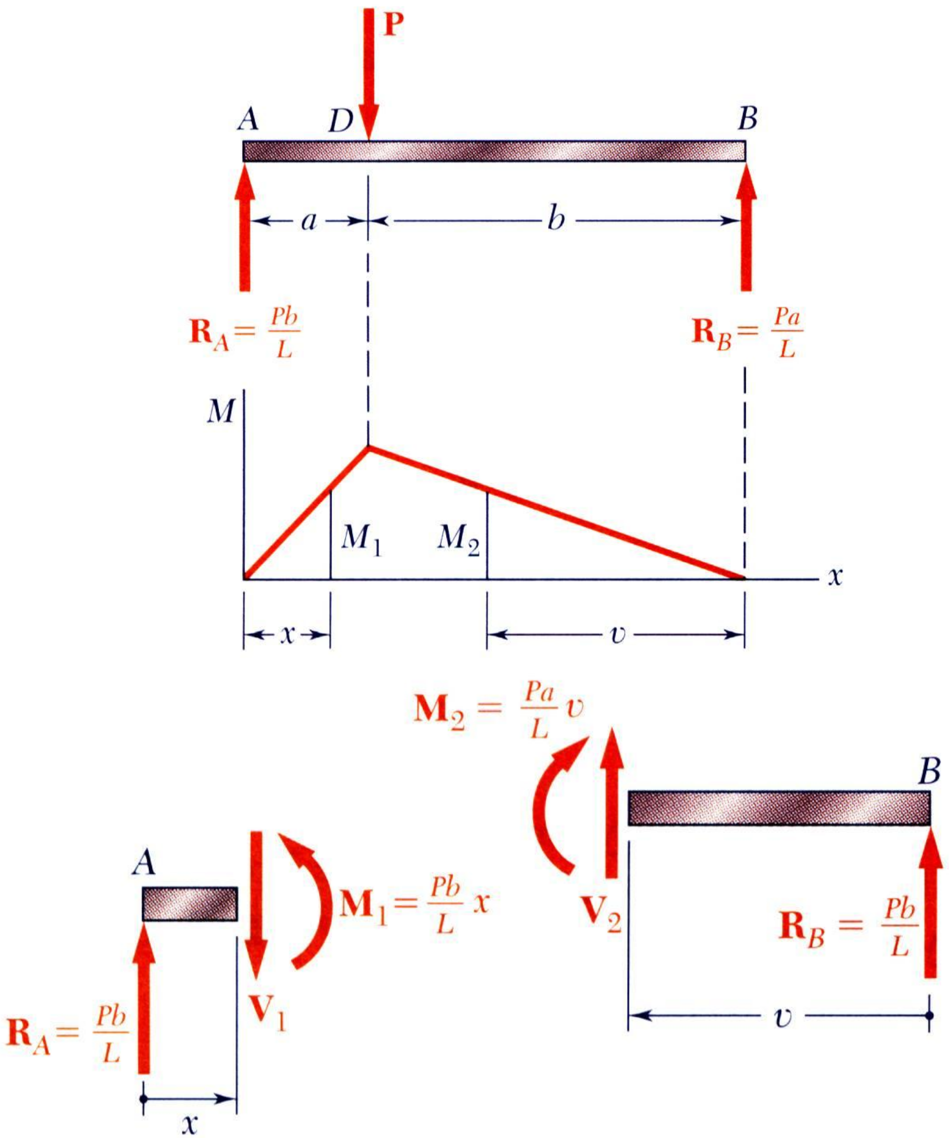
a) Taking into account only the normal stresses due to bending, determine the strain energy of the beam for the loading shown.

b) Evaluate the strain energy knowing that the beam is a W10x45, P = 40 kips, L = 12 ft, a = 3 ft, b = 9 ft, and E = 29x10^6 psi.

SOLUTION:

- Determine the reactions at A and B from a free-body diagram of the complete beam.
- Develop a diagram of the bending moment distribution.
- Integrate over the volume of the beam to find the strain energy.
- Apply the particular given conditions to evaluate the strain energy.

Sample Problem 11.2



SOLUTION:

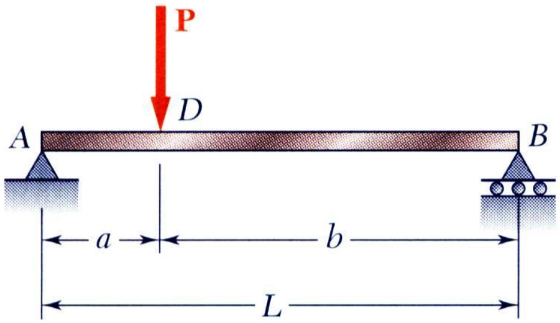
- Determine the reactions at *A* and *B* from a free-body diagram of the complete beam.

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

- Develop a diagram of the bending moment distribution.

$$M_1 = \frac{Pb}{L} x \qquad M_2 = \frac{Pa}{L} v$$

Sample Problem 11.2



Over the portion AD,

$$M_1 = \frac{Pb}{L} x$$

Over the portion BD,

$$M_2 = \frac{Pa}{L} v$$

$P = 45\,\text{kips}$ $\quad L = 144\,\text{in.}$

$a = 36\,\text{in.}$ $\quad b = 108\,\text{in.}$

$E = 29 \times 10^3\,\text{ksi}$ $\quad I = 248\,\text{in}^4$

- Integrate over the volume of the beam to find the strain energy.

$$U = \int_0^a \frac{M_1^2}{2EI} dx + \int_0^b \frac{M_2^2}{2EI} dv$$

$$= \frac{1}{2EI} \int_0^a \left(\frac{Pb}{L} x \right)^2 dx + \frac{1}{2EI} \int_0^b \left(\frac{Pa}{L} x \right)^2 dx$$

$$= \frac{1}{2EI} \frac{P^2}{L^2} \left(\frac{b^2 a^3}{3} + \frac{a^2 b^3}{3} \right) = \frac{P^2 a^2 b^2}{6EIL^2} (a + b)$$

$$U = \frac{P^2 a^2 b^2}{6EIL}$$

$$U = \frac{(40\,\text{kips})^2 (36\,\text{in})^2 (108\,\text{in})^2}{6 (29 \times 10^3\,\text{ksi})(248\,\text{in}^4)(144\,\text{in})}$$

$$U = 3.89\,\text{in} \cdot \text{kips}$$

Strain Energy for a General State of Stress

- Previously found strain energy due to uniaxial stress and plane shearing stress. For a general state of stress,

$$u = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

- With respect to the principal axes for an elastic, isotropic body,

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right]$$

$$= u_v + u_d$$

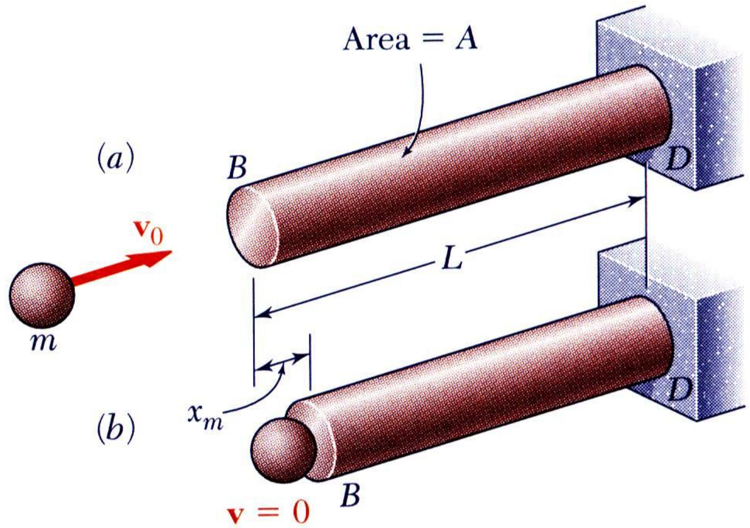
$$u_v = \frac{1-2\nu}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2 = \text{due to volume change}$$

$$u_d = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right] = \text{due to distortion}$$

- Basis for the *maximum distortion energy* failure criteria,

$$u_d < \left(u_d\right)_Y = \frac{\sigma_Y^2}{6G} \quad \text{for a tensile test specimen}$$

Impact Loading



- Consider a rod which is hit at its end with a body of mass *m* moving with a velocity v_0.
- Rod deforms under impact. Stresses reach a maximum value σ_m and then disappear.

- To determine the maximum stress σ_m
  - Assume that the kinetic energy is transferred entirely to the structure,

$$U_m = \frac{1}{2} m v_0^2$$

  - Assume that the stress-strain diagram obtained from a static test is also valid under impact loading.

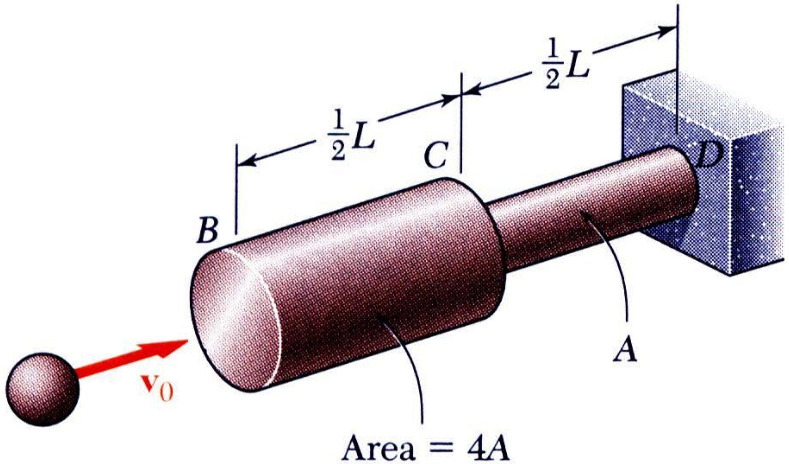
- Maximum value of the strain energy,

$$U_m = \int \frac{\sigma_m^2}{2E} dV$$

- For the case of a uniform rod,

$$\sigma_m = \sqrt{\frac{2U_m E}{V}} = \sqrt{\frac{m v_0^2 E}{V}}$$

Example 11.06

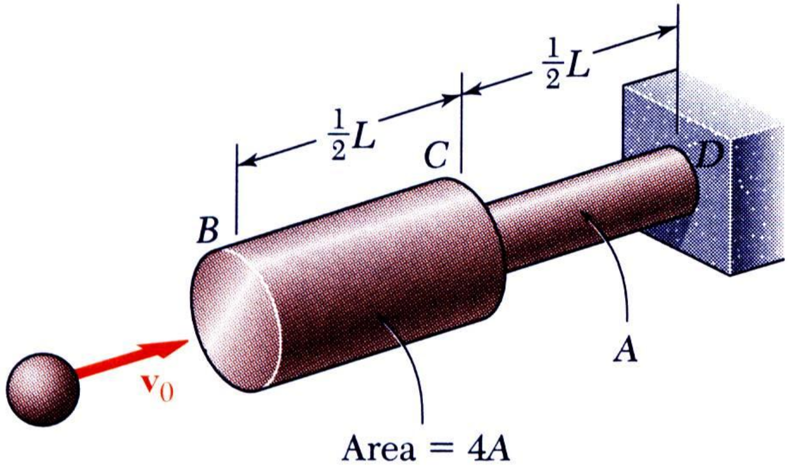


Body of mass m with velocity v_0 hits the end of the nonuniform rod BCD. Knowing that the diameter of the portion BC is twice the diameter of portion CD, determine the maximum value of the normal stress in the rod.

SOLUTION:

- Due to the change in diameter, the normal stress distribution is nonuniform.
- Find the static load P_m which produces the same strain energy as the impact.
- Evaluate the maximum stress resulting from the static load P_m

Example 11.06



SOLUTION:

- Due to the change in diameter, the normal stress distribution is nonuniform.

$$U_m = \tfrac{1}{2} m v_0^2$$

$$= \int \frac{\sigma_m^2}{2E} dV \neq \frac{\sigma_m^2 V}{2E}$$

- Find the static load P_m which produces the same strain energy as the impact.

$$U_m = \frac{P_m^2 (L/2)}{AE} + \frac{P_m^2 (L/2)}{4AE} = \frac{5}{16} \frac{P_m^2 L}{AE}$$

$$P_m = \sqrt{\frac{16}{5} \frac{U_m AE}{L}}$$

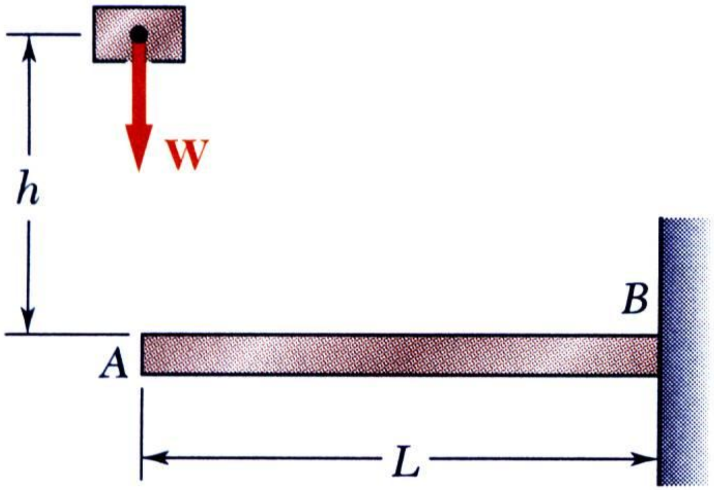
- Evaluate the maximum stress resulting from the static load P_m

$$\sigma_m = \frac{P_m}{A}$$

$$= \sqrt{\frac{16}{5} \frac{U_m E}{AL}}$$

$$= \sqrt{\frac{8}{5} \frac{m v_0^2 E}{AL}}$$

Example 11.07

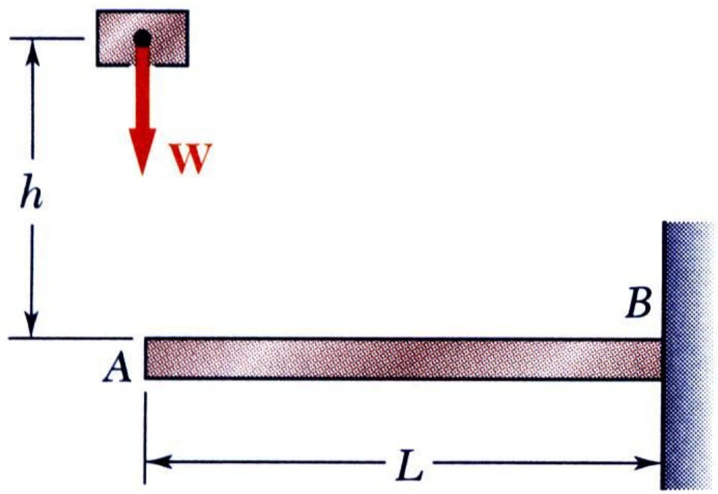


A block of weight W is dropped from a height h onto the free end of the cantilever beam. Determine the maximum value of the stresses in the beam.

SOLUTION:

- The normal stress varies linearly along the length of the beam as across a transverse section.
- Find the static load P_m which produces the same strain energy as the impact.
- Evaluate the maximum stress resulting from the static load P_m

Example 11.07



SOLUTION:

- The normal stress varies linearly along the length of the beam as across a transverse section.

$$U_m = Wh$$

$$= \int \frac{\sigma_m^2}{2E} dV \neq \frac{\sigma_m^2 V}{2E}$$

- Find the static load P_m which produces the same strain energy as the impact.

  For an end-loaded cantilever beam,

$$U_m = \frac{P_m^2 L^3}{6EI}$$

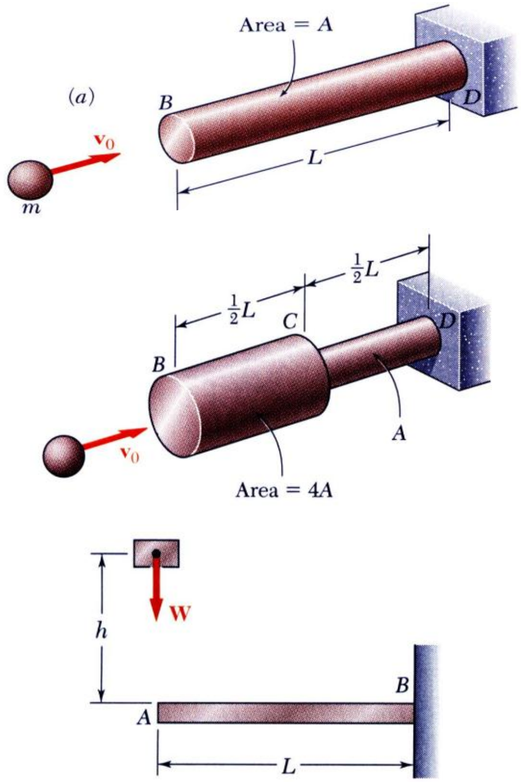
$$P_m = \sqrt{\frac{6 U_m EI}{L^3}}$$

- Evaluate the maximum stress resulting from the static load P_m

$$\sigma_m = \frac{|M|_m c}{I} = \frac{P_m L c}{I}$$

$$= \sqrt{\frac{6 U_m E}{L\left(I/c^2\right)}} = \sqrt{\frac{6WhE}{L\left(I/c^2\right)}}$$

Design for Impact Loads



Maximum stress reduced by:

- uniformity of stress
- low modulus of elasticity with high yield strength
- high volume

- For the case of a uniform rod,

$$\sigma_m = \sqrt{\frac{2U_m E}{V}}$$

- For the case of the nonuniform rod,

$$\sigma_m = \sqrt{\frac{16}{5}\frac{U_m E}{AL}}$$

$$V = 4A(L/2) + A(L/2) = 5AL/2$$

$$\sigma_m = \sqrt{\frac{8U_m E}{V}}$$

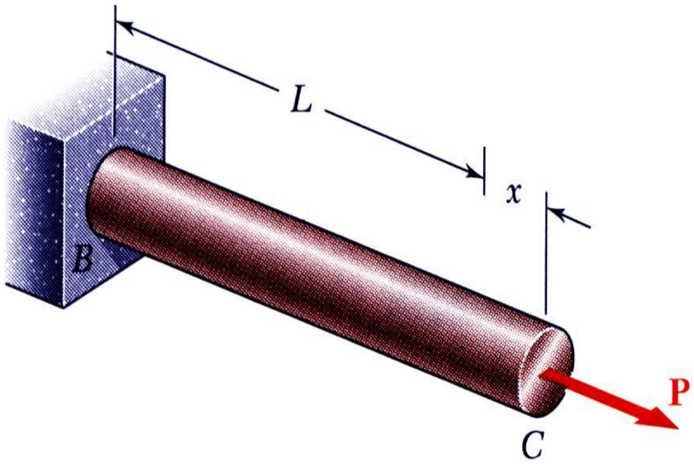
- For the case of the cantilever beam

$$\sigma_m = \sqrt{\frac{6U_m E}{L\left(I/c^2\right)}}$$

$$L\left(I/c^2\right) = L\left(\tfrac{1}{4}\pi c^4/c^2\right) = \tfrac{1}{4}\left(\pi c^2 L\right) = \tfrac{1}{4}V$$

$$\sigma_m = \sqrt{\frac{24U_m E}{V}}$$

Work and Energy Under a Single Load



- Previously, we found the strain energy by integrating the energy density over the volume. For a uniform rod,

$$U = \int u\,dV = \int \frac{\sigma^2}{2E}\,dV$$

$$= \int_0^L \frac{(P_1/A)^2}{2E}\,A\,dx = \frac{P_1^2 L}{2AE}$$

- Strain energy may also be found from the work of the single load P_1,

$$U = \int_0^{x_1} P\,dx$$

- For an elastic deformation,

$$U = \int_0^{x_1} P\,dx = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}P_1 x_1$$

- Knowing the relationship between force and displacement,

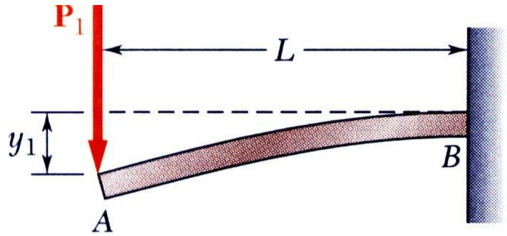
$$x_1 = \frac{P_1 L}{AE}$$

$$U = \tfrac{1}{2}P_1\left(\frac{P_1 L}{AE}\right) = \frac{P_1^2 L}{2AE}$$

Work and Energy Under a Single Load

- Strain energy may be found from the work of other types of single concentrated loads.

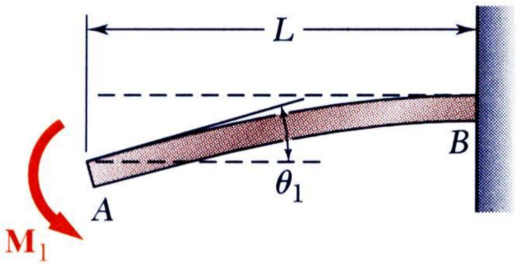
- Transverse load



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2} P_1 y_1$$

$$= \tfrac{1}{2} P_1 \left(\frac{P_1 L^3}{3EI} \right) = \frac{P_1^2 L^3}{6EI}$$

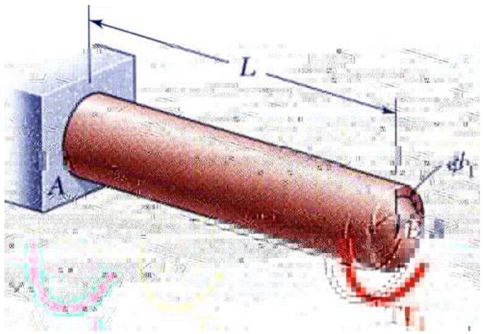
- Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2} M_1 \theta_1$$

$$= \tfrac{1}{2} M_1 \left(\frac{M_1 L}{EI} \right) = \frac{M_1^2 L}{2EI}$$

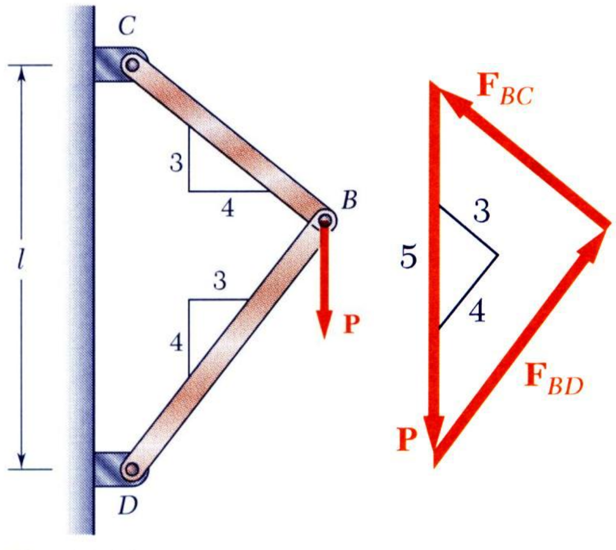
- Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2} T_1 \phi_1$$

$$= \tfrac{1}{2} T_1 \left(\frac{T_1 L}{JG} \right) = \frac{T_1^2 L}{2JG}$$

Deflection Under a Single Load



From the given geometry,

$$L_{BC} = 0.6l \quad L_{BD} = 0.8l$$

From statics,

$$F_{BC} = +0.6P \quad F_{BD} = -0.8P$$

- If the strain energy of a structure due to a single concentrated load is known, then the equality between the work of the load and energy may be used to find the deflection.

- Strain energy of the structure,

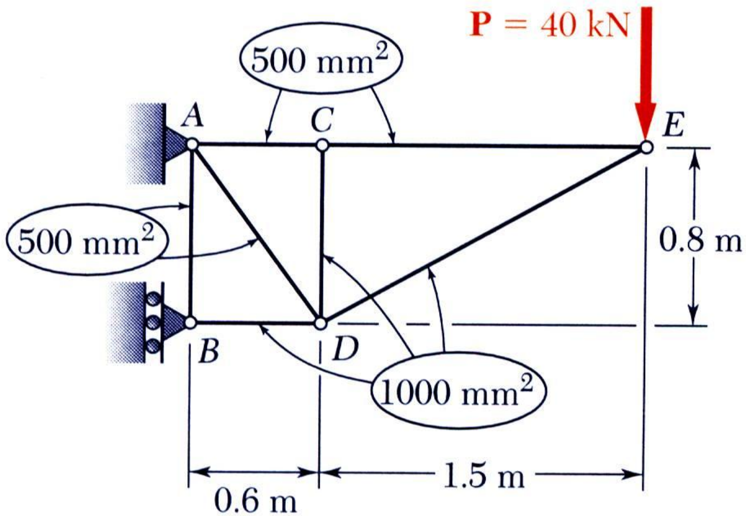
$$U = \frac{F_{BC}^2 L_{BC}}{2AE} + \frac{F_{BD}^2 L_{BD}}{2AE}$$
$$= \frac{P^2 l\left[(0.6)^3 + (0.8)^3\right]}{2AE} = 0.364\frac{P^2 l}{AE}$$

- Equating work and strain energy,

$$U = 0.364\frac{P^2 L}{AE} = \tfrac{1}{2} P y_B$$

$$y_B = 0.728\frac{Pl}{AE}$$

Sample Problem 11.4

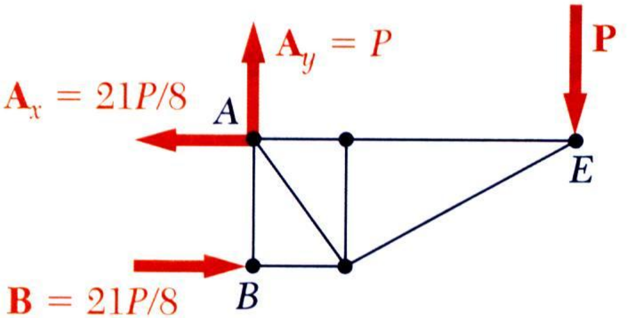


Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using E = 73 GPa, determine the vertical deflection of the point E caused by the load P.

SOLUTION:

- Find the reactions at A and B from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member.
- Evaluate the strain energy of the truss due to the load P.
- Equate the strain energy to the work of P and solve for the displacement.

Sample Problem 11.4

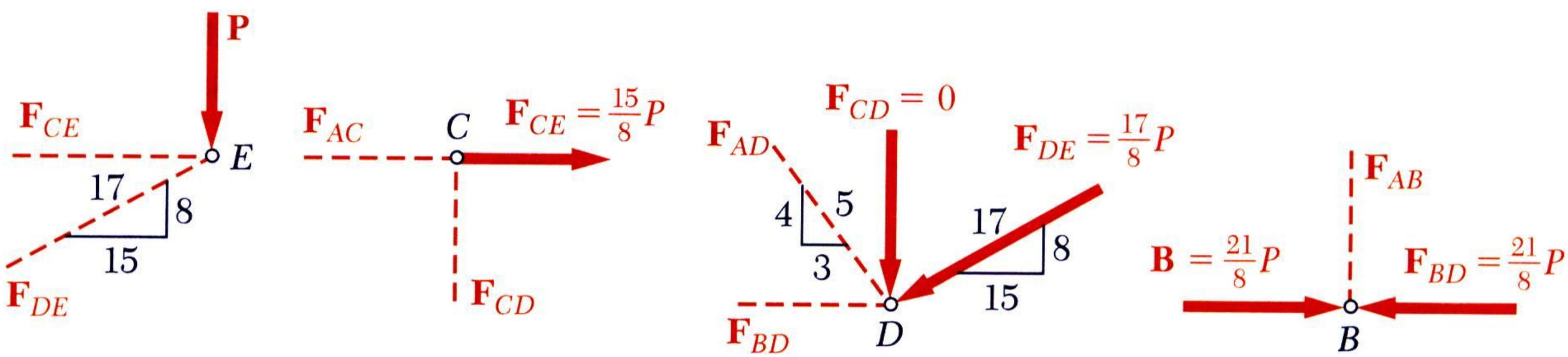


SOLUTION:

- Find the reactions at A and B from a free-body diagram of the entire truss.

$$A_x = -21P/8 \qquad A_y = P \qquad B = 21P/8$$

- Apply the method of joints to determine the axial force in each member.



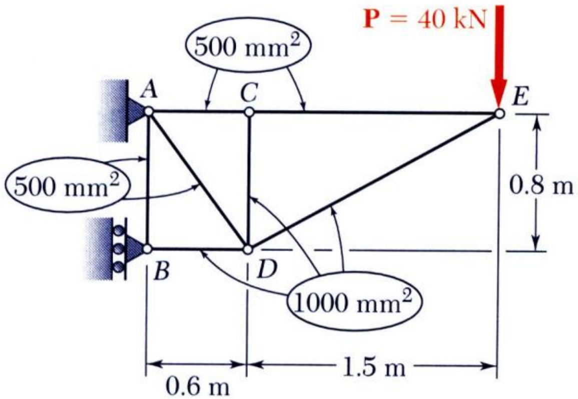
$$F_{DE} = -\frac{17}{8}P \qquad F_{CE} = +\frac{15}{8}P$$

$$F_{AC} = +\frac{15}{8}P \qquad F_{CD} = 0$$

$$F_{DE} = \frac{5}{4}P \qquad F_{CE} = -\frac{21}{8}P$$

$$F_{AB} = 0$$

Sample Problem 11.4



Member	F_i	L_i, m	A_i, m²	$\frac{F_i^2 L_i}{A_i}$
AB	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0.6	500×10^{-6}	$4\,219P^2$
AD	$+5P/4$	1.0	500×10^{-6}	$3\,125P^2$
BD	$-21P/8$	0.6	1000×10^{-6}	$4\,134P^2$
CD	0	0.8	1000×10^{-6}	0
CE	$+15P/8$	1.5	500×10^{-6}	$10\,547P^2$
DE	$-17P/8$	1.7	1000×10^{-6}	$7\,677P^2$

- Evaluate the strain energy of the truss due to the load *P*.

$$U = \sum \frac{F_i^2 L_i}{2A_i E} = \frac{1}{2E} \sum \frac{F_i^2 L_i}{A_i}$$

$$= \frac{1}{2E}\left(29700P^2\right)$$

- Equate the strain energy to the work by *P* and solve for the displacement.

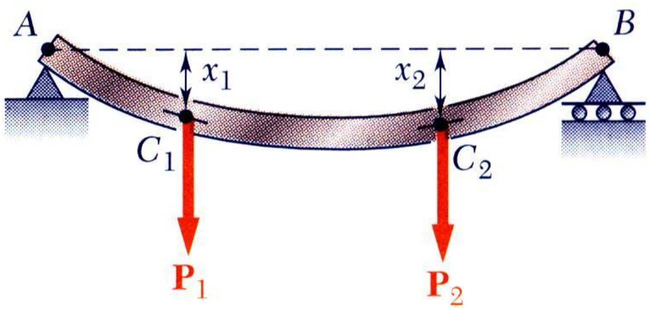
$$\tfrac{1}{2} P y_E = U$$

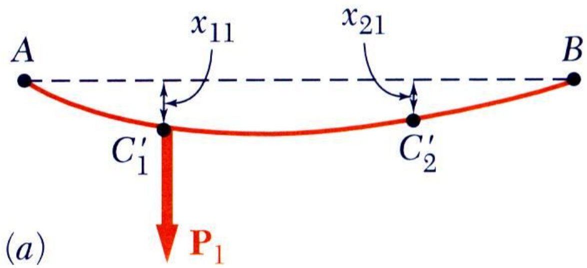
$$y_E = \frac{2U}{P} = \frac{2}{P}\left(\frac{29700P^2}{2E}\right)$$

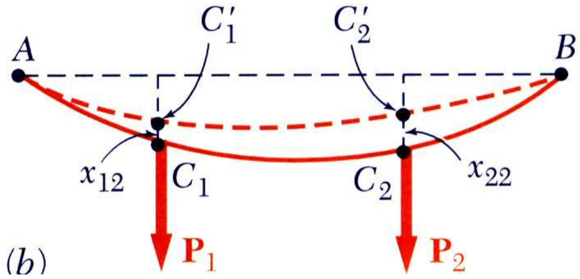
$$y_E = \frac{\left(29.7\times10^3\right)\left(40\times10^3\right)}{73\times10^9}$$

$$y_E = 16.27\,\text{mm} \downarrow$$

Work and Energy Under Several Loads







- Deflections of an elastic beam subjected to two concentrated loads,

$$x_1 = x_{11} + x_{12} = \alpha_{11} P_1 + \alpha_{12} P_2$$

$$x_2 = x_{21} + x_{22} = \alpha_{21} P_1 + \alpha_{22} P_2$$

- Compute the strain energy in the beam by evaluating the work done by slowly applying P_1 followed by P_2,

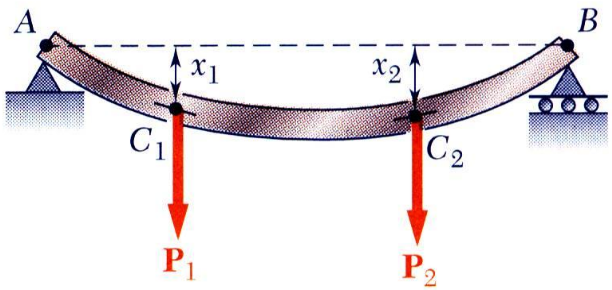
$$U = \frac{1}{2}\left(\alpha_{11} P_1^2 + 2\alpha_{12} P_1 P_2 + \alpha_{22} P_2^2\right)$$

- Reversing the application sequence yields

$$U = \frac{1}{2}\left(\alpha_{22} P_2^2 + 2\alpha_{21} P_2 P_1 + \alpha_{11} P_1^2\right)$$

- Strain energy expressions must be equivalent. It follows that $\alpha_{12}=\alpha_{21}$ (*Maxwell's reciprocal theorem*).

Castigliano's Theorem



- Strain energy for any elastic structure subjected to two concentrated loads,

$$U = \tfrac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

- Differentiating with respect to the loads,

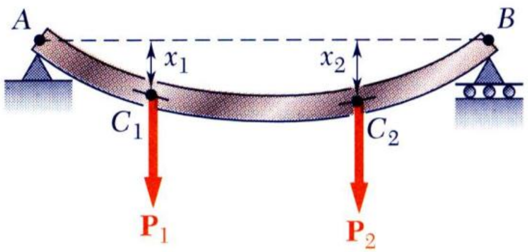
$$\frac{\partial U}{\partial P_1} = \alpha_{11}P_1 + \alpha_{12}P_2 = x_1$$

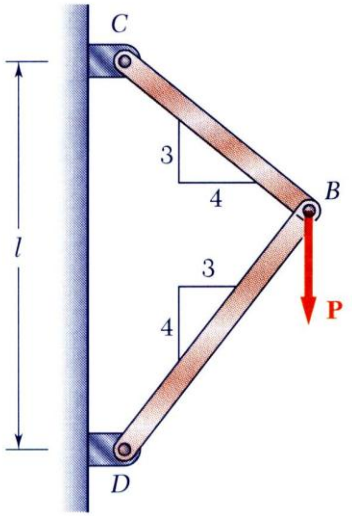
$$\frac{\partial U}{\partial P_2} = \alpha_{12}P_1 + \alpha_{22}P_2 = x_2$$

- *Castigliano's theorem*: For an elastic structure subjected to n loads, the deflection x_j of the point of application of P_j can be expressed as

$$x_j = \frac{\partial U}{\partial P_j} \quad \text{and} \quad \theta_j = \frac{\partial U}{\partial M_j} \qquad \phi_j = \frac{\partial U}{\partial T_j}$$

Deflections by Castigliano's Theorem





- Application of Castigliano's theorem is simplified if the differentiation with respect to the load P_j is performed before the integration or summation to obtain the strain energy U.

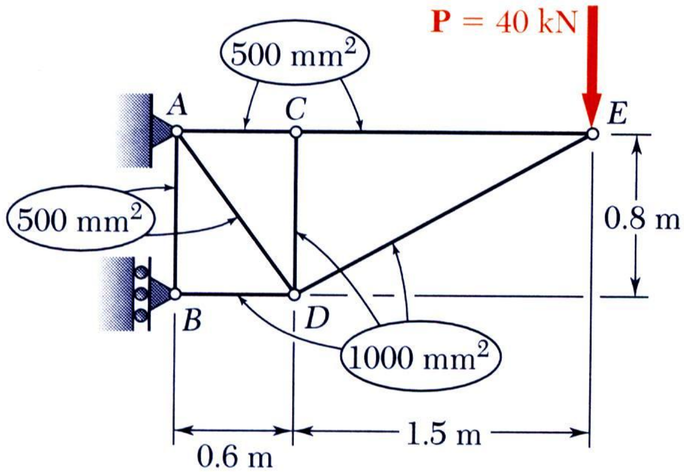
- In the case of a beam,

$$U = \int_0^L \frac{M^2}{2EI}\,dx \qquad x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_j}\,dx$$

- For a truss,

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E} \qquad x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E}\frac{\partial F_i}{\partial P_j}$$

Sample Problem 11.5

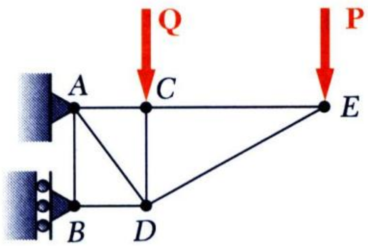


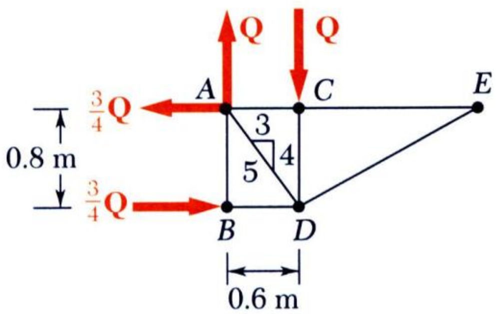
Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using $E = 73$ GPa, determine the vertical deflection of the joint C caused by the load P.

SOLUTION:

- For application of Castigliano's theorem, introduce a dummy vertical load Q at C. Find the reactions at A and B due to the dummy load from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member due to Q.
- Combine with the results of Sample Problem 11.4 to evaluate the derivative with respect to Q of the strain energy of the truss due to the loads P and Q.
- Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at C.

Sample Problem 11.5



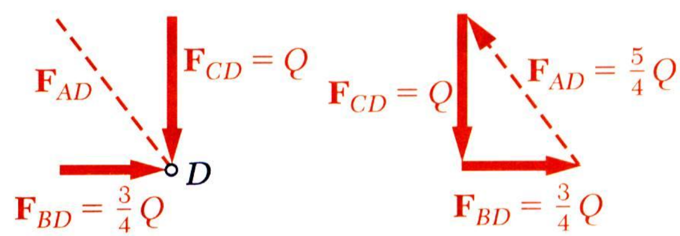


SOLUTION:

- Find the reactions at A and B due to a dummy load Q at C from a free-body diagram of the entire truss.

$$A_x = -\frac{3}{4}Q \qquad A_y = Q \qquad B = \frac{3}{4}Q$$

- Apply the method of joints to determine the axial force in each member due to Q.

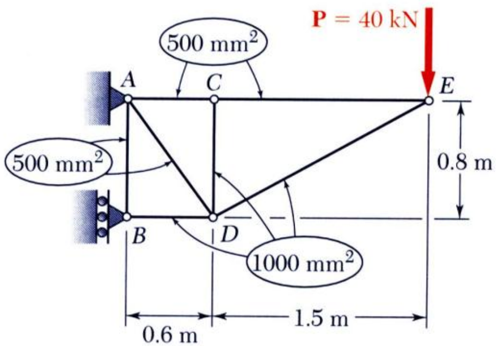


$$F_{CE} = F_{DE} = 0$$

$$F_{AC} = 0;\ F_{CD} = -Q$$

$$F_{AB} = 0;\ F_{BD} = -\frac{3}{4}Q$$

Sample Problem 11.5



Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_i L_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4 + 5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P + 3125Q$
BD	$-21P/8 - 3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P + 338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+ 800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

- Combine with the results of Sample Problem 11.4 to evaluate the derivative with respect to Q of the strain energy of the truss due to the loads P and Q.

$$y_C = \sum\left(\frac{F_i L_i}{A_i E}\right)\frac{\partial F_i}{\partial Q} = \frac{1}{E}(4306P + 4263Q)$$

- Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at C.

$$y_C = \frac{4306\left(40 \times 10^3 N\right)}{73 \times 10^9 \text{Pa}}$$

$$\boxed{y_C = 2.36\ \text{mm} \downarrow}$$