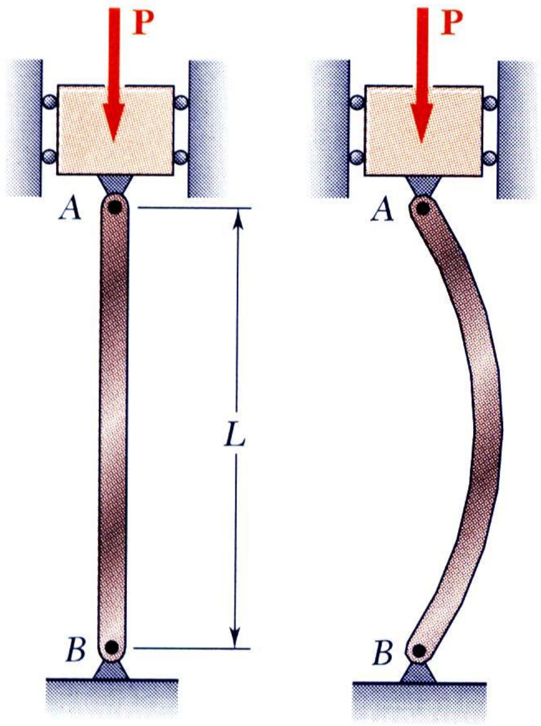
Chapter # 10

Columns

Columns

Stability of Structures



- In the design of columns, cross-sectional area is selected such that
  - allowable stress is not exceeded

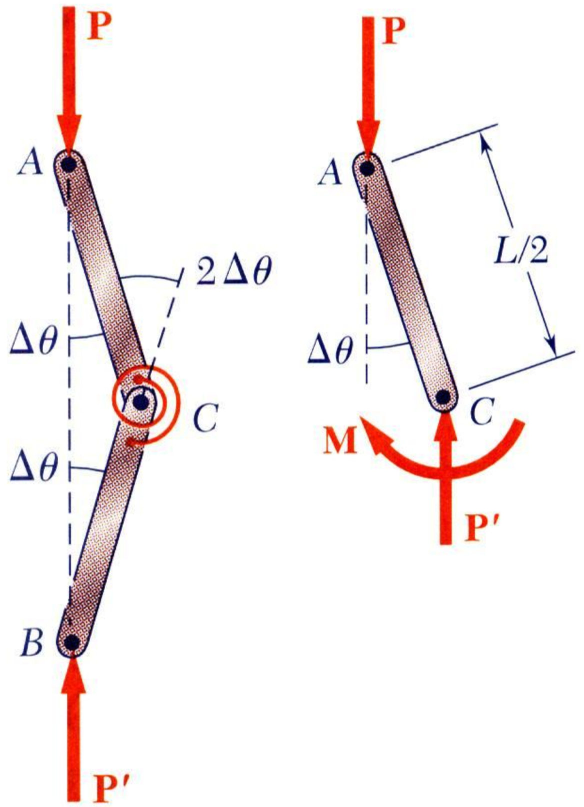
$$\sigma = \frac{P}{A} \leq \sigma_{all}$$

  - deformation falls within specifications

$$\delta = \frac{PL}{AE} \leq \delta_{spec}$$

- After these design calculations, may discover that the column is unstable under loading and that it suddenly becomes sharply curved or buckles.

Stability of Structures



- Consider model with two rods and torsional spring. After a small perturbation,

$$K(2\Delta\theta) = \text{restoring moment}$$

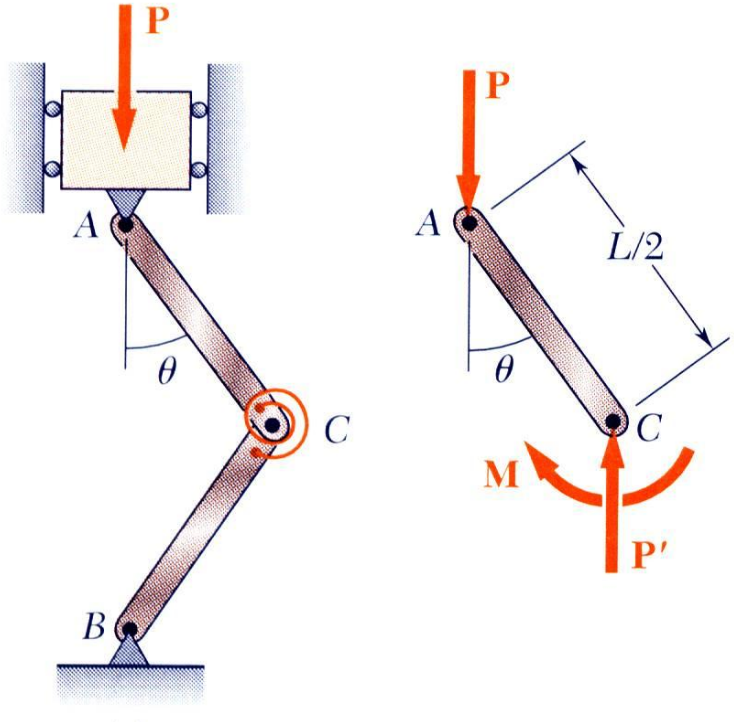
$$P\frac{L}{2}\sin\Delta\theta = P\frac{L}{2}\Delta\theta = \text{destabilizing moment}$$

- Column is stable (tends to return to aligned orientation) if

$$P\frac{L}{2}\Delta\theta < K(2\Delta\theta)$$

$$P < P_{cr} = \frac{4K}{L}$$

Stability of Structures



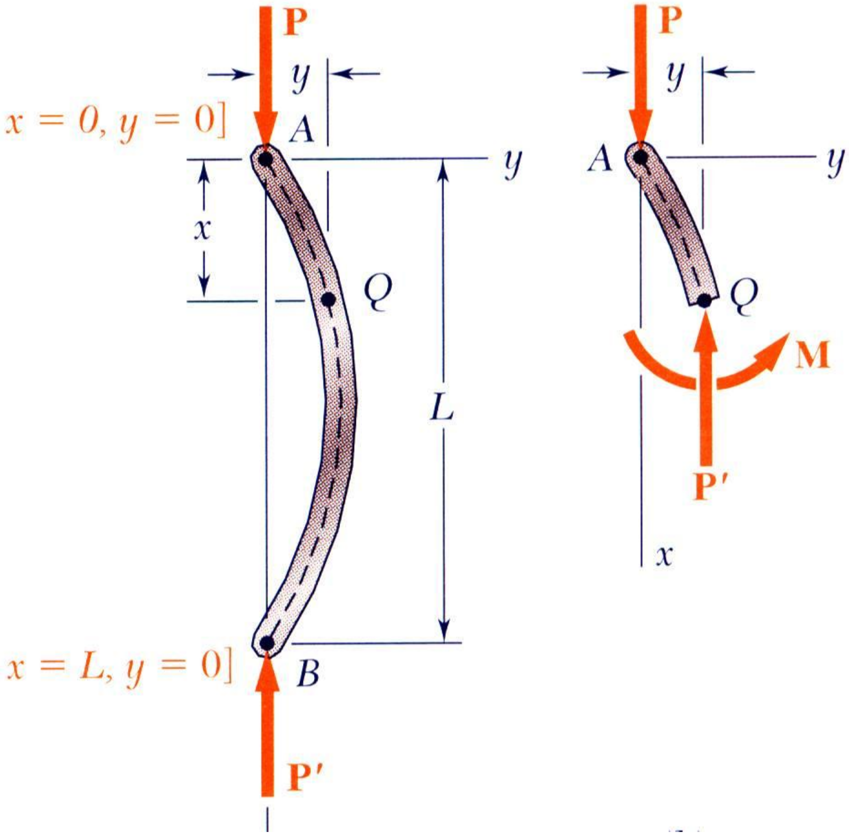
- Assume that a load P is applied. After a perturbation, the system settles to a new equilibrium configuration at a finite deflection angle.

$$P\frac{L}{2}\sin\theta = K(2\theta)$$

$$\frac{PL}{4K} = \frac{P}{P_{cr}} = \frac{\theta}{\sin\theta}$$

- Noting that $\sin\theta < \theta$, the assumed configuration is only possible if $P > P_{cr}$.

Euler's Formula for Pin-Ended Beams



- Consider an axially loaded beam. After a small perturbation, the system reaches an equilibrium configuration such that

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y$$

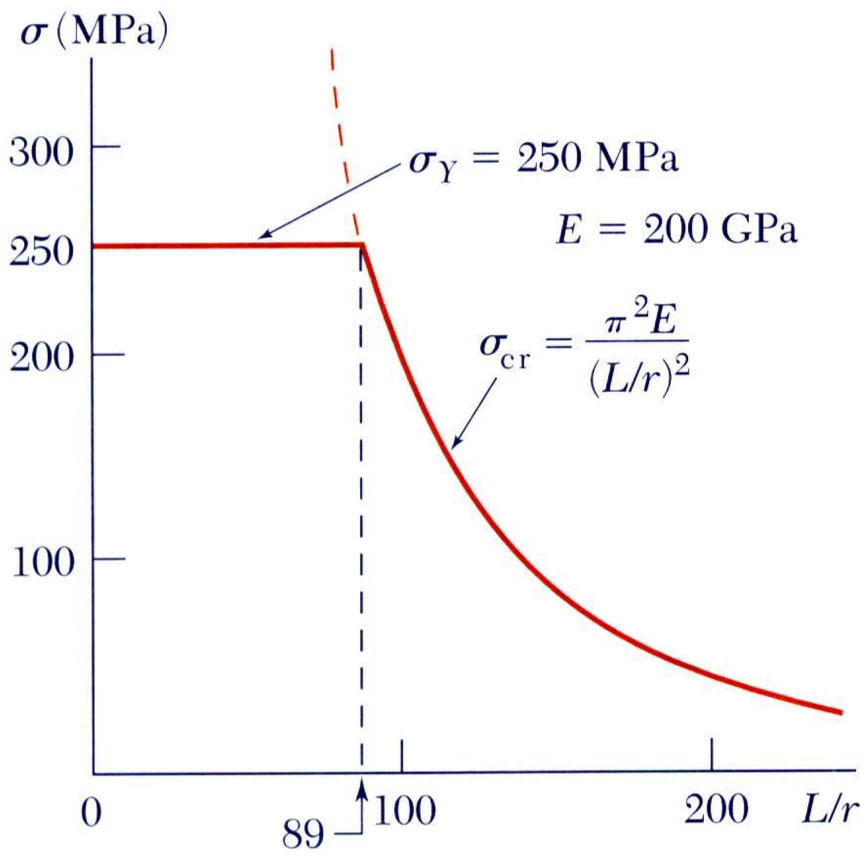
$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

- Solution with assumed configuration can only be obtained if

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2}$$

Euler's Formula for Pin-Ended Beams



- The value of stress corresponding to the critical load,

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{P_{cr}}{A}$$

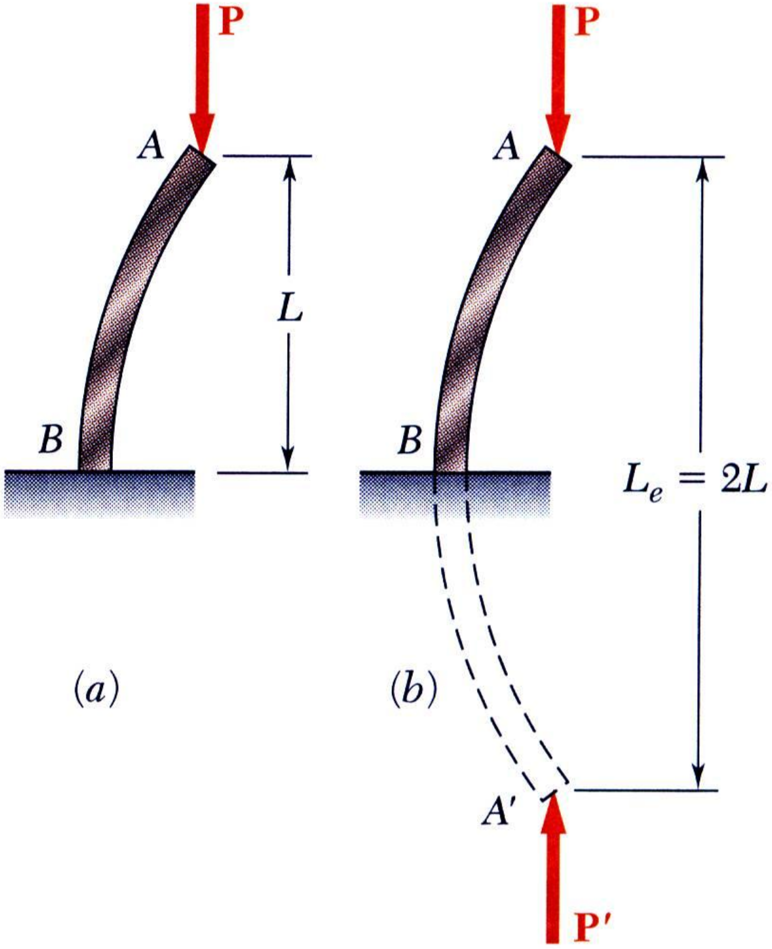
$$\sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A}$$

$$= \frac{\pi^2 E}{(L/r)^2} = \textit{critical stress}$$

$$\frac{L}{r} = \textit{slenderness ratio}$$

- Preceding analysis is limited to centric loadings.

Extension of Euler's Formula



- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.

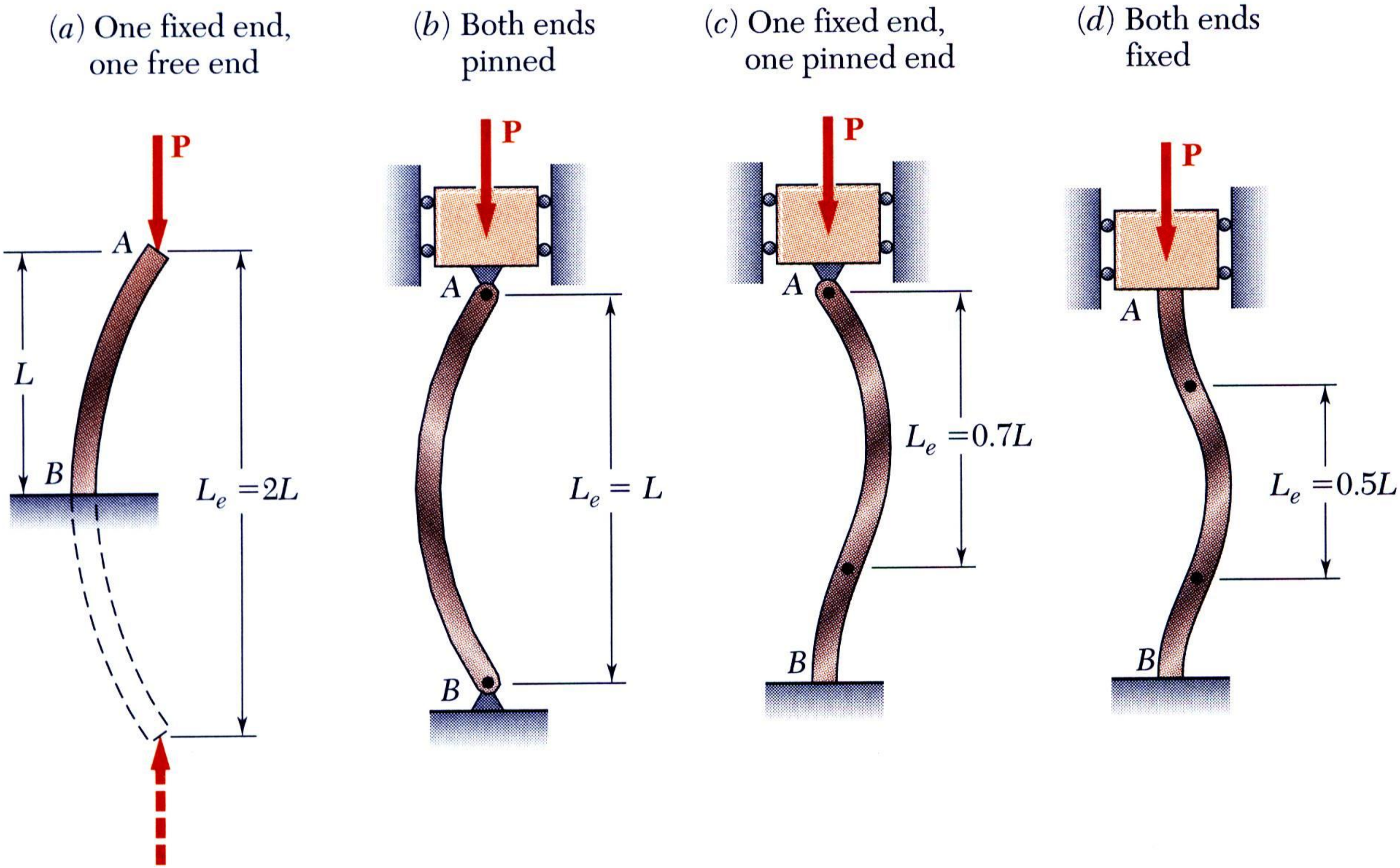
- The critical loading is calculated from Euler's formula,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

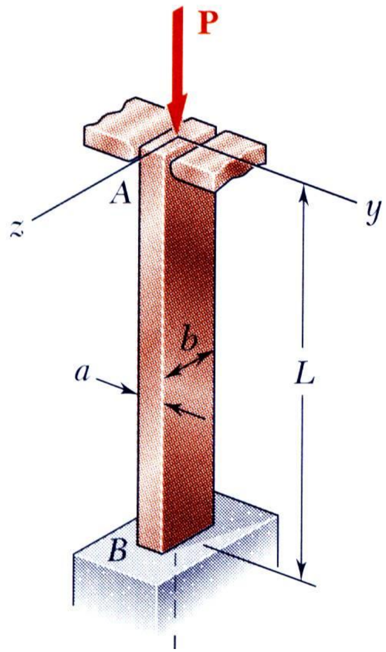
$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

$$L_e = 2L = \text{equivalent length}$$

Extension of Euler's Formula



Sample Problem 10.1



L = 20 in.

E = 10.1 x 10^6 psi

P = 5 kips

FS = 2.5

An aluminum column of length L and rectangular cross-section has a fixed end at B and supports a centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

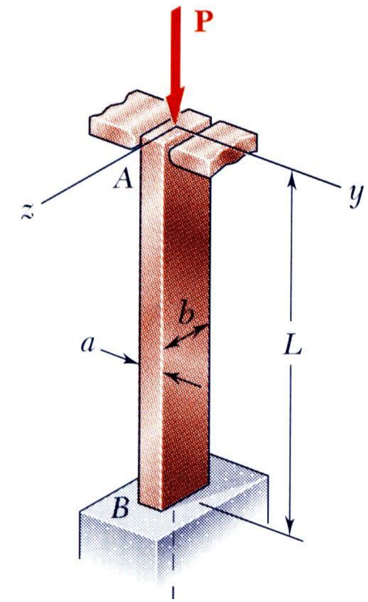
a) Determine the ratio a/b of the two sides of the cross-section corresponding to the most efficient design against buckling.

b) Design the most efficient cross-section for the column.

Sample Problem 10.1

SOLUTION:

The most efficient design occurs when the resistance to buckling is equal in both planes of symmetry. This occurs when the slenderness ratios are equal.



- Buckling in xy Plane:

$$r_z^2 = \frac{I_z}{A} = \frac{\frac{1}{12}ba^3}{ab} = \frac{a^2}{12} \qquad r_z = \frac{a}{\sqrt{12}}$$

$$\frac{L_{e,z}}{r_z} = \frac{0.7L}{a/\sqrt{12}}$$

- Buckling in xz Plane:

$$r_y^2 = \frac{I_y}{A} = \frac{\frac{1}{12}ab^3}{ab} = \frac{b^2}{12} \qquad r_y = \frac{b}{\sqrt{12}}$$

$$\frac{L_{e,y}}{r_y} = \frac{2L}{b/\sqrt{12}}$$

- Most efficient design:

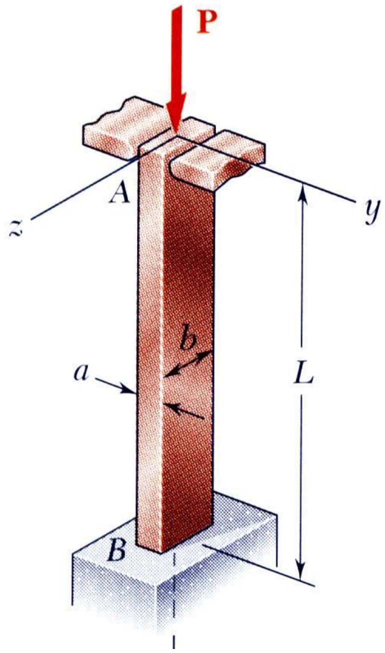
$$\frac{L_{e,z}}{r_z} = \frac{L_{e,y}}{r_y}$$

$$\frac{0.7L}{a/\sqrt{12}} = \frac{2L}{b/\sqrt{12}}$$

$$\frac{a}{b} = \frac{0.7}{2}$$

$$\boxed{\frac{a}{b} = 0.35}$$

Sample Problem 10.1



L = 20 in.

E = 10.1 x 10^6 psi

P = 5 kips

FS = 2.5

a/b = 0.35

- Design:

$$\frac{L_e}{r_y} = \frac{2L}{b/\sqrt{12}} = \frac{2(20\text{ in})}{b/\sqrt{12}} = \frac{138.6}{b}$$

$$P_{cr} = (FS)P = (2.5)(5\text{ kips}) = 12.5\text{ kips}$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{12500\text{ lbs}}{(0.35b)b}$$

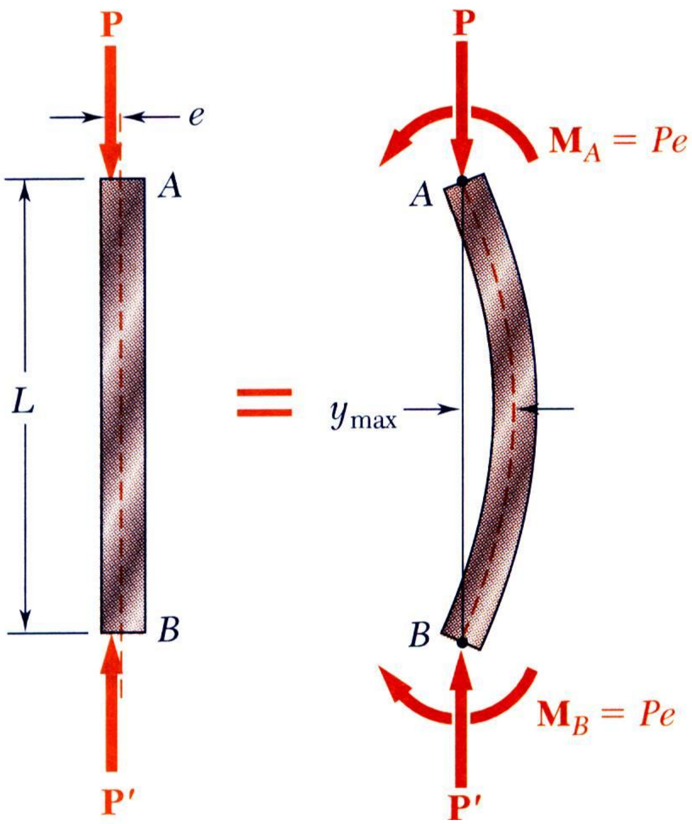
$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} = \frac{\pi^2(10.1\times10^6\text{ psi})}{(138.6/b)^2}$$

$$\frac{12500\text{ lbs}}{(0.35b)b} = \frac{\pi^2(10.1\times10^6\text{ psi})}{(138.6/b)^2}$$

$$b = 1.620\text{ in.}$$

$$a = 0.35b = 0.567\text{ in.}$$

Eccentric Loading; The Secant Formula



- Eccentric loading is equivalent to a centric load and a couple.
- Bending occurs for any nonzero eccentricity. Question of buckling becomes whether the resulting deflection is excessive.
- The deflection become infinite when $P = P_{cr}$

$$\frac{d^2y}{dx^2} = \frac{-Py - Pe}{EI}$$

$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

- Maximum stress

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{(y_{max} + e)c}{r^2}\right]$$

$$= \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L_e}{r}\right)\right]$$

Eccentric Loading; The Secant Formula

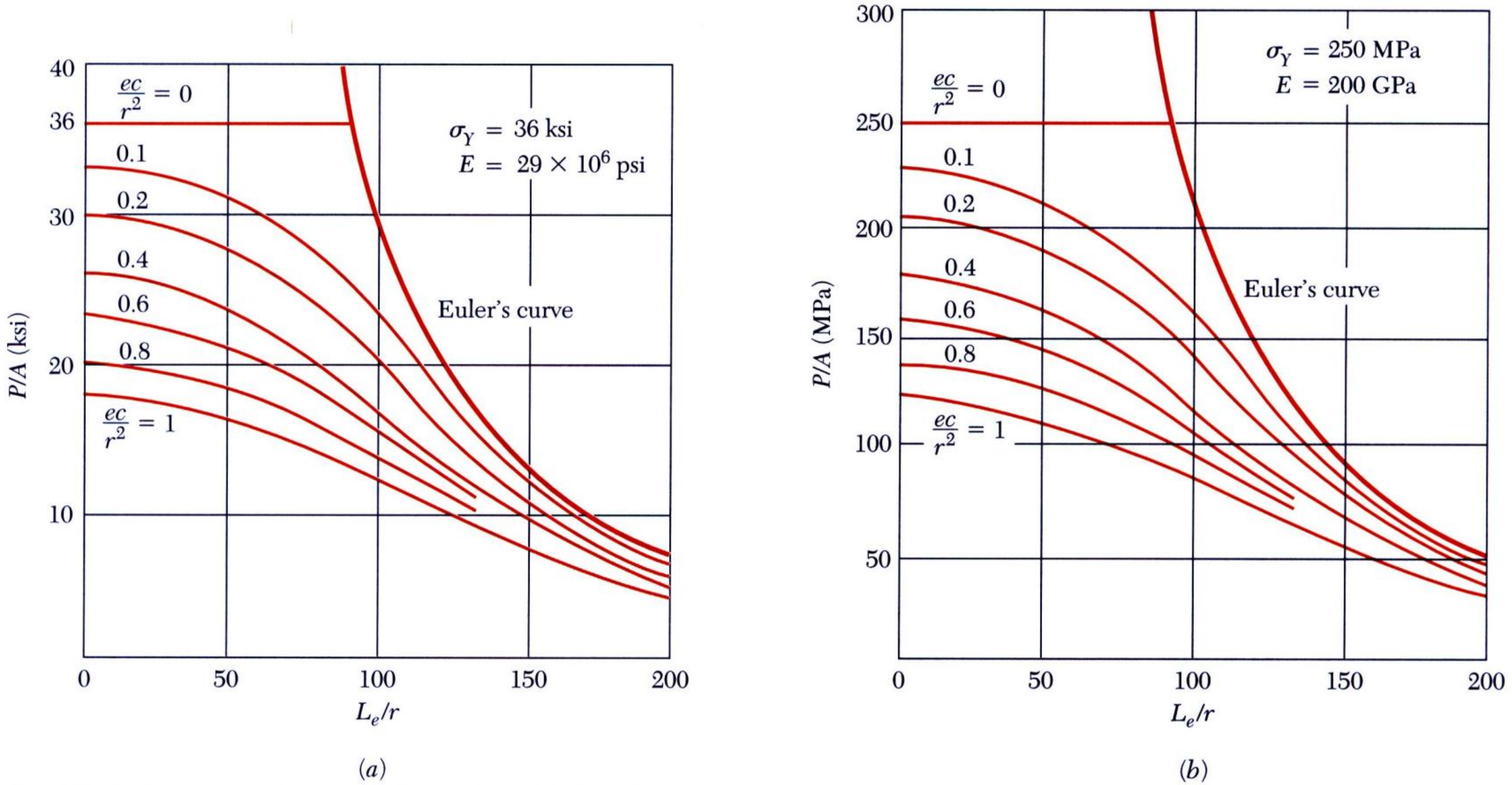
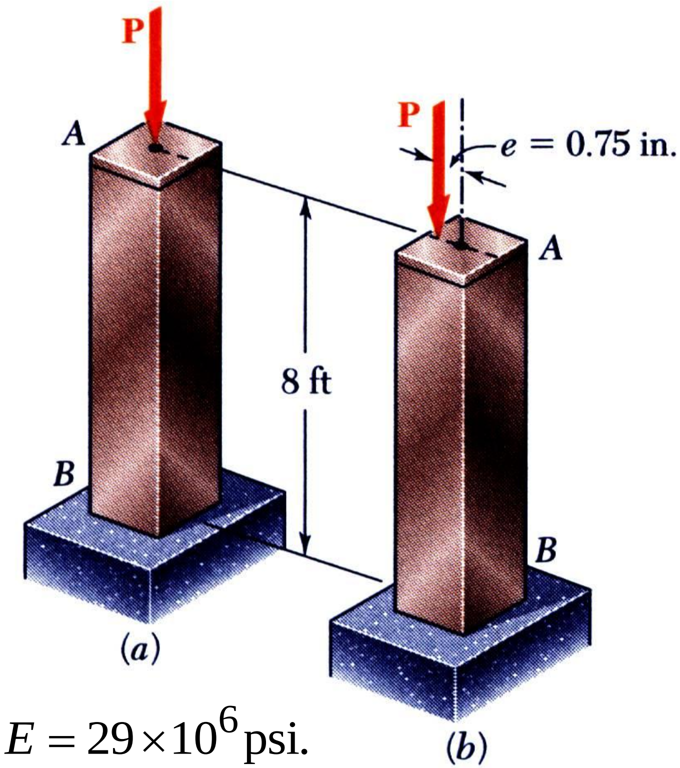


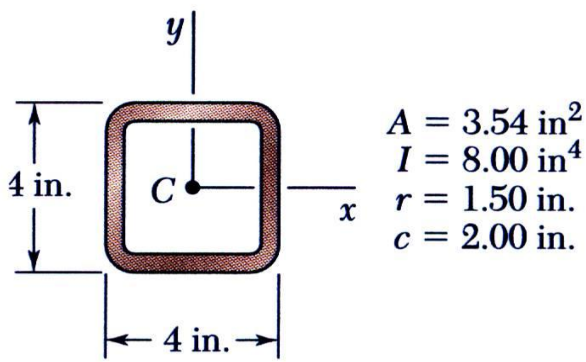
Fig. 10.24 Load per unit area, P/A, causing yield in column.

$$\sigma_{max} = \sigma_Y = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L_e}{r}\right)\right]$$

Sample Problem 10.2



$E = 29 \times 10^6$ psi.

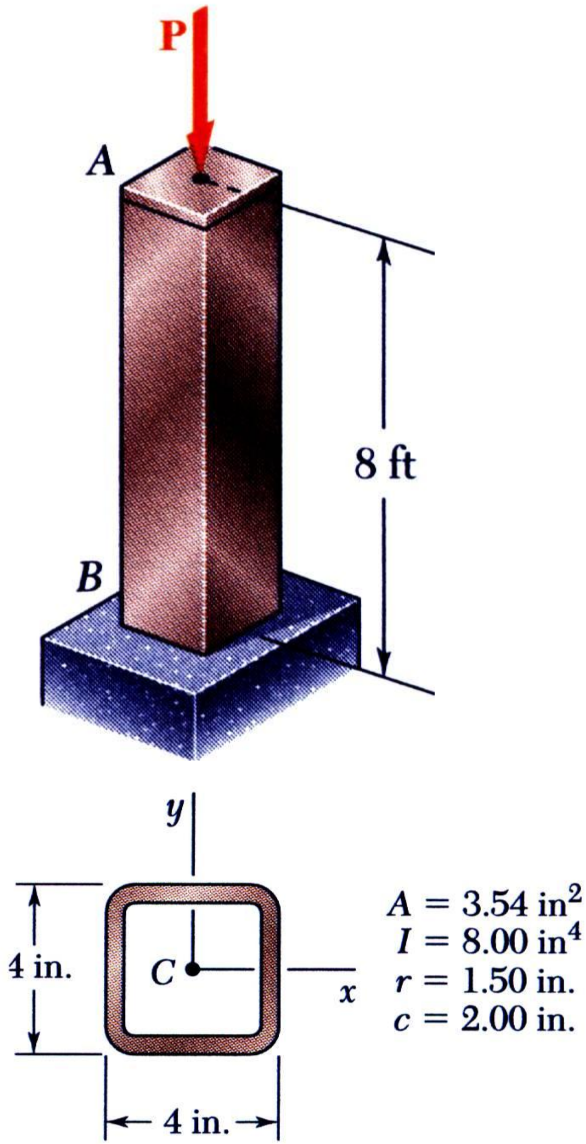


The uniform column consists of an 8-ft section of structural tubing having the cross-section shown.

a) Using Euler's formula and a factor of safety of two, determine the allowable centric load for the column and the corresponding normal stress.

b) Assuming that the allowable load, found in part *a*, is applied at a point 0.75 in. from the geometric axis of the column, determine the horizontal deflection of the top of the column and the maximum normal stress in the column.

Sample Problem 10.2



SOLUTION:

- Maximum allowable centric load:

- Effective length,

$$L_e = 2(8\text{ ft}) = 16\text{ ft} = 192\text{ in.}$$

- Critical load,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2\left(29\times10^6\text{ psi}\right)\left(8.0\text{ in}^4\right)}{(192\text{ in})^2}$$

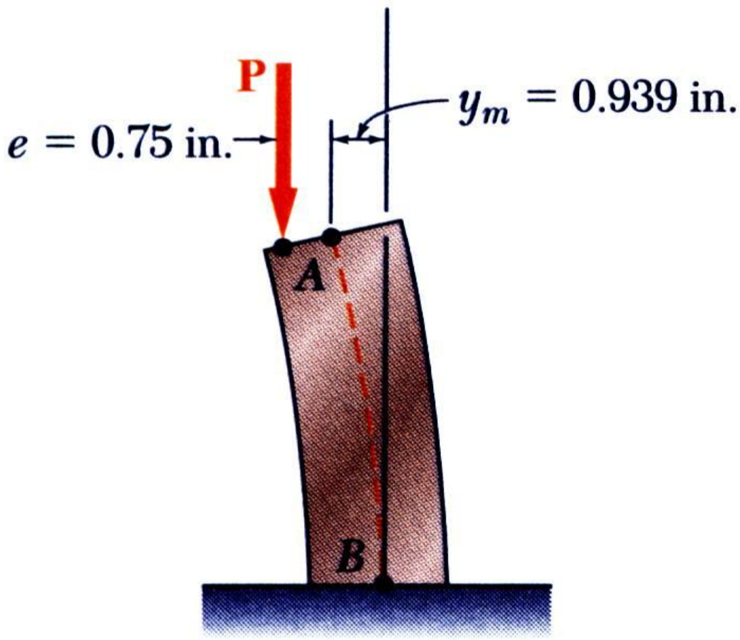
$$= 62.1\text{ kips}$$

- Allowable load,

$$P_{all} = \frac{P_{cr}}{FS} = \frac{62.1\text{ kips}}{2} \qquad \boxed{P_{all} = 31.1\text{ kips}}$$

$$\sigma = \frac{P_{all}}{A} = \frac{31.1\text{ kips}}{3.54\text{ in}^2} \qquad \boxed{\sigma = 8.79\text{ ksi}}$$

Sample Problem 10.2



- Eccentric load:
  - End deflection,

$$y_m = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]$$

$$= (0.075\text{ in})\left[\sec\left(\frac{\pi}{2\sqrt{2}}\right) - 1\right]$$

$$\boxed{y_m = 0.939\text{ in.}}$$

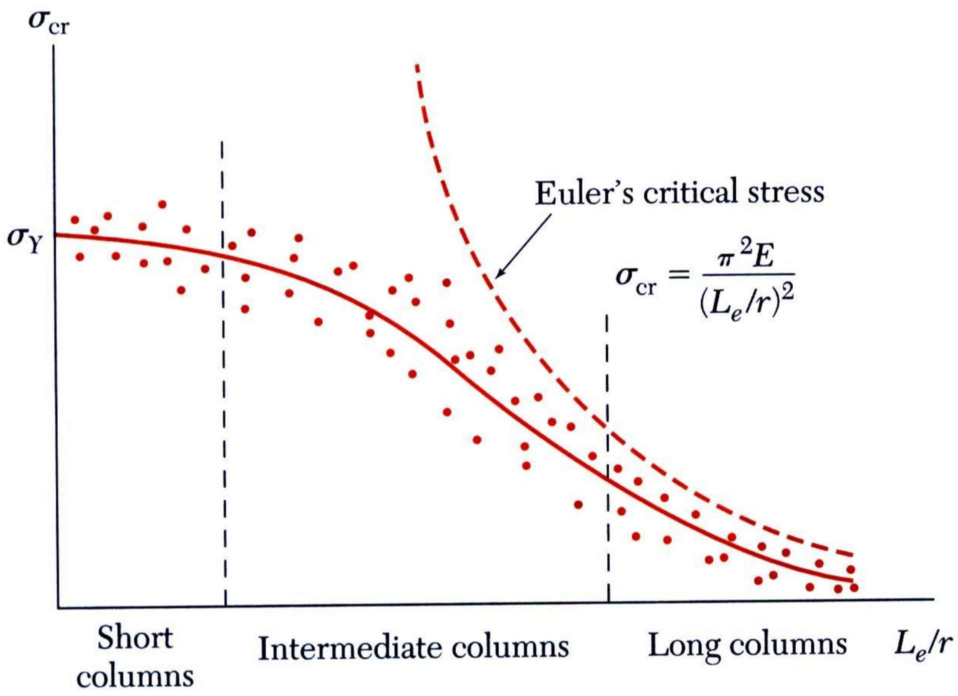
  - Maximum normal stress,

$$\sigma_m = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

$$= \frac{31.1\text{ kips}}{3.54\text{ in}^2}\left[1 + \frac{(0.75\text{ in})(2\text{ in})}{(1.50\text{ in})^2}\sec\left(\frac{\pi}{2\sqrt{2}}\right)\right]$$

$$\boxed{\sigma_m = 22.0\text{ ksi}}$$

Design of Columns Under Centric Load



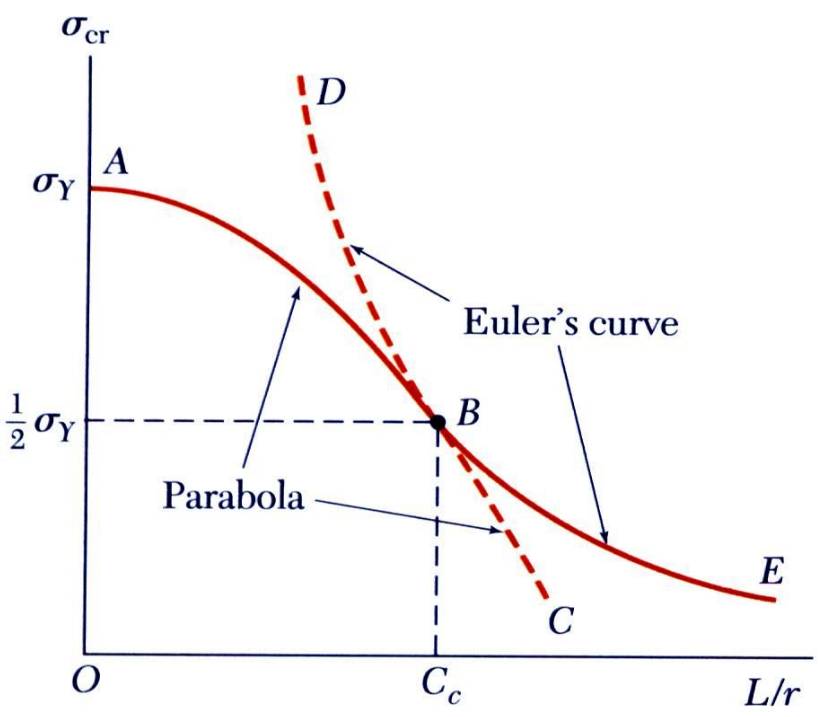
- Previous analyses assumed stresses below the proportional limit and initially straight, homogeneous columns

- Experimental data demonstrate
  - for large L_e/r, σ_{cr} follows Euler's formula and depends upon E but not σ_Y.
  - for small L_e/r, σ_{cr} is determined by the yield strength σ_Y and not E.
  - for intermediate L_e/r, σ_{cr} depends on both σ_Y and E.

Design of Columns Under Centric Load

Structural Steel

American Inst. of Steel Construction



- For $L_e/r > C_c$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \qquad \sigma_{all} = \frac{\sigma_{cr}}{FS}$$

$$FS = 1.92$$

- For $L_e/r > C_c$

$$\sigma_{cr} = \sigma_Y \left[1 - \frac{(L_e/r)^2}{2C_c^2}\right] \qquad \sigma_{all} = \frac{\sigma_{cr}}{FS}$$

$$FS = \frac{5}{3} + \frac{3}{8}\frac{L_e/r}{C_c} - \frac{1}{8}\left(\frac{L_e/r}{C_c}\right)^3$$

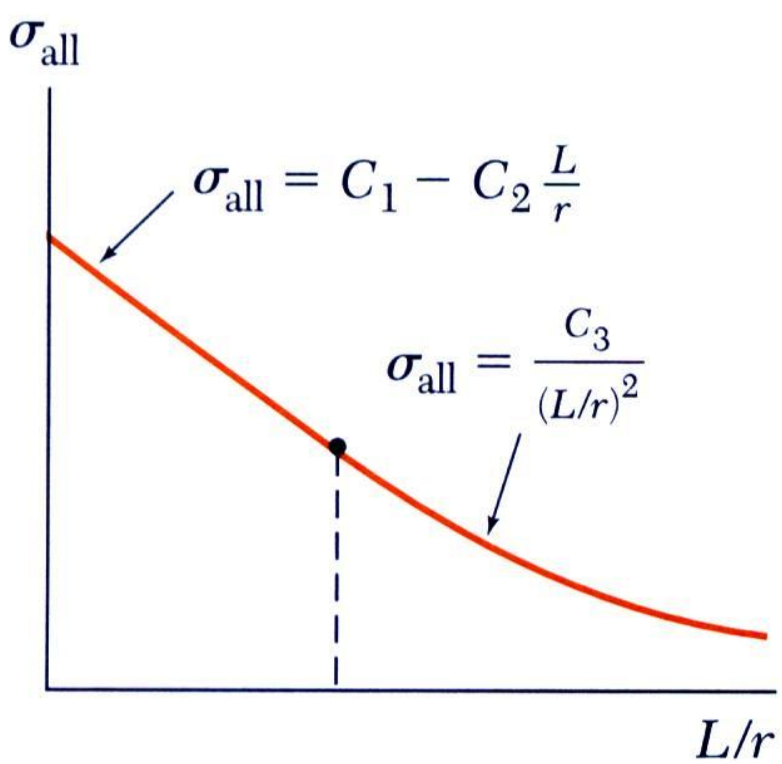
- At $L_e/r = C_c$

$$\sigma_{cr} = \tfrac{1}{2}\sigma_Y \qquad C_c^2 = \frac{2\pi^2 E}{\sigma_Y}$$

Design of Columns Under Centric Load

Aluminum

Aluminum Association, Inc.



- Alloy 6061-T6

  $L_e/r < 66$:

  $$\sigma_{all} = \left[20.2 - 0.126(L_e/r)\right]\text{ksi}$$
  $$= \left[139 - 0.868(L_e/r)\right]\text{MPa}$$

  $L_e/r \geq 66$:

  $$\sigma_{all} = \frac{51000\text{ ksi}}{(L_e/r)^2} = \frac{351 \times 10^3\text{ MPa}}{(L_e/r)^2}$$

- Alloy 2014-T6

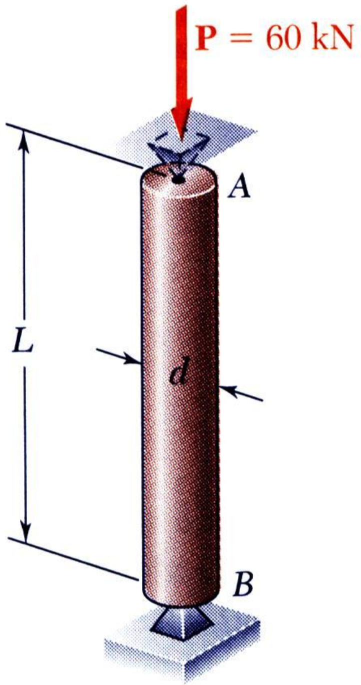
  $L_e/r < 55$:

  $$\sigma_{all} = \left[30.7 - 0.23(L_e/r)\right]\text{ksi}$$
  $$= \left[212 - 1.585(L_e/r)\right]\text{MPa}$$

  $L_e/r \geq 66$:

  $$\sigma_{all} = \frac{54000\text{ ksi}}{(L_e/r)^2} = \frac{372 \times 10^3\text{ MPa}}{(L_e/r)^2}$$

Sample Problem 10.4

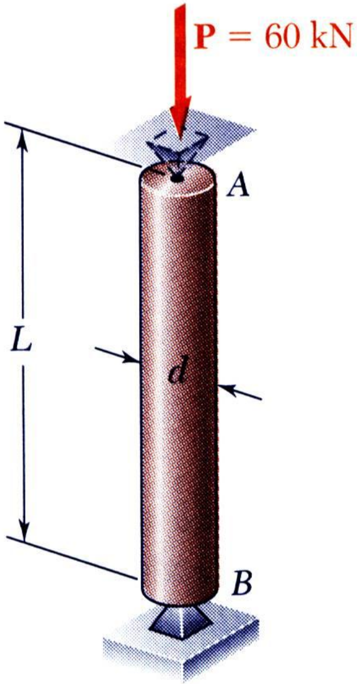


Using the aluminum alloy2014-T6, determine the smallest diameter rod which can be used to support the centric load P = 60 kN if a) L = 750 mm, b) L = 300 mm

SOLUTION:

- With the diameter unknown, the slenderness ration can not be evaluated. Must make an assumption on which slenderness ratio regime to utilize.
- Calculate required diameter for assumed slenderness ratio regime.
- Evaluate slenderness ratio and verify initial assumption. Repeat if necessary.

Sample Problem 10.4



$c =$ cylinder radius

$r =$ radius of gyration

$$= \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi c^4 / 4}{\pi c^2}} = \frac{c}{2}$$

- For L = 750 mm, assume L/r > 55

- Determine cylinder radius:

$$\sigma_{all} = \frac{P}{A} = \frac{372 \times 10^3 \text{ MPa}}{(L/r)^2}$$

$$\frac{60 \times 10^3 N}{\pi c^2} = \frac{372 \times 10^3 \text{ MPa}}{\left(\frac{0.750 \text{ m}}{c/2}\right)^2} \qquad c = 18.44 \text{ mm}$$

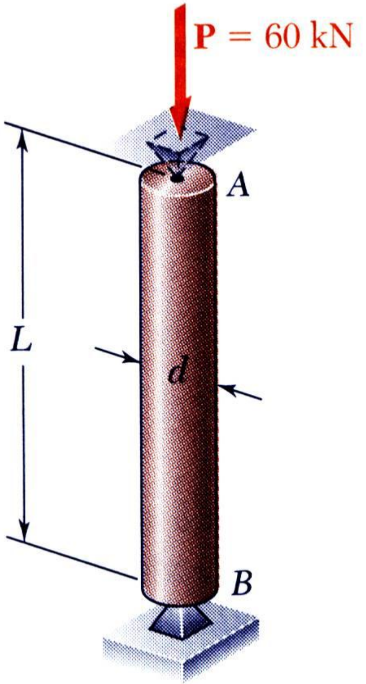
- Check slenderness ratio assumption:

$$\frac{L}{r} = \frac{L}{c/2} = \frac{750\text{mm}}{(18.44 \text{ mm})} = 81.3 > 55$$

assumption was correct

$$\boxed{d = 2c = 36.9 \text{ mm}}$$

Sample Problem 10.4



- For L = 300 mm, assume L/r < 55

- Determine cylinder radius:

$$\sigma_{all} = \frac{P}{A} = \left[212 - 1.585\left(\frac{L}{r}\right)\right] \text{MPa}$$

$$\frac{60 \times 10^3 N}{\pi c^2} = \left[212 - 1.585\left(\frac{0.3\,\text{m}}{c/2}\right)\right] \times 10^6 \text{Pa}$$

$$c = 12.00\,\text{mm}$$

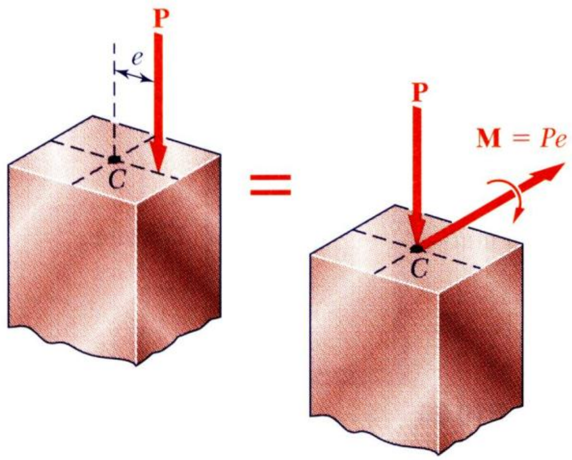
- Check slenderness ratio assumption:

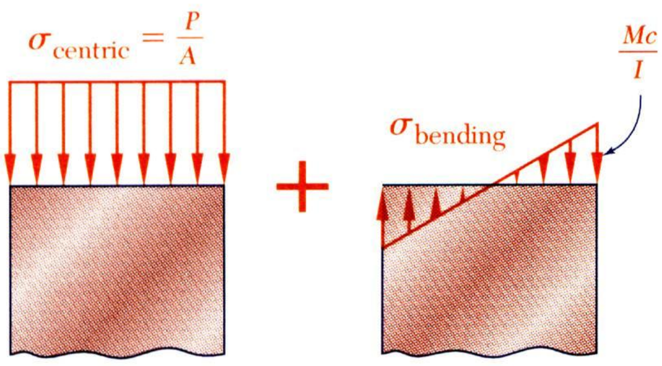
$$\frac{L}{r} = \frac{L}{c/2} = \frac{300\,\text{mm}}{(12.00\,\text{mm})} = 50 < 55$$

assumption was correct

$$\boxed{d = 2c = 24.0\,\text{mm}}$$

Design of Columns Under an Eccentric Load





- An eccentric load P can be replaced by a centric load P and a couple $M = Pe$.

- Normal stresses can be found from superposing the stresses due to the centric load and couple,

$$\sigma = \sigma_{centric} + \sigma_{bending}$$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I}$$

- Allowable stress method:

$$\frac{P}{A} + \frac{Mc}{I} \le \sigma_{all}$$

- Interaction method:

$$\frac{P/A}{(\sigma_{all})_{centric}} + \frac{Mc/I}{(\sigma_{all})_{bending}} \le 1$$