1. Introduction: Waves and Phasors

1.1. Dimensions, units and notation

Basis: International System of Units (SI). Table 1 summarizes fundamental units (note that others can be derived from these):

Dimension	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol

Table 1: Fundamental SI units.

For a range of values from 10^{-18} to 10^{18} a set of prefixes is use atto (a), femto (f), pico (p), nano (n), micro (μ), milli (m), kilo (k), mega (M), giga (G), tera (T), peta (P), exa (E) which increase by three orders of magnitude.

1.2. Electromagnetism

Electromagnetic force is one of the four fundamental forces in nature: nuclear, weak-interaction, and gravitational. Gravitational is the weakest at 10^{-41} that of the nuclear force. EM force exists between charged particles. It is the dominant force in microscopic systems (i.e. atoms and molecules). EM force is about 10^{-2} that of the nuclear force.

- **Notation Summary**
  - **Scalar quantities:** Medium weight italic in print and just the letter when written. As in C for capacitance.

- **Units:** medium-weight roman letters but looks the same as scalars when written and hopefully obvious from context.
- **Vector quantities:** boldface roman in the book as in **E** for electric field vector. But, when written I will put a little arrow over the letter.
- **Unit vectors:** boldface roman with circumflex (hat) ˆ over the letter as in $\hat{x}$.
- **Phasors:** a tilde ˜ over the letter as in $\tilde{E}$ for the phasor quantity of a time-harmonic scalar quantity $E(t)$. If it's a vector phasor it is boldface with a tilde over it. When writing on the board I will try to always use $E(t)$ and if I leave off the explicit time dependent (t) then a phasor is assumed (an arrow over the top for vector phasor).

• Gravitational force analogue

Newton's law of gravity states:

$$\mathbf{F}_{g21} = -\hat{\mathbf{R}}_{12}\frac{Gm_1m_2}{R_{12}^2} \quad \text{(N)} \tag{1}$$

which expresses the dependence of the gravitational force F acting on mass m_2 due to a mass m_1 at distance R_{12}. (see Fig. 1). G is the universal gravitational constant and $\hat{\mathbf{R}}_{12}$ is a unit vector pointing from m_1 to m_2.

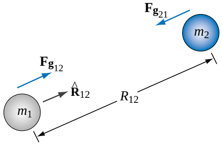


Figure 1: Gravitational forces between two masses.

- Force acts at a distance $\Rightarrow$ concept of fields
- Each mass m_1 induces a gravitational field Ψ_1 around it so that if another mass m_2 is introduced at some point, it will experience force equal to eq. 1
- The field does not physically eminate from the object but its influence exists at every point in space. The field is defined as:

$$\psi_1 = -\hat{\mathbf{R}}\frac{Gm_1}{R^2} \quad \text{(N/kg)} \tag{2}$$

where $\hat{\mathbf{R}}$ is a unit vector that points radially away from m_1 ($-\hat{\mathbf{R}}$ points towards m_1).

- The field is shown in Fig. 2.
- How do we find the force if the field is known?

$$\mathbf{F}_{g21} = \psi_1 m_2 = -\hat{\mathbf{R}}_{12}\frac{Gm_1 m_2}{R_{12}^2} \tag{3}$$

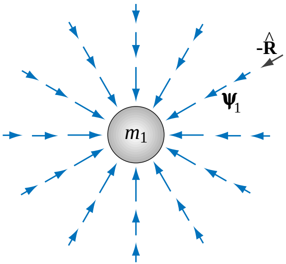


Figure 2: Gravitational field Ψ_1 induced by m_1.

or, if the force on a test mass m is known, then

$$\mathbf{\Psi} = \frac{\mathbf{F}_g}{m} \tag{4}$$

• Electric fields

This sets the stage for introducing electric field. Unlike the gravity field, its source is not mass, but charge, which can be either positive or negative. Both fields vary inversely with the square of distance.

- Charge has a minimum value: one electronic charge (e) and is measured in coulombs (C).

- Electron charge magnitude is

$$e = 1.602 \times 10^{-19} \quad \text{(C)} \tag{5}$$

- Actually, the charge of an electron is considered *negative* (as opposed to, e.g., protons) so that electron charge is $q_e = -e$ and the proton in $q_p = e$.

- Two charges of the same polarity repel each other, while those of opposite polarity attract each other,

- The force acts along the lines joining the charges,
- The force is proportional to the charges and inversely proportional to the square of the distance between them.

These properties summarized into Coulomb's law:

$$\mathbf{F}_{e21} = \hat{\mathbf{R}}_{12} \frac{q_1 q_2}{4\pi\epsilon_0 R_{12}^2} \quad \text{(N) in free space} \tag{6}$$

Symbols are similar to the gravity case, except now we have $\epsilon_0 = 8.854 \times 10^{-12}$ (F/m) which is *electrical permittivity of free space* and is measured in *Farads/meter* (F/m). Fig. 3 illustrates the two point charge case.

Electrical force also acts over distance and we again define electric field intensity $\mathbf{E}$ due to charge q:

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\epsilon_0 R^2} \quad \text{(V/m) in free space} \tag{7}$$

as illustrated in fig. 4 for a positive charge

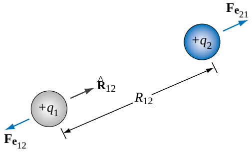


Figure 3: Electric forces between two positive point charges in free space.

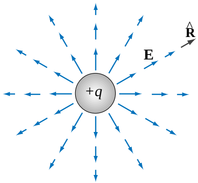


Figure 4: Electric field **E** due to charge q.

Two important observations regarding charges:

1. **Conservation of charge**: *(net) electric charge can neither be created nor destroyed.* Given n_p positive and n_n negative charges, the total is

$$q = n_p e - n_n e = (n_p - n_n)e \quad \text{(C)} \tag{8}$$

2. **Principle of linear superposition**: the total vector electric field at a point in space due to a system of point charges is equal to the vector sum of the electric fields at that point due to the individual charges.

- As noted, eq. 7 is valid for free space; what happens if an electron is introduced inside electrically neutral material?
- This situation is illustrated in Fig. 5.
- Each atom has an electrically positive nucleus and an electron "cloud" surrounding it.

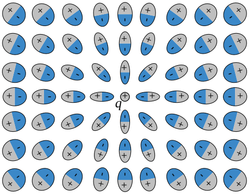


Figure 5: Polarization of atoms in a dielectric material.

- Each atom is electrically neutral.
- After introducing positive charge ⇒ different forces cause the atoms to become distorted.
- Now we can talk about one end (pole) of atom becoming more positive while the other becomes more negative ⇒ this effect is called polarization and the atom is now a electric dipole.

Observe that:

- The amount of polarization depends on the distance between the atom and the point charge,
- The orientation of the dipole is such that the dipole axis is directed toward the point charge, as illustrated in fig. 5.
- The electric dipoles tend to *counteract* the electric field from the point charge
- The electric field *inside* the material is different than in free space

The form of eq. 7 describing electric field is still valid, but we need to change the electrical permittivity so that

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\epsilon R^2} \quad \text{(V/m) where } \epsilon = \epsilon_r \epsilon_0 \quad \text{(F/m)} \tag{9}$$

ϵ_r is dimensionless quantity called relative permittivity or dielectric constant of the material. What are some typical values? See Appendix B.

Later on we will use additional quantity electric flux density $\mathbf{D} = \epsilon\mathbf{E}$ (C/m^2). Electric field $\mathbf{E}$ and electric flux density $\mathbf{D}$ constitute one of the two fundamental pairs of quantities in electromagnetics.

- **Magnetic fields**
  - Experimental observation of magnetism goes way back in history (Greeks 800 B.C.).
  - Observation of the magnetic force direction led to realization that magnetic field lines enter magnets at two points: *north and south poles*, independent of magnet's shape.
  - Magnetic field lines for a bar magnet are given in fig. 6.

  In addition:

  - "Like" poles repel each other, while "unlike" ones attract
  - While electric charges can be isolated, *magnetic poles always exist in pairs* (i.e. no magnetic monopoles)
  - When magnets are cut up, the pieces still have poles
  - Magnetic lines encircling a magnet are called magnetic field lines and represent magnetic flux density $\mathbf{B}$.

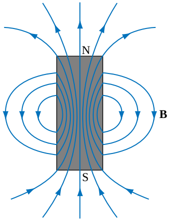


Figure 6: Magnetic field lines around a bar magnet.

- Magnetic fields need not come from permanent magnets; electrical current also causes it.

- Oersted made an important observation: a magnetic needle deflects when placed near wire carrying current.

- Needle's direction is always perpendicular to the the wire and the radial line connecting the wire to the needle.

- $\Rightarrow$ Current in a wire induces a magnetic field that forms closed circular loops around the wire, as shown in Fig. 7.

Biot and Savart developed a mathematical relationship between electric current and magnetic flux density, called, not surprisingly, Biot-Savart law. For a very long wire in free space, the magnetic flux density that is induced by a constant current is,

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r} \quad \text{(T)} \tag{10}$$

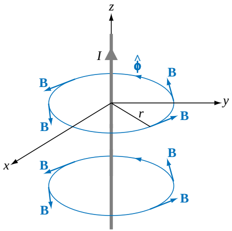


Figure 7: The magnetic field induced by a steady current flowing in the z-direction.

as illustrated in Fig. 7. Units are given in Tesla. $\mu_0 = 4\pi \times 10^{-7}$ (H/m) is called magnetic permeability of free space.

If you think that μ_0 is somehow analogous to ϵ_0, you are right; in fact the speed of light in free space c is (Chapter 2):

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \times 10^8 \text{ (m/s.)} \tag{11}$$

- Some material can have permeability $\mu \gg \mu_0 \Rightarrow$ ferrmagnetic material (such as iron and nickel)
- The majority of materials are nonmagnetic (i.e. $\mu = \mu_o$)
- μ accounts for the magnetization of a material and can be defined as

  $$\mu = \mu_r \mu_0 \text{ (H/m)} \tag{12}$$

  where μ_r is dimensionless quantity called relative magnetic permeability and H is Henries.

The second fundamental pair of electromagnetic quantities are magnetic flux density **B** and magnetic field intensity **H**, which are related by

$$\mathbf{B} = \mu \mathbf{H} \tag{13}$$

- **Static and dynamic fields**
  - Electric field (intensity) depends on charge q while magnetic field (intensity) depends on the current, i.e. the rate of change of charge flowing through some material.
  - So long as current is constant, the two will be independent (i.e. $dI/dt = 0$).
  - In this situation, electromagnetics divides into: **electrostatics** and **magnetostatics**.
  - The former requires stationary charges (i.e. $dq/dt = 0$) while the last requires constant currents (i.e. $dI/dt = 0$).

- More general: Dynamics, which deals with time-varying fields that are caused by time varying currents and charge densities (i.e. $dI/dt \neq 0$).

- In fact, time varying electric field generates a time varying magnetic field and vice-versa.

- In addition to ϵ and μ, we also need a conductivity measured in Siemens/meter (S/m) denoted σ.

- It describes how "freely" electrons can move around material.

- For $\sigma = 0$ material is perfect dielectric and for $\sigma = \infty$ it is said to be a perfect conductor.

- σ, ϵ, μ are called constitutive parameters. If constituent parameters are constant throughout the material, it is said to be homogeneous.

Branch	Condition	Field Quantities (Units)
Electrostatics	Stationary charges ($\partial q/\partial t = 0$)	Elec. field intensity $\mathbf{E}$ (V/m) Elec. flux density $\mathbf{D}$ (C/m^2) $\mathbf{D} = \epsilon\mathbf{E}$
Magnetostatics	Steady currents ($\partial I/\partial t = 0$)	Magnetic flux density $\mathbf{B}$ (T) Mag. field intensity $\mathbf{H}$ (A/m) $\mathbf{B} = \mu\mathbf{H}$
Dynamics	Time-varying currents ($\partial I/\partial t \neq 0$)	$\mathbf{E}, \mathbf{D}, \mathbf{B}$ and $\mathbf{H}$ $(\mathbf{E}, \mathbf{D})$ coupled to $(\mathbf{B}, \mathbf{H})$

Table 2: Branches of electromagnetics

1.3. Traveling Waves

There are many different kinds of waves—mechanical ones are easily observable (like stretched strings). Waves share some common properties:

- Moving waves carry energy from one point to another
- Waves have velocity, e.g. for EM waves $c = 3 \times 10^8$m/s
- Some waves are linear, i.e. they do not affect each other when passing through each other, e.g. EM and sound waves.
- Waves can be transient (caused by short duration disturbances) or continuous harmonic waves (generated by oscillating source).
- A wave is a self-sustaining disturbance of the medium in which it travels.

Take Fig. 8 as an example of 1-D wave.

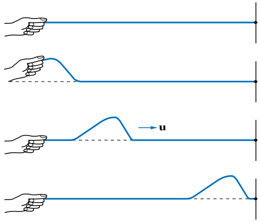


Figure 8: 1-D wave traveling on a string.

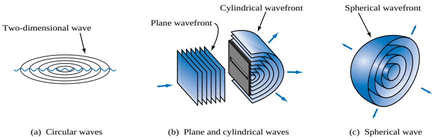


Figure 9: 2- and 3-D waves.

Extensions to 2-D and 3-D are shown in Fig. 9. 2-D waves are illustrated with surface waves (e.g. water). 3-D ones may have different shapes, e.g. plane waves, cylindrical, spherical. Nice pictures, but to do anything useful we need mathematical description! Look at simplest case first, i.e. sinusoidal waves in 1-D.

• Sinusoidal wave in lossless medium

Lossless medium: It does not attenuate the amplitude of the wave traveling within it or on its surface. Take water surface waves, where y denotes the height of water relative to unperturbed state, then

$$y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right) \text{ (m)} \tag{14}$$

A is **amplitude** of the wave, T is its **time period** , λ is **spatial wavelength**, and ϕ_0 is **reference phase**.

Even simpler form is obtained if the argument of the cosine term is called the **phase** of the wave (not to be confused with the reference phase ϕ_0):

$$\phi(x,t) = \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right) \tag{15}$$

which is measured in radians or degrees (rad = ? degrees?). The quantity $y(x,t)$ can be written,

$$y(x,t) = \cos\phi(x,t) \tag{16}$$

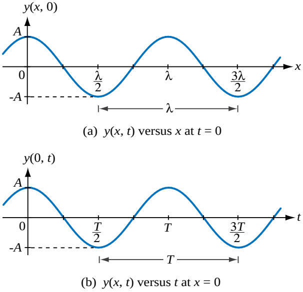


Figure 10: 1-D wave "snapshots".

Let's do some plotting of the wave $y(x, t)$. Math tells us that this is a periodic function. First look at $y(x, t)$ by fixing time to $t = 0$ and then by fixing position $x = 0$. The wave repeats itself with a spatial period λ and time period T. This is shown in Fig. 10.

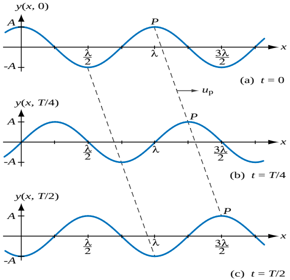


Figure 11: Wave snapshots illustrating wave travel.

What happens if we take a snapshot at different times (within one time period T)? That is shown in Fig. 11. If we look at position of, e.g., peak value P, we notice that it moves in the +x direction. If we can find what distance P travels in a given time then we can calculate phase velocity.

- To have a peak, phase must be zero or multiples of 2π (setting relative phase $\phi_0 = 0$)

$$\phi(x,t) = \frac{2\pi t}{T} - \frac{2\pi x}{\lambda} = 2n\pi,\ n = 0, 1, 2 \ldots \tag{17}$$

- For places other than peaks, this can be generalized for any point on the wave

$$\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} = \text{const.} \tag{18}$$

- Take time derivative of above eq. to get velocity

$$\frac{2\pi}{T} - \frac{2\pi}{\lambda}\frac{dx}{dt} = 0 \Rightarrow u_p = \frac{dx}{dt} = \frac{\lambda}{T}\ \text{(m/s)} \tag{19}$$

- Phase velocity = propagation velocity (u_p), is velocity of the wave pattern. Consider water waves, if you follow one part of the wave it moves at the phase velocity, however, the water itself is moving up and down only.

- What about direction of propagation? If the signs of the terms in the phase,

$$\phi(x,t) = \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right) \tag{20}$$

are different $\Rightarrow$ wave travels in $+x$ direction, otherwise in $-x$ direction.

- Frequency is the reciprocal of time period T: $f = 1/T$ (Hz)

- $\Rightarrow u_p = \lambda/T = f\lambda$.

- Things are "simplified" further by defining:

  1. Angular frequency: $\omega = 2\pi f$(rad/s)
  2. Phase constant (also called the wavenumber): $\beta = 2\pi/\lambda$ so substituting and taking the propagation direction in the positive x direction:

$$y(x,t) = A\cos\left(2\pi ft - \frac{2\pi}{\lambda}x\right) = A\cos(\omega t - \beta x) \tag{21}$$

- For the $-x$ direction:

$$y(x,t) = A\cos(\omega t + \beta x) \tag{22}$$

What about the phase reference ϕ_0? If it is not zero, then we have

$$y(x,t) = A\cos(\omega t - \beta x + \phi_0) \tag{23}$$

Fig. 12 shows what happens for different phase references at a fixed position $x = 0$. Note that,

- **Negative** ϕ_0 results in a lag behind the reference wave,
- While **positive** ϕ_0 leads the reference wave.

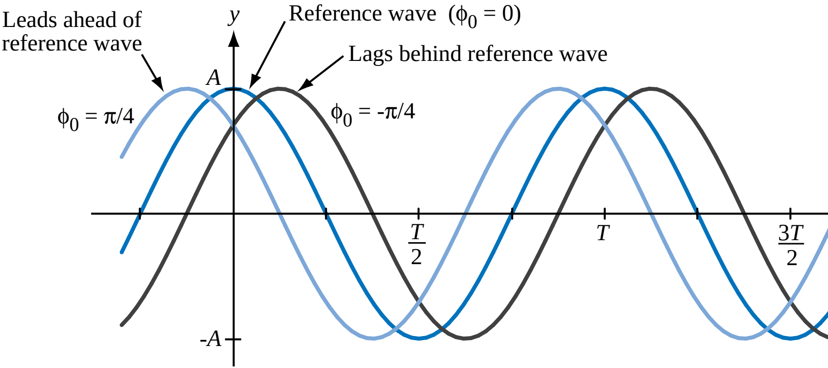


Figure 12: Effect of different phase reference on $y(0, t)$.

• Sinusoidal wave in a lossy medium

So far, wave's amplitude did not change with distance ⇒ lossless case. If it changes (decreases) ⇒ lossy case (lossy medium).

- **Attenuation constant** α characterizes how lossy the medium is
- α measured in Np/m
- Fall-off given by an exponential function $\exp(-\alpha x)$ so that full wave is given by

$$y(x,t) = Ae^{-\alpha x} \cos\left(\omega t - \beta x + \phi_0\right) \tag{24}$$

- Example of such function given in Fig. 13

To get the feel for the numbers and their meaning, do examples.

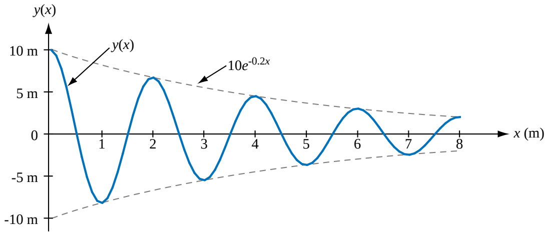


Figure 13: Exponentially attenuated wave.

1.4. EM spectrum

What are *electromagnetic waves*?

- EM waves consist of electric and magnetic field components of the same frequency (to be discussed further later)
- EM wave phase velocity in vacuum is constant and is the so-called velocity of light in vacuum (or free space) c recall,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \times 10^8 \text{ (m/s.)} \tag{25}$$

- Wavelength and frequency are related through

$$\lambda = \frac{c}{f} \tag{26}$$

Note different uses for different frequencies/wavelengths. Switch over from frequency to wavelength at around 300 GHz. Microwave and millimeter bands (used loosely).

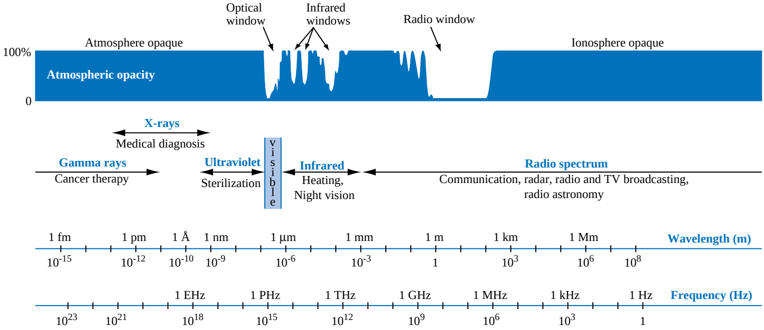
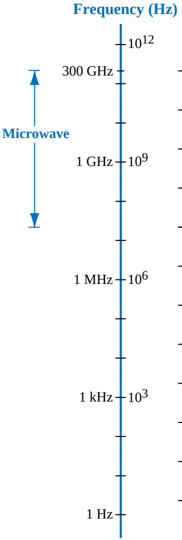


Figure 14: EM spectrum.



Band	Applications
Extremely High Frequency EHF (30 - 300 GHz)	Radar, advanced communication systems, remote sensing, radio astronomy
Super High Frequency SHF (3 - 30 GHz)	Radar, satellite communication systems, aircraft navigation, radio astronomy, remote sensing
Ultra High Frequency UHF (300 MHz - 3 GHz)	TV broadcasting, radar, radio astronomy, microwave ovens, cellular telephone
Very High Frequency VHF (30 - 300 MHz)	TV and FM broadcasting, mobile radio communication, air traffic control
High Frequency HF (3 - 30 MHz)	Short wave broadcasting
Medium Frequency MF (300 kHz - 3 MHz)	AM broadcasting
Low Frequency LF (30 - 300 kHz)	Radio beacons, weather broadcast stations for air navigation
Very Low Frequency VLF (3 - 30 kHz)	Navigation and position location
Ultra Low Frequency ULF (300 Hz - 3 kHz)	Audio signals on telephone
Super Low Frequency SLF (30 - 300 Hz)	Ionospheric sensing, electric power distribution, submarine communication
Extremely Low Frequency ELF (3 - 30 Hz)	Detection of buried metal objects
$f < 3$ Hz	Magnetotelluric sensing of the earth's structure

Figure 15: RF spectrum.

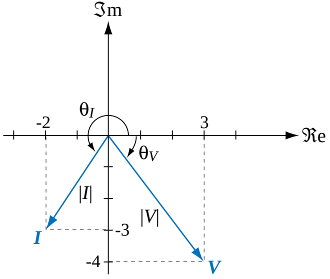


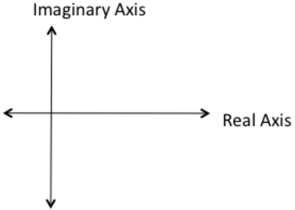
Figure 16: Complex V and I.

1.5. Review of Complex Numbers

We need to cover the basics: different forms and conversions between, Euler's identity, complex conjugate, identity, addition, multiplication, division, powers and some other useful relations.

Complex Numbers Review

We have a coordinate system that uses a real axis and imaginary axis

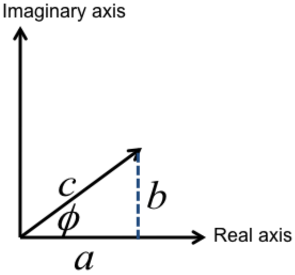


Complex number → $c = \underbrace{a}_{\text{Real}} + \underbrace{jb}_{\text{Imaginary}}$ ← Rectangular form

Note, *a*, *b* are real.

$$j^2 = -1$$

$$c = a + jb$$



Polar phasor form:.

$$c = |c| e^{j\phi}$$

Magnitude of c (or absolute value):.

$$|c| = \sqrt{a^2 + b^2}$$

Phase of c (or phase angle):.

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$

Euler's Formula: (with real ϕ)

$$e^{j\phi} = \cos\phi + j\sin\phi$$

Where does this come from?

One way to see this is using Taylor series. Even if you don't prove it, you can convince yourself that these series hold. Take the Taylor series representation for sin and cos:

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \frac{\phi^8}{8!} - \ldots$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \frac{\phi^9}{9!} - \ldots$$

What about for the exponential function?

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} \ldots$$

We write as *z* since this can be a complex number

What about if we take z to be $j\phi$?

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} \ldots$$

$$e^{j\phi} = 1 + j\phi - \frac{\phi^2}{2!} - \frac{j\phi^3}{3!} + \frac{\phi^4}{4!} + \frac{j\phi^5}{5!} - \frac{\phi^6}{6!} - \frac{j\phi^7}{7!} \ldots$$

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \frac{\phi^8}{8!} - \ldots$$

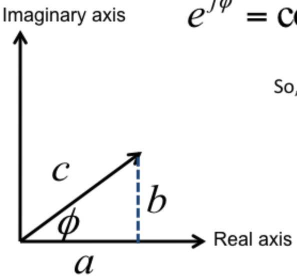
$$j\sin\phi = j\phi - \frac{j\phi^3}{3!} + \frac{j\phi^5}{5!} - \frac{j\phi^7}{7!} + \frac{j\phi^9}{9!} - \ldots$$

So, we have:

$$e^{j\phi} = \cos\phi + j\sin\phi$$

Euler's Formula: (with real ϕ)

$$e^{j\phi} = \cos\phi + j\sin\phi$$



So, we can see that: $a = \cos\phi$

$$b = \sin\phi$$

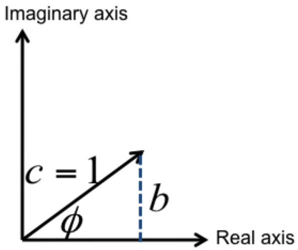
In general, when magnitude is not 1:

$$|c| = \sqrt{a^2 + b^2}$$

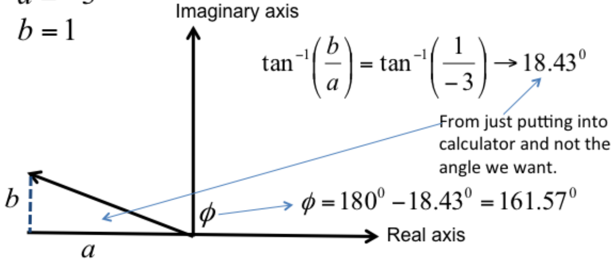
$$c = |c|e^{j\phi} = \underbrace{|c|\cos\phi}_{\text{Real}} + j\underbrace{|c|\sin\phi}_{\text{Imaginary}}$$

Polar to rectangular phasor form

Be careful when getting the phase angle from *a* and *b* using arctan. In first quadrant it's not a problem.



Take: $a = -3$
$b = 1$



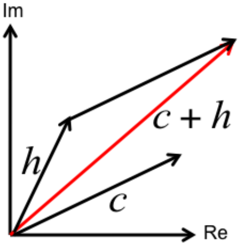
Equality of complex numbers:

$$c = a + jb \qquad h = f + jg$$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal

If: $h = c$ Then: $a = f$ And: $b = g$

Addition of complex numbers

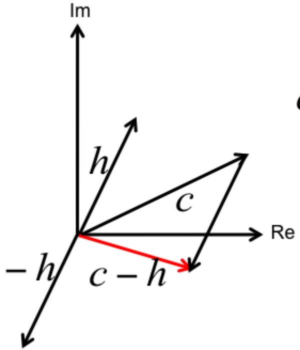


$$c = a + jb$$
$$h = f + jg$$

$$c + h = (a + f) + j(b + g)$$

Real Imaginary

Subtraction of complex numbers



$$c - h = (a - f) + j(b - g)$$

Multiplication of complex numbers

$$ch = (a + jb)(f + jg) = (af - bg) + j(bf + ag)$$

This is a bit of work and prone to mistakes.

A more convenient approach is to use the phasor form:

$$c = |c|e^{j\phi_1} \qquad h = |h|e^{j\phi_2}$$

$$ch = |c|e^{j\phi_1}|h|e^{j\phi_2} = |c||h|e^{j(\phi_1 + \phi_2)}$$

Division of complex numbers

$$\frac{c}{h} = \frac{(a + jb)}{(f + jg)} = \frac{(a + jb)}{(f + jg)} \frac{(f - jg)}{(f - jg)} =$$

$$= \frac{af + bg + jbf - jag}{f^2 + jgf - jgf + g^2}$$

$$= \frac{(af + bg) + j(bf - ag)}{f^2 + g^2}$$

Again, a more convenient approach is to use the polar phasor form:

$$\frac{c}{h} = \frac{|c|e^{j\phi_1}}{|h|e^{j\phi_2}} = \frac{|c|}{|h|} e^{j(\phi_1 - \phi_2)}$$

Complex Conjugation

$$c = a + jb$$

The complex conjugate of complex number c is denoted c^* and has the effect of changing the sign of the imaginary part:

$$c^* = a - jb$$

Example:

$$c + c^* = (a + jb) + (a - jb) = 2a \quad \text{Always real}$$

$$c - c^* = (a + jb) - (a - jb) = 2jb \quad \text{Always Imag.}$$

Example continued:

$$\left|\frac{c}{c^*}\right| = \frac{|(a+jb)|}{|(a-jb)|} = \frac{a^2+b^2}{a^2+b^2} = 1$$

Or, using polar phasor form:

$$\left|\frac{c}{c^*}\right| = \frac{\left||c|e^{j\phi}\right|}{\left||c|e^{-j\phi}\right|} = \frac{|c|}{|c|} = 1$$

$$c^*c = (a+jb)(a-jb) = a^2+b^2$$ Always real and positive (unless *c* is zero)

Square Root of a Complex Number

Take a complex number: $z = 1 + j2$

How do you take the square root? $\sqrt{z} = \sqrt{1 + j2}$

In polar form: $z = \sqrt{5}e^{j\phi}$ $\phi = \tan^{-1}\left(\frac{2}{1}\right)$

$$\sqrt{z} = \sqrt{\sqrt{5}e^{j\phi}} = \sqrt{\sqrt{5}}e^{j\frac{\phi}{2}}, \sqrt{\sqrt{5}}e^{j(\phi/2+\pi)}$$

1.6. Review of Phasors

- Why use phasors? Enable solutions to integro-differential equations by transforming them into linear equations.
- It is often easier to solve problems as time-harmonic. That means the source excitation– or forcing function– varies sinusoidally in time.
- A large class of problems are defined this way, for example 60 Hz power lines.
- For more general signals (e.g., broadband) we can decompose other (periodic) functions — using Fourier methods. That is, we can compute the time harmonic response for each of the frequency components of a signal and recombine using Fourier synthesis (principle of superposition).
- We will describe here how to represent time-harmonic quantities as phasors (complex quanties) and how to solve problems.

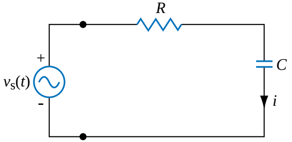


Figure 17: R-C circuit for phasor analysis.

- Let's have a look at simple RC circuit example in Fig. 17
- The source function varies sinusoidally as

$$v_s(t) = V_0 \sin(\omega t + \phi_0) \tag{27}$$

- Our goal is to find the current as a function of time $i(t)$. It could be done in the time domain but that is somewhat more difficult than using the phasor domain.

The starting equation is from Kirchhoff's voltage law:

$$Ri(t) + \frac{1}{C}\int i(t)dt = v_s(t) \tag{28}$$

which is in time domain. The steps are outlined as follows:

Adopt a cosine reference: We use a convention here so that our phase reference is consistent. We choose to use a cosine reference. Therefore forcing function must be cast into cosine form when it is not given that way. Remember that $\sin x = \cos(\pi/2 - x)$, $\cos(-x) = \cos(x)$. So, here we have,

$$v_s(t) = V_0 \sin(\omega t + \phi_0) = V_0 \cos(\pi/2 - \omega t - \phi_0) \tag{29}$$

$$v_s(t) = V_0 \cos(\omega t + \phi_0 - \pi/2) \tag{30}$$

Introduce phasors: We can write a cosine function as the real part of a complex number. Take the above cosine function, we can use Euler's identity,

$$e^{j(\omega t+\phi_0-\pi/2)} = \cos(\omega t + \phi_0 - \pi/2) + j\sin(\omega t + \phi_0 - \pi/2) \tag{31}$$

so that we can say that,

$$\cos(\omega t + \phi_0 - \pi/2) = \Re[e^{j(\omega t+\phi_0-\pi/2)}] \tag{32}$$

Where, the symbol $\Re$ indicates taking the real part. This could have an amplitude in front as well, and we can group that with the other terms in the exponent to get,

$$A\cos(\omega t + \phi_0 - \pi/2) = \Re[Ae^{j(\omega t+\phi_0-\pi/2)}] = \Re[Ae^{j(\phi_0-\pi/2)}e^{j\omega t}] \tag{33}$$

And, we can call all of the first parts of the term some complex number $\tilde{Z}$,

$$A\cos(\omega t + \phi_0 - \pi/2) = \Re[\tilde{Z}e^{j\omega t}] \tag{34}$$

So we can write $z(t) = \Re[\tilde{Z}e^{j\omega t}]$ to express any cosinusoidal time function. $\tilde{Z}$ is the **phasor**. So that voltage becomes,

$$v_s(t) = V_0 \sin(\omega t + \phi_0) = V_0 \cos(\omega t + \phi_0 - \pi/2) = \Re[V_0 e^{j(\omega t + \phi_0 - \pi/2)}] \tag{35}$$

$$v_s(t) = \Re[\tilde{V}_s e^{j\omega t}],\ \tilde{V}_s = V_0 e^{j(\phi_0 - \pi/2)} \tag{36}$$

Remember: integration in the time domain becomes division by $j\omega$ in phasor domain, and differentiation becomes multiplication by $j\omega$.

Similarly, we will can write the unknown current in the same form,

$$i(t) = \Re[\tilde{I} e^{j\omega t}] \tag{37}$$

Recast the equation in phasor form: Using the phasor transformations we get:

$$R\Re(\tilde{I}e^{j\omega t}) + \frac{1}{C}\Re\left(\frac{\tilde{I}}{j\omega}e^{j\omega t}\right) = \Re(\tilde{V}_s e^{j\omega t}) \tag{38}$$

For the phasor equation we drop the $e^{j\omega t}$ terms and know that to get back to the time domain we need to put that back and take the real part. The phasor equation becomes just,

$$R\tilde{I} + \frac{\tilde{I}}{j\omega C} = \tilde{V}_s \tag{39}$$

$$\tilde{I}\left(R + \frac{1}{j\omega C}\right) = \tilde{V}_s \tag{40}$$

Solution in phasor domain: We can solve for the current phasor,

$$\tilde{I} = \frac{\tilde{V}_s}{\left(R + \frac{1}{j\omega C}\right)} \tag{41}$$

Substitute in for the source voltage phasor, $\tilde{V}_s = V_0 e^{j(\phi_0 - \pi/2)}$,

$$\tilde{I} = \frac{V_0 e^{j(\phi_0 - \pi/2)}}{\left(R + \frac{1}{j\omega C}\right)} \tag{42}$$

Multiply out denominator term,

$$\tilde{I} = \frac{V_0 e^{j(\phi_0 - \pi/2)} j\omega C}{(1 + j\omega RC)} = \frac{V_0 \omega C e^{j\phi_0}}{(1 + j\omega RC)} \tag{43}$$

We can write the denominator term as a magnitude and phase,

$$1 + j\omega RC = \sqrt{1 + \omega^2 R^2 C^2} e^{j\phi_1} \tag{44}$$

Where the phase is $\phi_1 = \tan^{-1}(\omega RC)$.

We can then write the current phasor as,

$$\tilde{I} = \frac{V_0 \omega C}{\sqrt{1+\omega^2 R^2 C^2}} e^{j(\phi_0 - \phi_1)} \tag{45}$$

Solution in time domain: The last step is that we need to go back to a time domain solution. To find $i(t)$, we multiply the phasor by $e^{j\omega t}$ and take the real part.

$$i(t) = \Re[\tilde{I} e^{j\omega t}] \tag{46}$$

$$i(t) = \Re[\frac{V_0 \omega C}{\sqrt{1+\omega^2 R^2 C^2}} e^{j(\phi_0 - \phi_1)} e^{j\omega t}] \tag{47}$$

$$i(t) = \frac{V_0 \omega C}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t + \phi_0 - \phi_1) \tag{48}$$

Table 1-5 in the book gives various time domain expressions and their corresponding phasor domain expression.

Traveling waves in the phasor domain

- A traveling wave has a dependency that is something of the form $\omega t \pm \beta x$.
- For a time harmonic signal,

$$A\cos(\omega t + \beta x) \Leftrightarrow Ae^{j\beta x} \tag{49}$$

- For a wave traveling in the positive x direction, the sign between ωt and βx are opposite (e.g., $\omega t - \beta x$).
- When the sign is the same the wave is moving in the negative direction.
- So, a wave traveling in the negative x direction has a phasor of the form $Ae^{j\beta x}$ and moving in the positive x direction is, $Ae^{-j\beta x}$.
- Therefore, the sign of the exponent is opposite to the direction of travel.