

Chapter 1

Introduction: Waves and Phasors

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1.1 Introduction

Chapter one serves as an introduction to the course technical content an review of concepts from math and physics courses outside the ECE department, and circuit theory with emphasis on sinusoidal steady-state analysis using *phasors*

1.2 Dimensions, Units, and Notation

- The *International System of Units* (SI) is used throughout this course
- When working problems always include units
- There are also many secondary units, such as resistance, current, voltage, ... (see text Appendix A)

Table 1.1: SI Units.

Dimension	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol

- Fundamental and secondary units frequently carry prefixes as shown below

Table 1.2: SI Unit Prefixes.

Prefix	Symbol	Magnitude
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^9
mega	M	10^6
kilo	k	10^3
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

- Notational conventions are also important as we have:
 - **Scalars:** Such as C for capacitance and q for charge
 - **Units:** Such as V/m for volts per meter or A for amps
 - **Vectors:** Such as \mathbf{E} for the electric field vector; in hand writing a reasonable convention might be $\underline{\mathbf{E}}$ or $\vec{\mathbf{E}}$
 - **Unit Vectors:** Place a *circumflex* or hat over a boldface letter, e.g., the unit vector in the x -direction is $\hat{\mathbf{x}}$
 - **Phasors:** In a circuit theory course you were introduced to the phasor component corresponding to a sinusoidally time-varying signal. Here we use the tilde (\sim) over the character:
 - * For a time-varying scalar field $E(t) \leftrightarrow \tilde{E}$ is the phasor

* For a time-varying vector field $\mathbf{E}(t) \leftrightarrow \tilde{\mathbf{E}}$ is the phasor

Remember that the use of phasor quantities always implies sinusoidal time variation (more on this in the phasor review later)

Example 1.1: Notation Combinations

Consider

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{x}}E \\ \mathbf{E}(t) &= \hat{\mathbf{x}}E(t) \\ \mathbf{E}(t) &= \mathbf{Re} \left[\tilde{\mathbf{E}}e^{j\omega t} \right] = \hat{\mathbf{x}} \mathbf{Re} \left[\tilde{E}e^{j\omega t} \right]\end{aligned}$$

where in the last line $\tilde{E} = \hat{\mathbf{x}}\tilde{E}$

1.3 The Nature of Electromagnetism

The big picture view of nature's forces places electromagnetic force in the company of three other forces:

- **Nuclear force:** The strongest, but only at subatomic scales
- **Electromagnetic:** Operates at around 2^{-2} that of nuclear force; it exists between all charge particles and is the dominant force in *microscopic* systems
- **Weak-interaction force:** 10^{-14} that of nuclear force and primarily involving radioactive particles

- **Gravitational force:** The weakest of all the forces, just 10^{-41} that of nuclear forces, but it operates on a grand scale, e.g., over a solar system; applies to *macroscopic* systems

1.3.1 Gravitational Force

The text chooses to ease into electromagnetic field quantities by starting with gravitational force. As a simple review recall that the force between two masses is product of the masses over square of the distance between the masses. Gravitation force is always attractive, right?

- Have you ever wished you could turn down the force of gravity on you or objects you are trying to work with?

1.3.2 Electric Fields (intro)

Finally getting into some real EM, we consider EM field forces. The electric field has associated force \mathbf{F}_e and the magnetic field has associated force \mathbf{F}_m

- By way of review from your physics background and analogous to the gravitational field, an electric field results from electric charge which may be positive or negative
- The force is again inverse with the distance squared between two charges and proportional to the product of the charges \longrightarrow the resulting force may attract or repel
- The fundamental unit of charge is that of the electron, taken to have absolute value e

$$e = 1.6 \times 10^{-19} \text{ (C)}$$

where the coulomb (C) is the unit of charge

Note: The charge of a single electron is $q_e = -e$

- Note: The charge on an electron is actually negative, that is $q_e = -e$
 - The charge on a proton is equal and opposite, that is $q_p = e$
- In Physics recall learning about *Coulomb's law* which states that the force between two point charges is

$$\mathbf{F}_{e21} = \hat{\mathbf{R}}_{12} \frac{q_1 q_2}{4\pi\epsilon_0 R_{12}^2} \text{ N}$$

in *free space*, where the subscript e_{12} denotes *electric force* on charge q_1 due to charge q_2

- To be clear R_{12} is the distance between the two charges and the precise orientation of the force requires the unit vector $\hat{\mathbf{R}}_{12}$ associated with the line joining the two charges
- To account for the free space environment the constant $\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$, the *permittivity of free space* is required

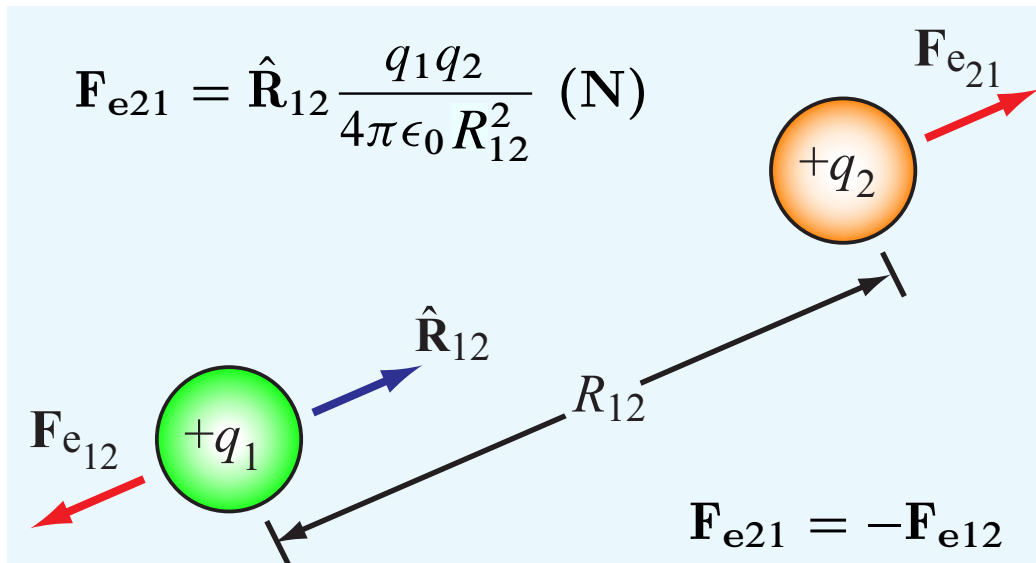
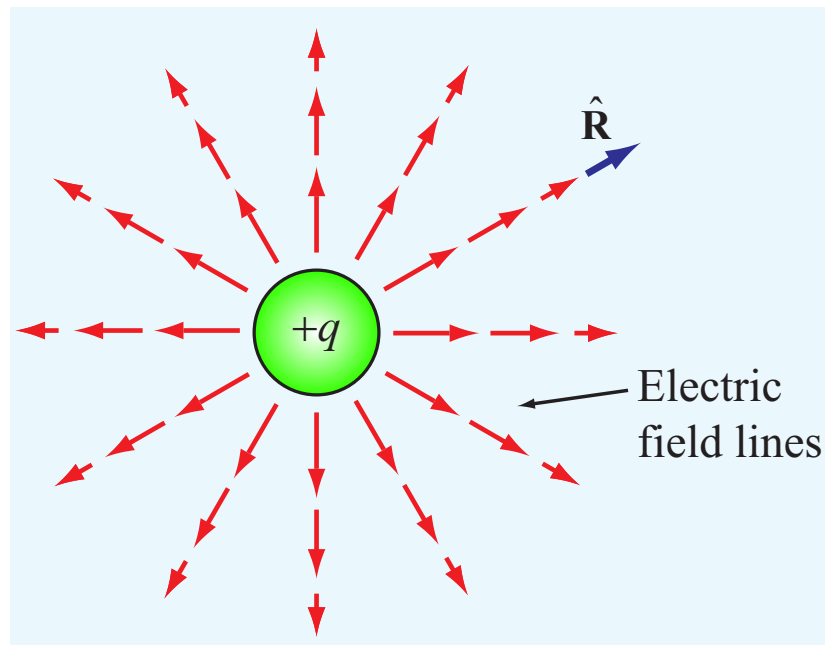


Figure 1.1: Coulomb's law pictorial

- The electric field intensity \mathbf{E} due to charge q can be defined as

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\epsilon_0 R^2} \text{ (V/m)}$$

in free space

Figure 1.2: Electric field \mathbf{E} due to point charge q

Note: The unit vector $\hat{\mathbf{R}}$ associated with \mathbf{E} depends upon the field observation point (more on this in Chapter 4)

- Using the above definition of electric field, \mathbf{E} , we can write that the force experienced by any point charge q is simply

$$\mathbf{F}_e = q\mathbf{E} \quad (\text{N})$$

- Relax a bit, since Chapter 4 of the text is dedicated to *Electrostatics*, i.e., more to come later

Example 1.2: Simulating Field Lines Using Python

- A quick Jupyter notebook example

Two Important Properties

1. *Conservation of charge:* Net electric charge is neither created or destroyed.

For a volume containing n_p protons (positive charge) and n_e electrons, the total charge is

$$q_{\text{vol}} = n_p \cdot e - n_e \cdot e = (n_p - n_e)e \quad (\text{C})$$

2. *Linear superposition:* The total vector field at a point is the sum of the electric field due to the individual charges (more in Chapter 4)

E Field in a Material or Medium

- A material, say water or a plastic, is composed of atoms electrically neutral (nucleus surrounded by an electron cloud) → the interior \mathbf{E} field is zero
- Now place a point charge inside the material and observe that the electron cloud becomes distorted due to the charge imbalance

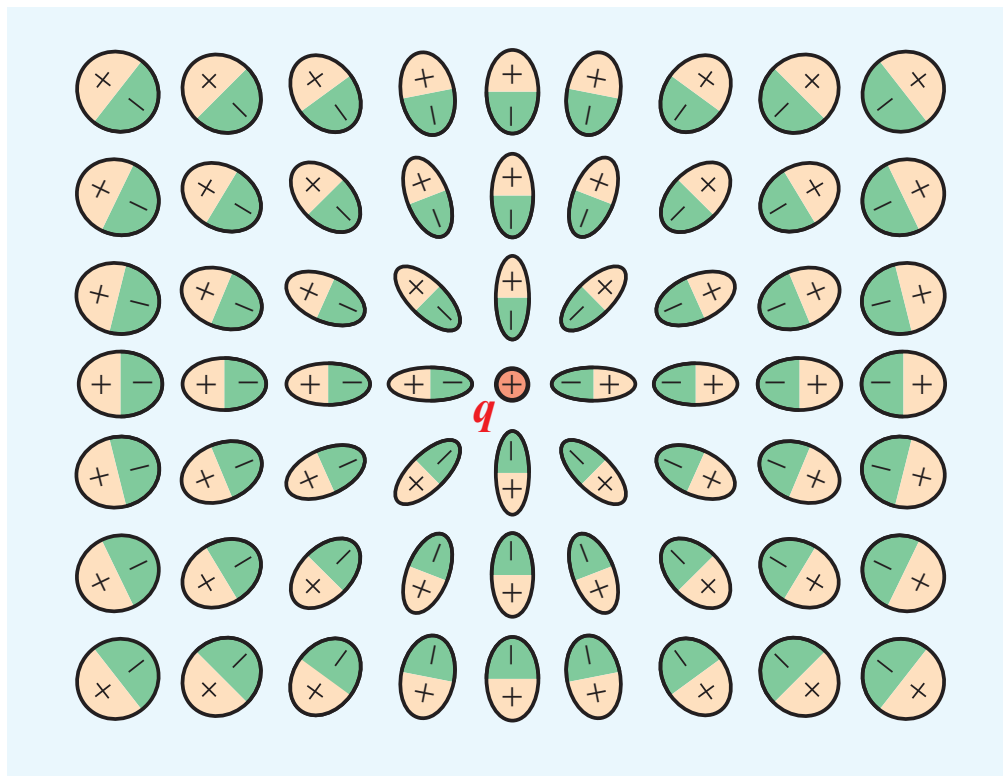


Figure 1.3: Material atoms become *polarized* when point charge q is placed inside

- The atoms align with the \mathbf{E} field forming *electric dipoles*
- The distortion process is known as *polarization*
- The \mathbf{E} is different with and without the material being present

- A polarizable material has a property known as the *relative permittivity* or *dielectric constant*, ϵ_r , which makes $\epsilon_0 \rightarrow \epsilon = \epsilon_r \epsilon_0$
- The electric field expression becomes

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\epsilon R^2} \quad (\text{V/m})$$

(see text Appendix B Table B-1 for material permittivities)

- In a vacuum $\epsilon_r = 1$ while at the surface $\epsilon_r = 1.0006$ Surprised?
- **Definition:** *Electric flux density* \mathbf{D} is given by

$$\mathbf{D} = \epsilon \mathbf{E} \quad (\text{C/m}^2)$$

Example 1.3: Mathematica Field Plotting in a Dielectric

- In class demo.

1.3.3 Magnetic Fields (intro)

- In physics you learn about *magnetism* and may have experimented with iron filings on a sheet of paper placed over a magnet
- The iron filings align in a pattern that matches the magnetic field
- Formally, the magnetic field lines, which are actually loops, are shown below for a bar magnet

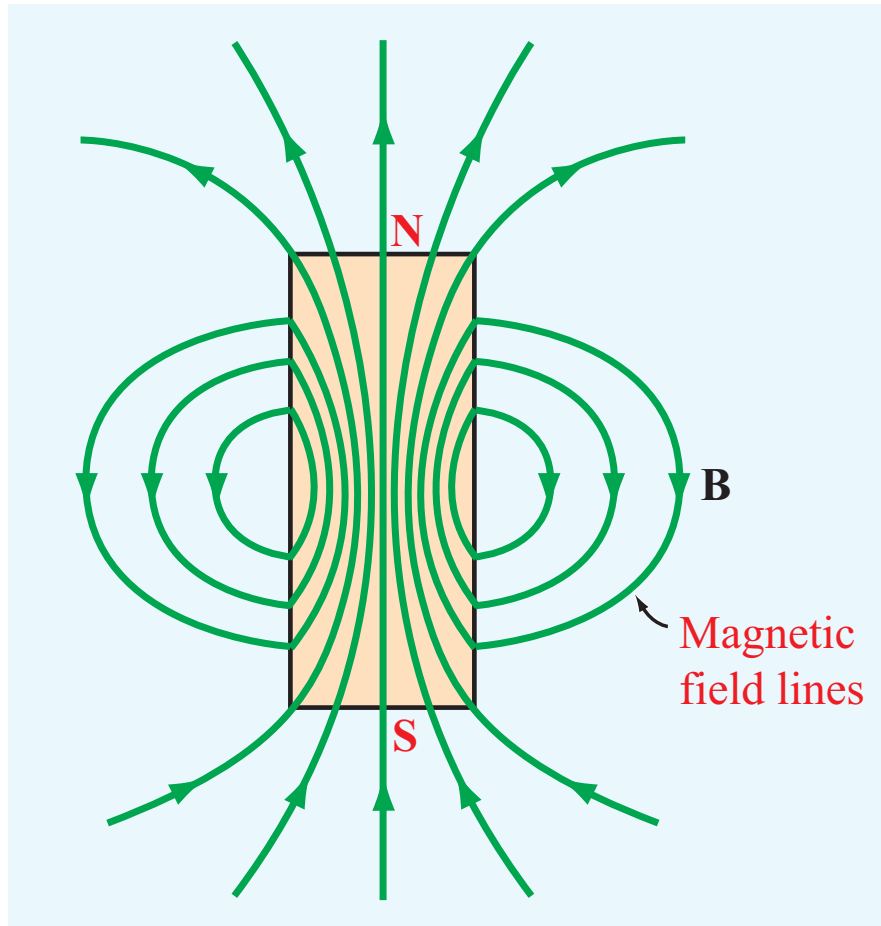


Figure 1.4: Magnetic field lines around a bar magnet

- Like poles of different magnets repel while unlike poles attract
- Attraction and repulsion exists with electric charge, but magnetic poles **must** exist in pairs (no magnetic *monopoles*)
- The magnetic field lines represent the *magnetic flux density* \mathbf{B}

Inducing a \mathbf{B} Field Using a Steady Current

- Electricity and magnetism are related as Oersted showed in 1819 that a magnetic field surrounds a wire carrying an electric current

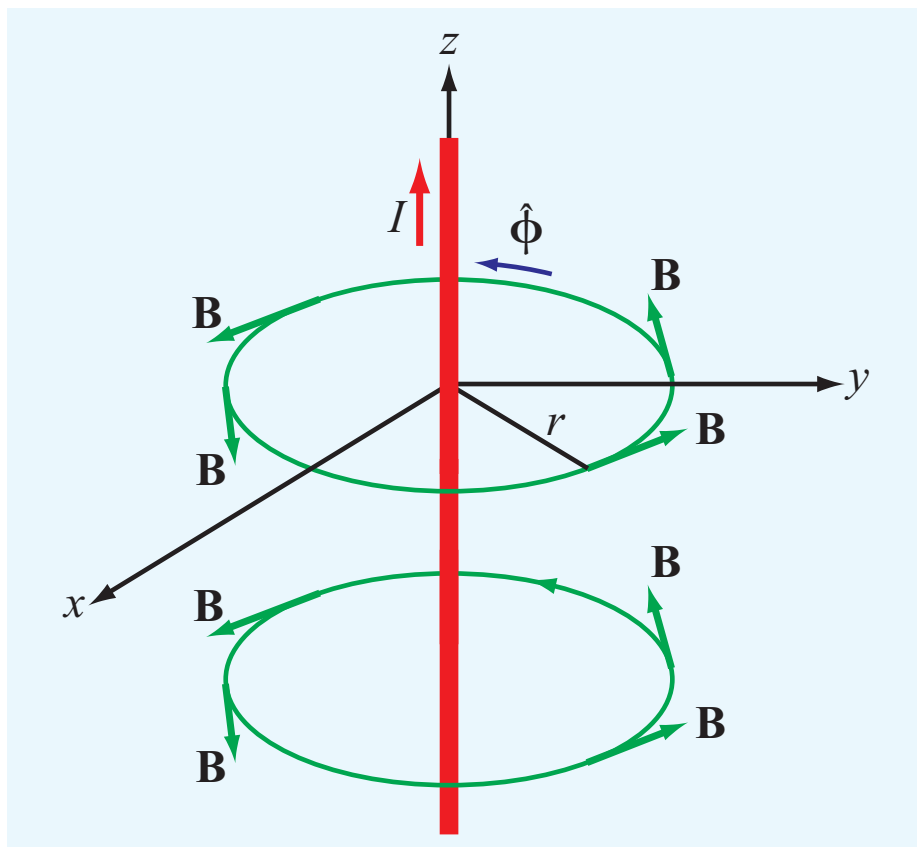


Figure 1.5: The \mathbf{B} field induced by a steady current in the z direction.

- Formally, the *Biot-Savart law* (more in text Chapter 5) says that

$$\mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} \quad (\text{T})$$

where r is the radial distance in the current in the (x, y) plane, $\hat{\phi}$ is the azimuthal unit vector circling the z -axis in the (x, y) plane, and μ_0 is *magnetic permeability of free space* ($\mu_0 = 4\pi \times 10^{-7}$ H/m, H = henry)

- The units for the flux density is tesla (T)
- Note that μ_0 is the magnetic equivalent of ϵ_0 for electric fields

- Also note that as we move into an introduction to traveling waves, we discover that μ_0 and ϵ_0 are related to the velocity of light waves in free space, c , via

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ (m/s)}$$

Magnetic Force (intro)

- We know that a charge q placed in an electric field \mathbf{E} receives a force \mathbf{F}_e
- A charge q that is *moving* with velocity \mathbf{u} through flux density \mathbf{B} , receives magnetic force \mathbf{F}_m perpendicular to both \mathbf{B} and \mathbf{u} via the vector *cross product* (more in Chapter 5)

B Field in a Material or Medium

- When magnetic flux is present in a medium μ_0 is replaced by μ , the magnetic permeability of the material
- Examples of ferromagnetic materials include iron and nickel (for more see text Appendix B Table B-3)
 - Note: Most materials found in nature are *nonmagnetic*
- Similar to the dielectric constant, we have the *relative permeability*, μ_r , defined as

$$\mu = \mu_r \mu_0 \text{ (H/m)}$$

- For pure iron μ_r ranges between 4000 to 5000

- Staying in step with \mathbf{E} and \mathbf{D} for electric fields, we introduce the *magnetic field intensity* \mathbf{H}

$$\mathbf{B} = \mu\mathbf{H} \quad (\text{A/m})$$

1.3.4 Static vs Dynamic Fields

- In the EM field definitions thus far, no mention has been made of *time-varying* fields (other than notation)
- In general fields fall into two classes: static and dynamic, but there are other descriptors used too:
 - **Static:** The quantity is steady or constant over time, as in dc for *direct current*
 - **Dynamic:** The quantity varies with time, in some fashion
 - **Periodic:** The time-varying nature repeats with some period, say T s; also think of a Fourier series decomposition into harmonically related sinusoids, e.g. $1/T = f_0$ with harmonics nf_0 with n an integer
 - **Sinusoidal:** It is very common for time varying quantities to be sinusoidal in time (the math works out nice and is practical, why?)
- The study of dynamic fields is limited in this course
- The departure from static fields is mostly found in Chapter 2 when we study transmission lines and consider *transverse* electromagnetic (TEM) fields

EM Branches

The study of static and dynamic fields breaks out into three branches:

Figure 1.6: The three branches of EM.

Branch	Condition	Field Quantities (Units)
Electrostatics	Stationary charges ($\partial q/\partial t = 0$)	Electric field intensity E (V/m) Electric flux density D (C/m ²) D = $\epsilon\mathbf{E}$
Magnetostatics	Steady currents ($\partial I/\partial t = 0$)	Magnetic flux density B (T) Magnetic field intensity H (A/m) B = $\mu\mathbf{H}$
Dynamics (time-varying fields)	Time-varying currents ($\partial I/\partial t \neq 0$)	E, D, B, and H (E, D) coupled to (B, H)

Fundamental Dynamic Fields Result

A time-varying electric field will generate a time-varying magnetic field (and vice versa)

More On Materials

- Beyond dielectric and magnetic properties of materials we also need to consider *conductivity*

Table 1.3: Material *constitutive* properties.

Parameter	Units	Free-Space Value
Electrical permittivity ϵ	F/m	$\epsilon_0 = 8.854 \times 10^{-12}$ $\approx \frac{1}{36\pi} \times 10^{-9}$
Magnetic permeability μ	H/m	$\mu_0 = 4\pi \times 10^{-7}$
Conductivity σ	S/m	0

1.4 Traveling Waves

Fundamental to dynamic fields and the upcoming study of transmissions line in Chapter 2, is the concept of a *traveling wave*. Both wired, think the internet backbone of fiber optics, and wireless, think WiFi, communications totally rely on propagating waves.

- Traveling waves occur in a variety of physical scenarios
- Waves are found:
 - In water
 - In air as sound pressure
 - As vibrations (earth quake)
 - Electric and magnetic fields traveling through free space or media (this includes radio, microwaves, X-rays, light, ...)
- Common properties:

- Carry energy
- Have velocity; EM waves in free space travel at 3×10^8 m/s
- Frequently are linear in the sense that *superposition* applies; sound pressure waves from multiple people talking simultaneously or the many electromagnetic waves we are immersed in *all the time*
- Waves may be *transient* when a sudden disturbance occurs *periodic*, or perhaps *random*
 - In this class the focus is on periodic sinusoidal waves with few exceptions (superposition is of course possible)
- The one dimensional wave is the easiest to understand, e.g., waves on a string or rope

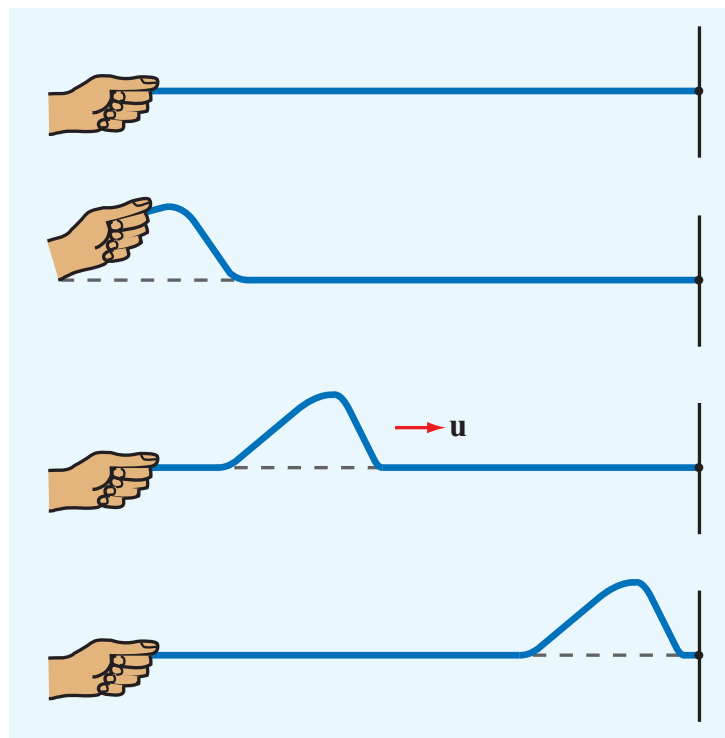


Figure 1.7: Waves in 1D on a string.

- In the Chapter 2 study of transmission lines the 1D wave will be featured
- 2D waves occur on surfaces such a lake or *surface acoustic waves* (SAW) in RF filter designs found in communications receivers
- When a wave propagates through a volume (e.g., air or water) a 3D wave model is appropriate
 - In the 3D case the type of wavefront is also important (see the figure below)

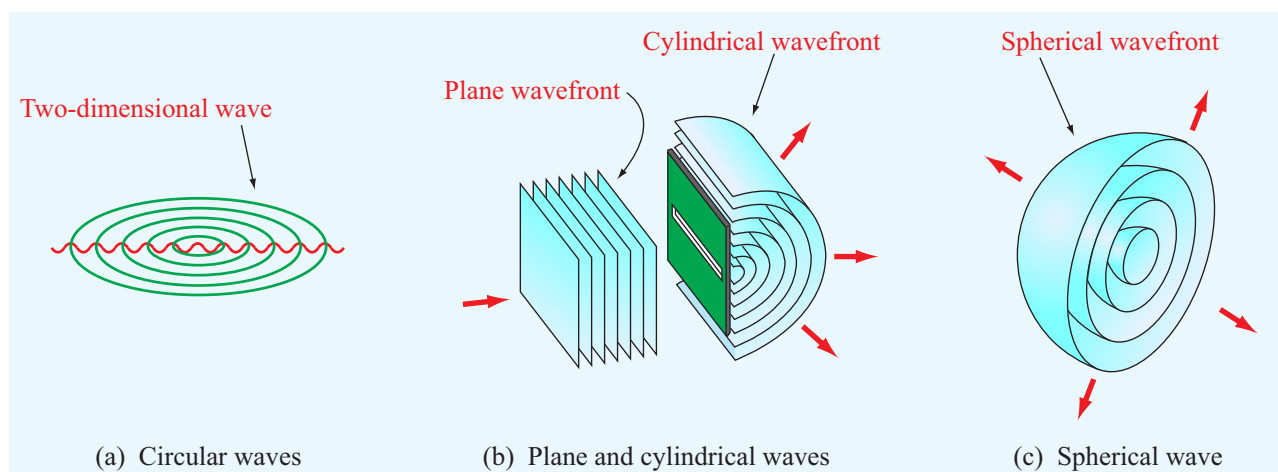


Figure 1.8: Waves in 2D and 3D.

- The remainder of the traveling wave discussion in this chapter is limited to the 1D case with sinusoidal time variation
- What you learn about 1D waves here will prove useful in Chapter 2 the study of transmission lines

1.4.1 Sinusoidal Waves in a Lossless Medium

- A medium is *lossless* if the wave amplitude does not *attenuate* or reduce in amplitude as the wave travels through space
- For simplicity the remainder of this discussion is limited to 1D model, with x the spatial variable
- The 1D general sinusoidal wave mathematical model takes the form

$$y(x, t) = A \cos \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0 \right) \quad (\text{m})$$

where A is the *amplitude* in m, T is the *period* in time, λ is the *spatial wavelength* in m, and ϕ_0 is a *reference phase*

- The action of this model lies in the argument of the cosine, so it is convenient to write

$$y(x, t) = A \cos (\phi(x, t))$$

where

$$\phi(x, t) = \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0 \right) \quad (\text{rad})$$

is the *phase* of the wave

- The wave amplitude is sinusoidal in both x and t , but the periodicity is obviously different, i.e., λ in space and T in time

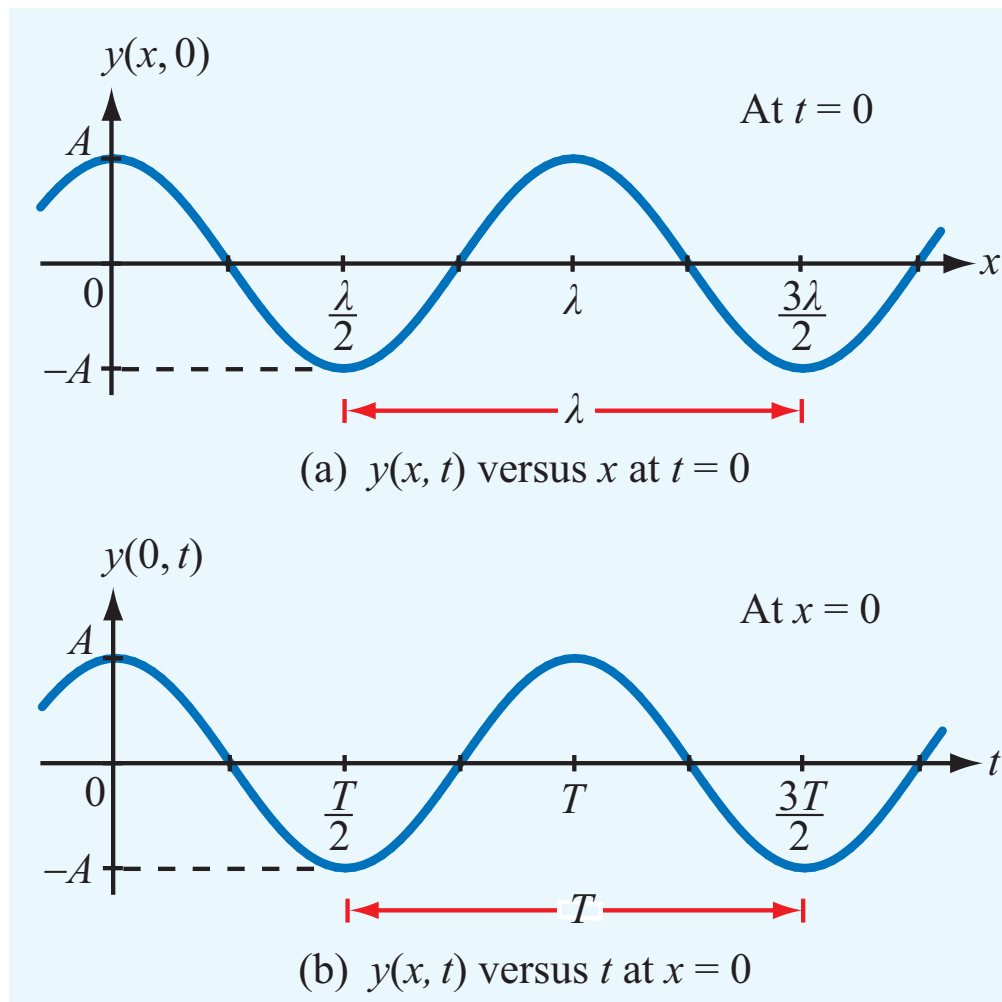


Figure 1.9: $y(x, t)$ versus x then t (assume $\phi_0 = 0$)

- As expected the spatial period is exactly one wavelength (λ) and the temporal period is exactly T s
- The wave travels in either the $+x$ or $-x$ directions, but which way for the form given here?
- Consider snapshots of $y(x, t)$ at increasing time instants versus x and follow the movement of a wave crest along the x -axis

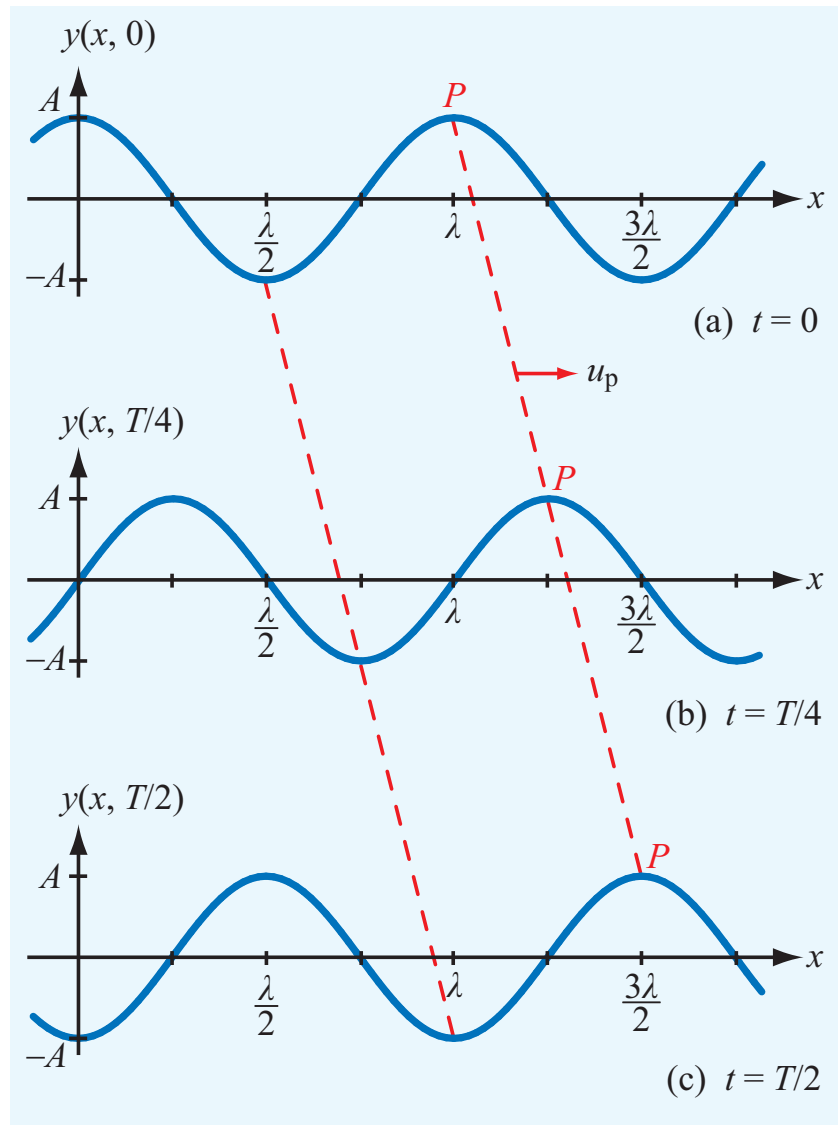


Figure 1.10: 1D wave time snapshots showing wave propagation in the $+x$ direction

- Now that we know the wave is moving in the $+x$ direction, we would like to know the *phase velocity*
- To get a handle on this we follow a particular wave amplitude, $y(x, t) = y_0$ as t increases (again keeping $\phi_0 = 0$) by writing

$$y(x, t) \equiv y_0 = A \cos \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} \right),$$

then solve for the argument of the cosine

$$\text{some constant} = \cos^{-1} \left(\frac{y_0}{A} \right) \stackrel{\text{must}}{=} \frac{2\pi t}{T} - \frac{2\pi x}{\lambda}$$

- The velocity of this constant amplitude can be found by taking the derivative with respect to t of both sides in the above, i.e.,

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} \right] &= \frac{\partial}{\partial t} \text{some constant} \\ \frac{2\pi}{T} - \frac{2\pi}{\lambda} \cdot \frac{dx}{dt} &= 0 \end{aligned}$$

- Finally, solving yields the phase velocity or *propagation velocity* u_p

$$u_p = \frac{dx}{dt} = \frac{\lambda}{T} \quad (\text{m/s})$$

- Since the frequency of the wave is just one over the period, i.e., $f = 1/T$ Hz, we can also write

$$u_p = f\lambda \quad (\text{m/s})$$

- By defining

$$\text{radian frequency} = \omega = 2\pi f \quad (\text{rad/s})$$

$$\text{wavenumber} = \beta = \frac{2\pi}{\lambda} \quad (\text{rad/m})$$

Note: Radian frequency is also denoted *angular velocity* and wavenumber is also denoted *phase constant*

We can alternatively express $y(x, t)$ as

$$\begin{aligned} y_+(x, t) &= A \cos \left(2\pi f t - \frac{2\pi}{\lambda} x \right) \\ &= A \cos (\omega t - \beta x) \end{aligned}$$

where now notation is added to be clear on the wave direction as a superscript $+$ to denote $+x$ propagation

- As a further observation we can alternatively write

$$u_p = f\lambda = \frac{\omega}{\beta}$$

Propagation in the $-x$ Direction

- By changing the sign on the βx we can make the wave propagation in the opposite direction (why?), i.e.,

$$\begin{aligned} y^-(x, t) &= A \cos \left(2\pi f t + \frac{2\pi}{\lambda} x \right) \\ &= A \cos (\omega t + \beta x) \end{aligned}$$

- If the sign in front of the t term is opposite the sign in front of the x term the propagation is $+x$
- If the sign of both terms is identical, plus or minus, the propagation is $-x$

What About ϕ_0 ?

- A while back we dropped ϕ_0 by setting it to zero
- Bringing it back onto the scene, we see that with ϕ_0 present

$$y^\pm(x, t) = A \cos (\omega t \mp \beta x + \phi_0)$$

- The wave either *leads* ($\phi_0 > 0$) or *lags* ($\phi_0 < 0$) the reference ($\phi_0 = 0$ case)

Superimposing $+x$ and $-x$ Waves

- In practical applications coming up in Chapter 2 we will routinely deal with a superposition of $+x$ and $-x$ waves
- Text problem 1.8 begins this exploration by considering

$$y_s(x, t) = y_+(x, t) + y_-(x, t)$$

- When the waves superimpose a *standing wave* pattern is created
 - The magnitude of the wave $y_s(x, t)$ will contain peaks and valleys where there is *constructive* and *destructive* interference occurring
- Chapter 2 will do a lot more with this topic
- There are plenty of practical applications of this happening in our lives, both for acoustical waves, radio waves, and if you are out on the water too

Example 1.4: 1D Electric Field

- Consider the electric field portion of a traveling electromagnetic wave

$$E(z, t) = 10 \cos(\pi \times 10^7 t + \pi z/15 + \pi/6) \quad \text{V/m}$$

- A lot can be said about this model: (a) propagation direction ($+z/-z$), (b) frequency (f) or period (T), (c) wavelength (λ) or wavenumber (β), (d) its phase velocity (u_p)

- So, what are the answers?
 - $f = ?$
 - $T = ?$
 - $\lambda = ?$
 - $\beta = ?$
 - $u_p = ?$
-

1.4.2 Sinusoidal Waves in a Lossy Medium

- Real materials have losses
- The physical details of how the losses come about is not a concern at the present time
- An appropriate model for the losses is a simple exponential decrease in wave amplitude (*attenuation factor*) over distance, i.e., $e^{-\alpha x}$ for a wave traveling in the x direction
- The parameter α is the *attenuation constant* of the medium having units of *neper per meter* (Np/m)
 - The term neper or NP is actually dimensionless, so $1/m$ is the active unit of α
 - See <https://en.wikipedia.org/wiki/Neper>

- The general 1D wave model now takes the form

$$y(x, t) = Ae^{-\alpha x} \cos(\omega t - \beta x + \phi_0)$$

making the effective amplitude $Ae^{-\alpha x}$

- As an example consider $y(x) = y(x, 0)$, where $y(x)$ is displacement in meters

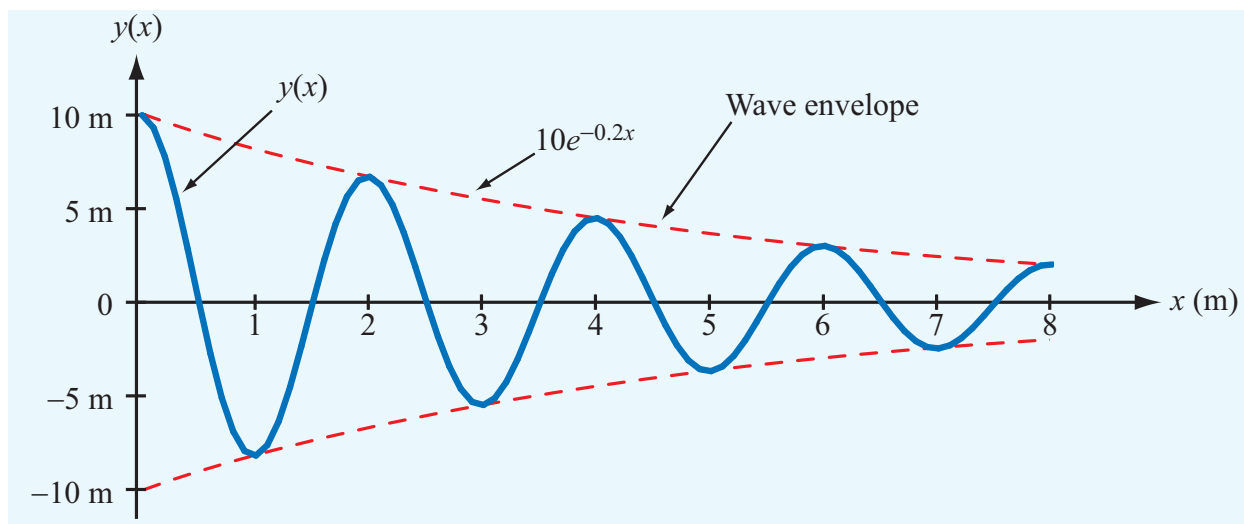


Figure 1.11: Displacement wave attenuation for $\alpha = 0.2$ Np/m.

- Notice that $Ae^{-\alpha x}$ forms an *envelope* about the actual sinusoidal wave
- In the study of transmission lines α play a role

Example 1.5: Power Loss

- Consider a laser beam propagating through the atmosphere with electric field

$$E(x, t) = 150e^{-0.003x} \cos(3 \times 10^{15} - 10^7 x) \quad (\text{V/m})$$

- The attenuation is due to absorption by atmospheric gases (see the next section)
- Of interest is: (a) the propagation direction, (b) frequency, (c) wavelength, (d) the wave velocity, and (e) the wave amplitude 200 m into the medium
- Answers:
 - Due to the sign differences in the cosine arguments the wave propagates in the $+x$ direction
 - The frequency is $3 \times 10^{15}/(2\pi) = 477.4$ THz
 - The wavelength is $2\pi/10^7 = 0.628 \mu\text{m}$
 - The wave velocity is $f\lambda \stackrel{\text{also}}{=} \omega/\beta$ where $\omega = 3 \times 10^{15}$ and $\beta = 10^7$, so

$$u_p = 3 \times 10^{15}/10^7 = 3 \times 10^8 \quad (\text{m/s}),$$

which is the speed of light in a vacuum; surprised?

- The wave amplitude of 150 V/m decays at 200 m to

$$150 \cdot e^{-0.003 \times 200} = 0.37 \quad (\text{V/m})$$

In Python with PyLab

```
In [3]: print('%3.2f' % (150*exp(-0.03*200),))
0.37
```

1.5 The EM Spectrum

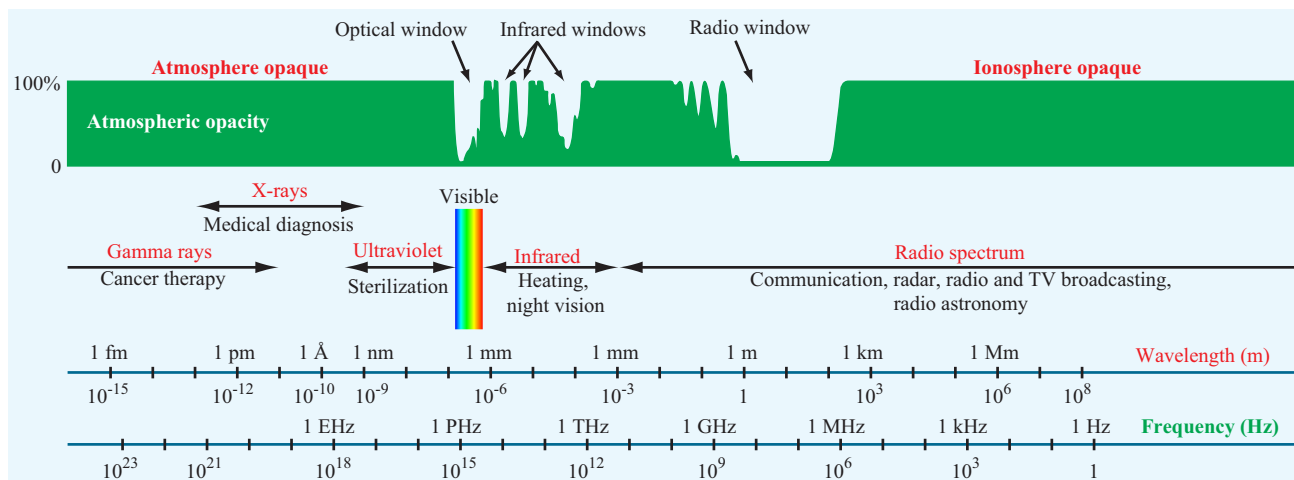


Figure 1.12: The electromagnetic spectrum.

- The electromagnetic spectrum covers DC to light, OK DC does not propagate in free space, but very low frequencies do, e.g. 60 Hz
- Single frequency (fancy name is *monochromatic*) EM waves have both an electric and magnetic field at the same frequency (dynamic field property)
- The phase velocity, u_p , in a vacuum is always c
- Thus the wavelength is always related to the frequency via

$$\lambda = \frac{c}{f} \quad \text{or} \quad f = \frac{c}{\lambda}$$

- At optical frequencies it is more convenient to wavelength, e.g.

1.5.1 Federal Communications Commission

- In the United States the Federal Communications Commission (FCC) is responsible for managing radio frequency spectrum

1.6 Review of Complex Numbers

- Do juniors and seniors in electrical engineering (EE) need a review of complex arithmetic?
- The text, pp. 34–36, contains a nice review that you may find helpful
- What is important is being able to:
 - Do pencil and paper analysis with complex variables in the expressions
 - Be able to do numerical calculations using a computer tool such as Python (PyLab) or MATLAB
 - Be able to do numerical calculations using a pocket calculator on quizzes and exams

1.6.1 A Few Basics

- To the EE we define $j = \sqrt{-1}$ so as not to confuse current, i , with imaginary numbers
- A general complex number is $z = x + jy$ where the real part of z is $x = \text{Re}(z)$ and imaginary part $y = \text{Im}(z)$
- The *rectangular form* has components x and y as already noted, while switching to polar form yields the magnitude and phase angle, i.e.,

$$z = x + jy = |z|e^{j\theta}$$

where $|z| = r = \sqrt{x^2 + y^2}$ and θ is the angle formed by the point (x, y) in the complex plane relative to the positive real axis

- **Note:** Finding θ via $\tan^{-1}(y/x)$ has issues that requires you pay attention to the quadrant where z lies
- In Python (numpy) `angle(3 + 1j*5)` is better
- On a the TI-89 `angle(3+5i)` is also available

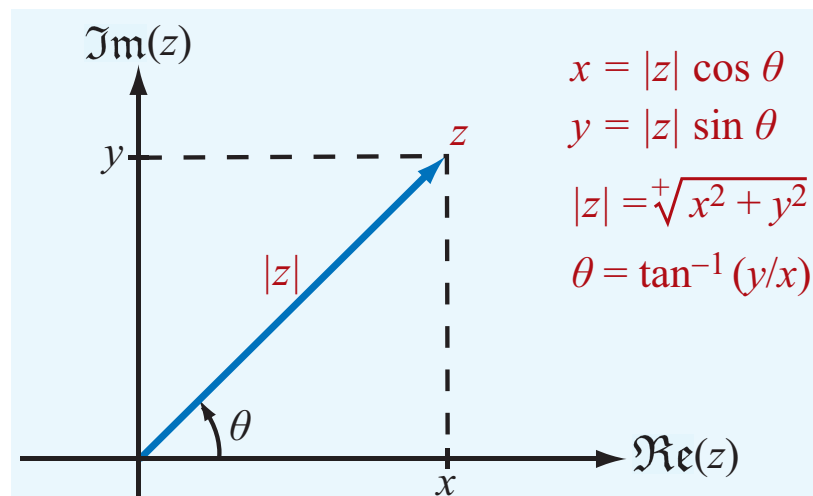


Figure 1.14: z in the complex plane

- Fundamental to the work of this course, especially Chapter 2, is Euler's identity

$$e^{j\theta} = \cos\theta + j \sin\theta$$

- The inverse forms are also useful:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Example 1.6: Working with Complex Numbers

- Consider $z_1 = 6 - j11$ and $z_2 = 5 + j17$
- The Python function `cpx_fmt(z, mode1, mode2, mode3, d1, d2)` is included in the Chapter 1 IPYNB file found on the Web Site
- This function allows formatting of complex arithmetic calculations into ‘rect’ and ‘polar’ forms (mode1) with either ‘f’loat or ‘e’xponential number format (mode2), and angle display in ‘rad’ians or ‘deg’rees (mode3), and decimal precision controlled by `d1=2` (default) and `d2=4` (default)

```
z1 = 6-11j
z2 = 5+17j
```

```
print('z1+z2 = ' + cpx_fmt(z1+z2, 'rect', 'f', 'deg', 3, 2))
print('z1+z2 = ' + cpx_fmt(z1+z2, 'polar', 'f', 'deg', 3, 2))
print('z1+z2 = ' + cpx_fmt(z1+z2, 'polar', 'f', 'rad', 3, 2))
print('z1-z2 = ' + cpx_fmt(z1-z2, 'rect', 'f', 'deg', 3, 3))
print('z1-z2 = ' + cpx_fmt(z1-z2, 'polar', 'f', 'deg', 3, 3))
print('z1-z2 = ' + cpx_fmt(z1-z2, 'polar', 'f', 'rad', 3, 3))
print('z1*z2 = ' + cpx_fmt(z1*z2, 'rect', 'f', 'deg', 3, 3))
print('z1*z2 = ' + cpx_fmt(z1*z2, 'polar', 'f', 'deg', 3, 3))
print('z1*z2 = ' + cpx_fmt(z1*z2, 'polar', 'f', 'rad', 3, 3))
print('z1/z2 = ' + cpx_fmt(z1/z2, 'rect', 'e', 'deg', 3, 3))
print('z1/z2 = ' + cpx_fmt(z1/z2, 'polar', 'e', 'deg', 3, 3))
print('z1/z2 = ' + cpx_fmt(z1/z2, 'polar', 'e', 'rad', 3, 3))
```

```
z1+z2 = 11.00 + j6.00
z1+z2 = 12.53 /_ 28.61 (deg)
z1+z2 = 12.53 /_ 0.50 (rad)
z1-z2 = 1.000 - j28.000
z1-z2 = 28.018 /_ -87.955 (deg)
z1-z2 = 28.018 /_ -1.535 (rad)
z1*z2 = 217.000 + j47.000
z1*z2 = 222.032 /_ 12.221 (deg)
z1*z2 = 222.032 /_ 0.213 (rad)
z1/z2 = -5.000e-01 - j5.000e-01
z1/z2 = 7.071e-01 /_ -1.350e+02 (deg)
z1/z2 = 7.071e-01 /_ -2.356e+00 (rad)
```

Figure 1.15: Complex arithmetic in Python (numpy).

- The calculations using a TI-89 are as follows

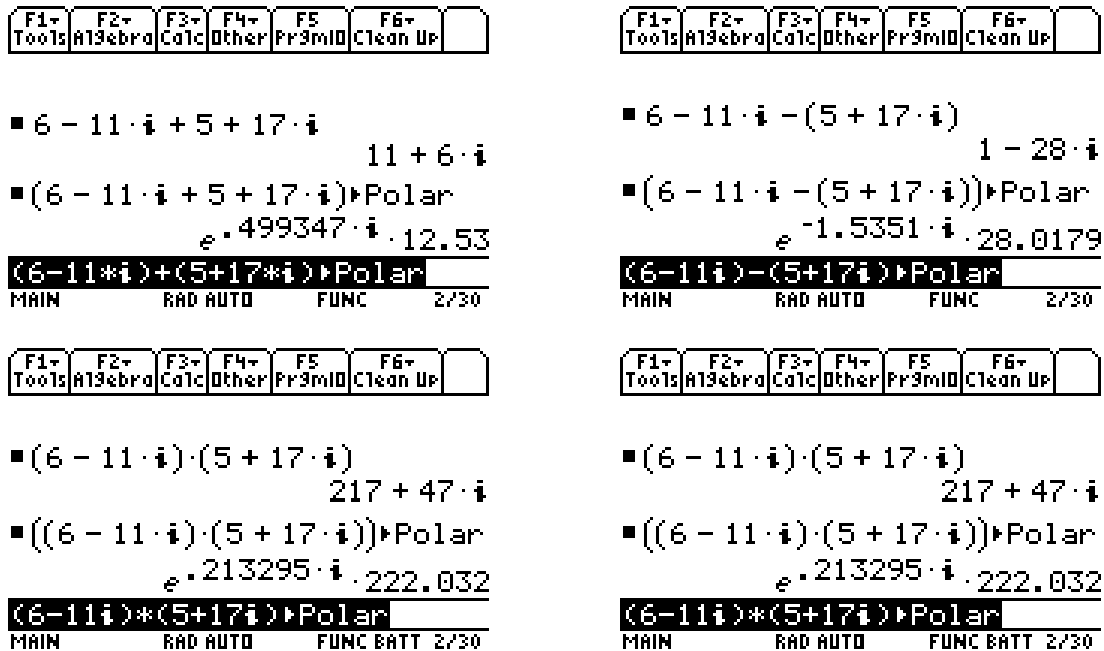


Figure 1.16: Complex arithmetic on a TI-89.

- The calculator agrees, but note the angles are in radians by default

1.7 Review of Phasors and S -Domain Analysis

- This topic is at the heart and soul of traditional circuits and systems EE
- In the study of electromagnetics, and in particular transmission lines, having this background is vital
- The phasor analysis techniques are founded in *steady-state* analysis of *linear* and *time-invariant* systems

- The input or *forcing function* is assumed to be periodic and most often sinusoidal, although with Fourier series concepts, any periodic waveform can be dealt with via superposition
- The text review focuses on phasor analysis, but the use of the Laplace transform and the s -domain, is also very useful
- When possible, the s -domain (steady-state for now) is favored as the notation is cleaner
- Moving in and out of the time-domain still requires an understanding of phasors

1.7.1 The Classical RC Lowpass Filter

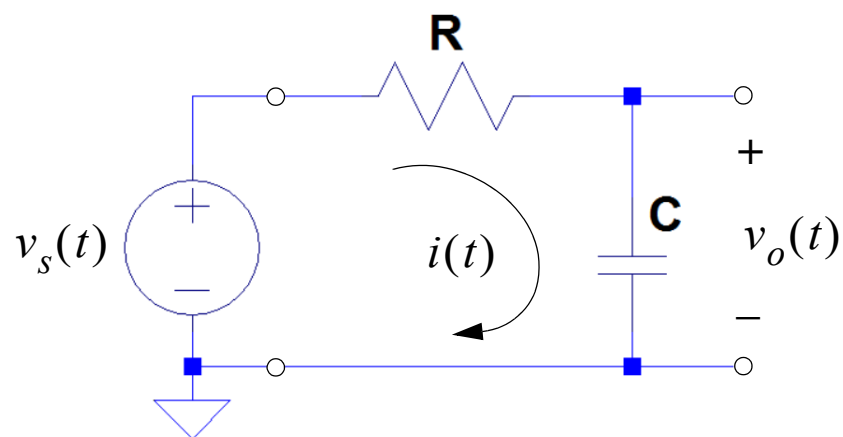


Figure 1.17: RC lowpass filter with input $v_s(t)$.

- Assume the input is of the form

$$v_s(t) = V_0 \cos(\omega t + \phi_0), \quad -\infty < t < \infty$$

- The input source or forcing function has phasor representation derived form

$$v_s(t) = \text{Re}[V_0 e^{j(\omega t + \phi_0)}] = \text{Re}[V_0 e^{j\phi_0} \cdot e^{j\omega t}]$$

- Specifically the phasor corresponding to $v_s(t)$ is given by

$$\tilde{V}_s = V_0 e^{j\phi}$$

Note: The time varying nature of the signal is now suppressed

- The s -domain analysis is very similar, except we would say

$$V_s(s) = V_0 e^{j\phi}$$

- Suppose we are interested in $i(t)$ and $v_o(t)$
- Using s -domain analysis for the circuit analysis of voltage and current, we can write (using Kirchoff's Laws)

$$I(s) = \frac{V_s(s)}{z(s)} = \frac{V_s(s)}{R + \frac{1}{sC}} = V_s(s) \cdot \frac{sC}{1 + sRC}$$

$$V_o(s) = V_s(s) \cdot \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = V_s(s) \cdot \frac{1}{1 + sRC}$$

- To obtain the phasor current and voltage we let $s \rightarrow j\omega$

$$\tilde{I} = V_0 e^{j\phi_0} \cdot \frac{j\omega C}{1 + j\omega RC}$$

$$\tilde{V}_o = V_0 e^{j\phi_0} \cdot \frac{1}{1 + j\omega RC}$$

- The system (s -domain based) or transfer functions ($j\omega$ -domain based) $j\omega C / (1 + j\omega RC)$ and $1 / (1 + j\omega RC)$ need to be further evaluated in polar form once values are given for ω , R , and C

- Continue to solve for the current and voltage waveforms assuming that $V_0 = 5$ V, $\phi_0 = \pi/4$, $R = 1000$ ohms, $C = 10$ nF, and $f = 2$ kHz ($\omega = 2\pi f$)
- We will use Python (PyLab) to obtain the numerical values for \tilde{I} and \tilde{V}_s

```

# Constants
f = 2e3 # Hz
Vs = 5*exp(1j*pi/4) # Input phasor voltage
R = 1000 # ohms
C = 500e-9 # F
# Transfer functions
s = 1j*2*pi*f
H_I = s*C/(1 + s*R*C) # Vs to I
print('H_I(w) = ' + cplx_fmt(H_I, 'polar', 'e', 'deg', 1, 2))
H_Vo = 1/(1 + s*R*C) # Vs to Vo
print('H_Vo(w) = ' + cplx_fmt(H_Vo, 'polar', 'e', 'deg', 1, 2))
# Phasor quantities
I = Vs*H_I # Loop current phasor
print('*****')
print('I(phasor) = ' + cplx_fmt(I, 'polar', 'e', 'deg', 1, 2))
Vo = Vs*H_Vo # Output voltage phasor
print('Vo(phasor) = ' + cplx_fmt(Vo, 'polar', 'e', 'deg', 1, 2))

H_I(w) = 9.88e-04 / _9.04e+00 (deg)
H_Vo(w) = 1.57e-01 / _-8.10e+01 (deg)
*****
I(phasor) = 4.94e-03 / _5.40e+01 (deg)
Vo(phasor) = 7.86e-01 / _-3.60e+01 (deg)

```

Figure 1.18: Lowpass circuit phasor analysis source and component values.

- The steady-state time domain waveforms can be written making use of the above phasor quantities

$$i(t) = 4.94 \times 10^{-3} \cos(2\pi(2000)t + 54.0^\circ) \quad (\text{A})$$

$$v_o(t) = 0.786 \cos(2\pi(2000)t - 36.0^\circ) \quad (\text{V})$$

- We can check the analysis by setting up an LTspice circuit simulation

- AC analysis is the most appropriate as it will give us phasor quantities directly (a frequency sweep which includes 2000 Hz will be needed)

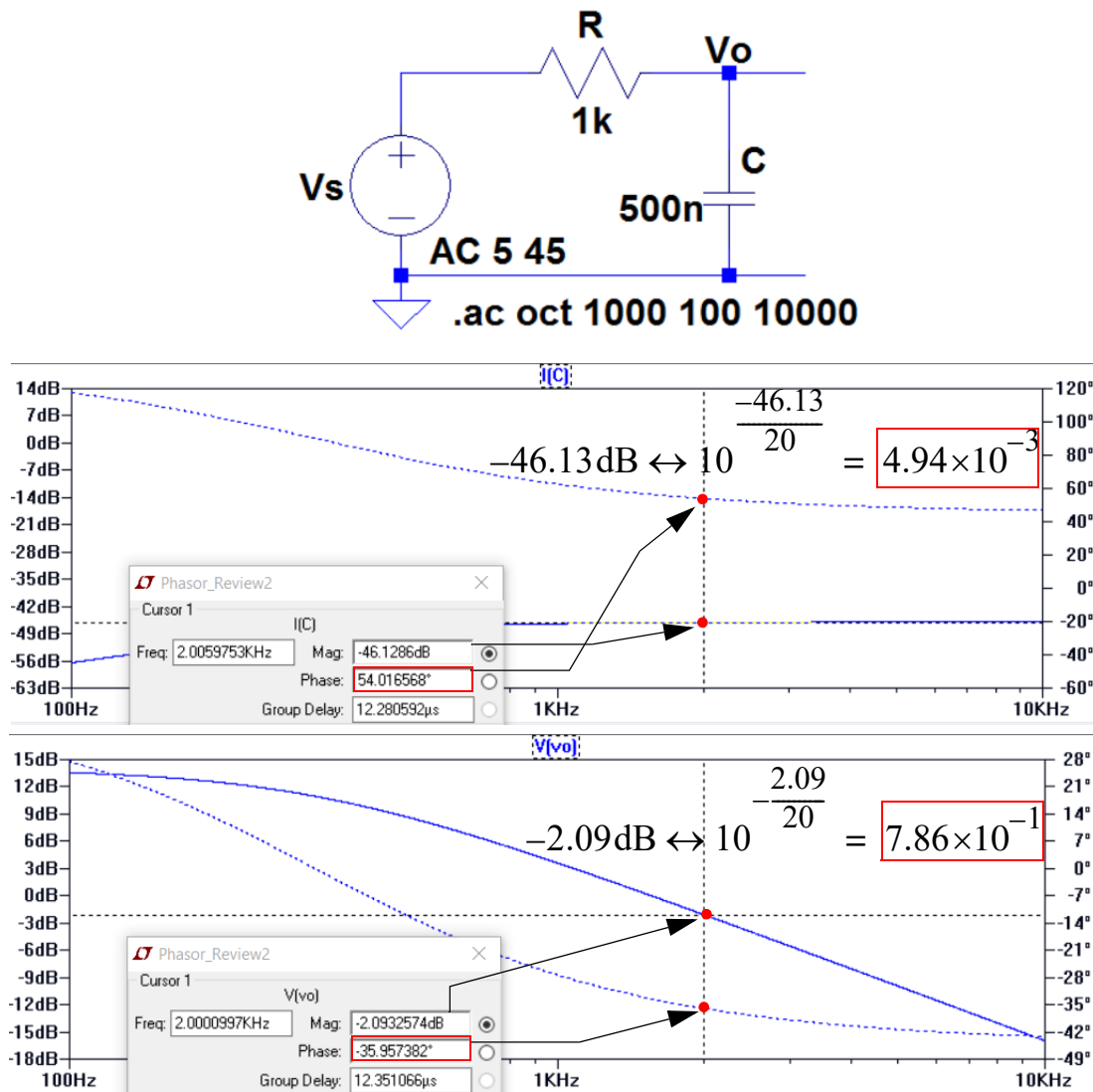


Figure 1.19: LTspice simulation of the RC lowpass system.

- The hand calculation (using Python) and the LTspice AC analysis are in agreement (no surprise)

1.7.2 Traveling Waves in the Phasor and s -Domains

- Consider the $+x$ traveling wave

$$v^+(x, t) = Ae^{-\alpha t} \cos(\omega t - \beta x + \phi_0)$$

- A phasor representation is possible if we consider x as a parameter, that is

$$v^+(x, t) = Ae^{-\alpha x} \operatorname{Re}[e^{-j(\beta x - \phi_0)} \cdot e^{j\omega t}]$$

so

$$\tilde{V}^+(x) = Ae^{-\alpha x} e^{-j(\beta x - \phi_0)}$$

- The s -domain view makes use of the Laplace transform delay theorem

$$x(t - t_0) \xleftrightarrow{\mathcal{L}} X(s)e^{-st_0}$$

- For this to work we need to make the cosine argument look like $\omega(t - t_0)$, so we write

$$\omega t - \beta x + \phi_0 = \omega \left(t - \frac{\beta x - \phi_0}{\omega} x \right)$$

which implies that

$$t_0 = \frac{x}{u_p} - \frac{\phi_0}{\omega} \quad (\text{s})$$

- The first term is a time delay due to the wavefront location along the x -axis
- The second term is a time delay/advance due to the phase shift ϕ_0 (you likely saw this in ECE2610)

- So, given that $\mathcal{L}\{v^+(x, t)\} = V_+(x, s)$, we have

$$\begin{aligned} V^+(x, s) &= V_+(0, s) \cdot e^{-\alpha x} e^{-s \cdot t_0} \\ &= V^+(0, s) \cdot e^{-\alpha x} \underbrace{e^{-sx/u_p} e^{s\phi_0/\omega}}_{s\text{-domain time delay}} \end{aligned}$$

- The phasor and perhaps the s -domain returns in Chapter 2

1.7.3 Sinusoids and Their Corresponding Cosine-Reference Phasors

Table 1.4: Useful phasor results for $z(t) = \text{Re}[\tilde{Z}e^{j\omega t}]$.

$z(t)$		\tilde{Z}
$A \cos \omega t$	\longleftrightarrow	A
$A \cos(\omega t + \phi_0)$	\longleftrightarrow	$Ae^{j\phi_0}$
$A \cos(\omega t + \beta x + \phi_0)$	\longleftrightarrow	$Ae^{j(\beta x + \phi_0)}$
$Ae^{-\alpha x} \cos(\omega t + \beta x + \phi_0)$	\longleftrightarrow	$Ae^{-\alpha x} e^{j(\beta x + \phi_0)}$
$A \sin \omega t$	\longleftrightarrow	$Ae^{-j\pi/2}$
$A \sin(\omega t + \phi_0)$	\longleftrightarrow	$Ae^{j(\phi_0 - \pi/2)}$
$\frac{d}{dt}(z(t))$	\longleftrightarrow	$j\omega\tilde{Z}$
$\frac{d}{dt}[A \cos(\omega t + \phi_0)]$	\longleftrightarrow	$j\omega Ae^{j\phi_0}$
$\int z(t) dt$	\longleftrightarrow	$\frac{1}{j\omega}\tilde{Z}$
$\int A \sin(\omega t + \phi_0) dt$	\longleftrightarrow	$\frac{1}{j\omega} Ae^{j(\phi_0 - \pi/2)}$