Chapter 7

Digital Filters

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- Designing of Digital Filters
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Digital Filters

Introduction

- Frequency Separator
- Efficient Tool
- Diversified Applications

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = -\sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \rightarrow \quad recursive IIR$$

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) \quad \rightarrow \quad non-recursive \ FIR$$

There are two major types of Digital filters

- IIR
- FIR

Design of Digital Filters

Design of Digital Filters

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = -\sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \rightarrow \quad recursive \;\; IIR$$

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) \quad \rightarrow \quad non-recursive \;\; FIR$$

If we calculate Impulse Response of the system we can design digital filter from transfer function of an analog filter, its one technique.

IIR filter

Analog to Digital filter

1. Impulse Invariant Method

Example:

$$H(s)=\frac{1}{1+sT} \quad , \quad T=0.5$$

1. Convert into time-domain
2. Sample the regions
3. Take z-transform
4. Find Difference Equation

$Step \quad 1:$

$$h(s) = \frac{1}{1 + sT}$$

$$h(s) = \frac{1}{1 + 0.5s}$$

$$h(s) = \frac{2}{2 + s}$$

$$h(t) = 2e^{-2t}$$

$Step \quad 2:$

$$h(nt) = 2e^{-2nT}$$

$$h(nt) = 2e^{-2n(1/2)}$$

$$h(nt) = 2e^{-n}$$

$Step\ 3:$

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}}$$

$$H(z) = \frac{2}{1 - 0.367\, z^{-1}} \quad \rightarrow \quad till \;\; here \;\; filter \;\; is \;\; design$$

$Step\ 3:$

$$a^{n+1} y(-1) + \sum_k a_k \, x(n-k)$$

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} = \frac{Y(z)}{X(z)}$$

$$2X(z) = Y(z)[1 - 0.367\, z^{-1}]$$

$$2X(z) = Y(z) - 0.367\, Y(z) z^{-1}$$

$$Y(z) = 0.367\, Y(z) z^{-1} + 2X(z)$$

$$y(n) = 0.367\; y(n-1) + 2x(n)$$

Example:

$$H(s)=\frac{1}{(s+1)(s+2)} \quad , \qquad T=0.5$$

$$(1)\, h(t)=e^{-t}-e^{-2t}$$

$$(2)\, h(nT)=e^{-ns}-e^{-2ns}$$

$$h(n)=e^{-n}-e^{-2n}$$

$$(3)\, Taking z-transform$$

$$H(z)=\frac{1}{1-e^{-1}z^{-1}}-\frac{1}{1-e^{-0.5}z^{-0.5}}$$

$$H(z)=\frac{1}{1-e^{-1}z^{-1}}-\frac{1}{1-0.367\, z^{-1}}$$

$$(4)\, Difference \quad Equation$$

$$H(s) = \frac{1}{(s+1)(s+2)}$$

$$h(t) = e^{-t} - e^{-2t}$$

$$z(h(t)) = z(e^{-t}) - z(e^{-2t})$$

$$H(z) = \frac{1}{1 - e^{-0.5} z^{-0.5}} - \frac{1}{1 - e^{-1} z^{-1}}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 0.606 \; z^{-1}} - \frac{1}{1 - 0.367 \; z^{-1}}$$

$$\frac{Y(z)}{X(z)} = \frac{1 - 0.367 \; z^{-1} - 1 + 0.606 \; z^{-1}}{(1 - 0.606 \; z^{-1})(1 - 0.367 \; z^{-1})}$$

$$Y(z) = 1 - 0.973 \; z^{-1} + 0.2224 \; z^{-2}$$

$$Y(z) = 0.239 \; z^{-1} X(z)$$

$$Y(z) = -0.9737 \; (z) z^{-1} + 0.2224 \; z^{-1}$$

$$Y(z) = 0.239 \; X(z) z^{-1}$$

$$y(n) = 0.9737 \; y(n-1) + 0.2224 \; y(n-2)$$

$$y(n) = 0.2372 \; (n-1)$$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$since \quad z = e^{st} \quad where \quad s = \sigma + j\omega$$

$$z = re^{j\omega} \quad (if \; polar \; form)$$

$$\therefore re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

$$where \quad r = e^{\sigma T} \quad ; \quad \omega = \Omega T$$

$$\sigma < 0 \Rightarrow 0 < r < 1 \quad \rightarrow \quad \textit{Inside the circle}$$

$$\sigma > 0 \Rightarrow r > 1 \quad \rightarrow \quad \textit{Outside the circle (Unstable)}$$

$$\sigma = 0 \Rightarrow r = 1$$

$$\textit{since} \quad \omega \ \textit{ranges} \ (-\pi, \pi)$$

$$-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T} \Rightarrow (-\pi, \pi)$$

$$-\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T} \Rightarrow (-\pi, \pi)$$

$$(2k-1)\frac{\pi}{T} \le \Omega \le (2k+1)\frac{\pi}{T} \Rightarrow (-\pi, \pi)$$

2. <u>Bilinear Transformation:</u>

IIR filter designing, Bilinear transformation

<u>Method:</u>

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \rightarrow \; analog\ filter\ locate\ this\ value$$

or

$$s = \frac{2}{T}\left(\frac{z - 1}{z + 1}\right) \rightarrow \; discrete$$

and

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad \rightarrow \quad continuous$$

or

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \quad \rightarrow \quad discrete$$

Example:

Design a single pole low pass digital filter with a 3dB bandwidth of 0.2 π,using bilinear transformation, with analog filter.

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Solution:

$$since\ \omega_c = 0.2\pi$$

$$\therefore \Omega_c = \frac{2}{T}\tan 0.1\pi = \frac{0.65}{T} \qquad let\ T = 1$$

$$H(s)=\frac{0.65/T}{s+o.65/T}$$

apply bilinear transformation

$let T=1$

$$H(z)=\frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$$

$$H(\omega)=\frac{0.245(1+e^{-j\omega})}{1-0.509e^{-j\omega}} \quad \rightarrow \quad frequency\ \ response$$

$\omega=0, \left|H(0)\right|=1$

$\omega=0.2\pi$

$H(0.2\pi)=0.707$

FIR Digital Filters

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) \qquad (Difference \quad Equation)$$

$$Y(z) = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \qquad (Transfer \quad Function)$$

$$H(\omega) = \sum_{k=0}^{M} b_k e^{-j\omega k} \qquad (z = e^{-j\omega})$$

b_k is the impulse response of the system when designing the FIR filter.

$$H(\omega) = \cdots b_{-2} e^{2j\omega} + b_{-1} e^{j\omega} + b_0 + b_1 e^{-j\omega} + b_2 e^{-2j\omega} + \cdots$$

$$H(\omega) = b_0 + 2b_2 \left(\frac{e^{2j\omega} + e^{-2j\omega}}{2} \right) + 2b_1 \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right)$$

$$H(\omega) = b_0 + 2b_2 \cos 2\omega + 2b_1 \cos \omega + \cdots$$

$$H(\omega) = b_0 + 2 \sum_{k=1}^{M} b_k \cos k\omega$$

Designing Method

Fourier Transform Method:

$$h(n) = \frac{1}{n\pi} \sin n\omega_0$$

$$h(n) = \frac{\omega_0}{\pi} \sin c(n\omega_0)$$

Example:

Find and sketch the impulse response of ideal zero phase, low-pass digital filter with cutoff frequency $\omega_0 = \pi/5$.

$$h(n) = \frac{1}{n\pi} \sin n\omega_0$$

$$h(n) = \frac{\omega_0}{\pi} \sin \frac{n\pi}{5}$$

$$h(1) = 0.18$$

$$h(n) = \frac{\sin\frac{n\pi}{5}}{n\pi}$$

$$h(n) = \frac{\frac{d}{dn}\sin\frac{n\pi}{5}}{\frac{d}{dn}n\pi}$$

$$h(n) = \frac{\frac{\pi}{5}\cos\frac{n\pi}{5}}{\pi}$$

$$h(n) = \frac{\cos\frac{n\pi}{5}}{5}$$

$$h(0) = 0.2$$