CHAPTER 23

GAUSS' LAW

23-1 WHAT IS PHYSICS?

One of the primary goals of physics is to find simple ways of solving seemingly complex problems. One of the main tools of physics in attaining this goal is the use of symmetry. For example, in finding the electric field $\vec{E}$ of the charged ring of Fig. 22-10 and the charged rod of Fig. 22-11, we considered the fields $d\vec{E}$ $(= k\,dq/r^2)$ of charge elements in the ring and rod. Then we simplified the calculation of $\vec{E}$ by using symmetry to discard the perpendicular components of the $d\vec{E}$ vectors. That saved us some work.

For certain charge distributions involving symmetry, we can save far more work by using a law called Gauss' law, developed by German mathematician and physicist Carl Friedrich Gauss (1777–1855). Instead of considering the fields $d\vec{E}$ of charge elements in a given charge distribution, Gauss' law considers a hypothetical (imaginary) closed surface enclosing the charge distribution. This **Gaussian surface,** as it is called, can have any shape, but the shape that minimizes our calculations of the electric field is one that mimics the symmetry of the charge distribution. For example, if the charge is spread uniformly over a sphere, we enclose the sphere with a spherical Gaussian surface, such as the one in Fig. 23-1, and then, as we discuss in this chapter, find the electric field on the surface by using the fact that

Gauss' law relates the electric fields at points on a (closed) Gaussian surface to the net charge enclosed by that surface.

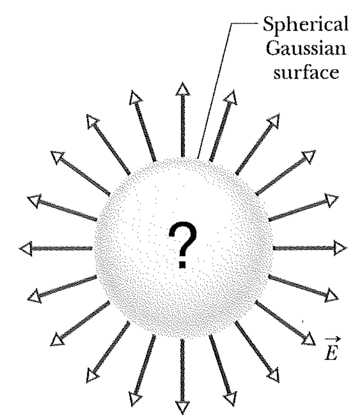

Fig. 23-1 A spherical Gaussian surface. If the electric field vectors are of uniform magnitude and point radially outward at all surface points, you can conclude that a net positive distribution of charge must lie within the surface and have spherical symmetry.

We can also use Gauss' law in reverse: If we know the electric field on a Gaussian surface, we can find the net charge enclosed by the surface. As a limited example, suppose that the electric field vectors in Fig. 23-1 all point radially outward from the center of the sphere and have equal magnitude. Gauss' law immediately tells us that the spherical surface must enclose a net positive charge that is either a particle or distributed spherically. However, to calculate how *much* charge is enclosed, we need a way of calculating how much electric field is intercepted by the Gaussian surface in Fig. 23-1. This measure of intercepted field is called *flux*, which we discuss next.

23-2 Flux

Suppose that, as in Fig. 23-2*a*, you aim a wide airstream of uniform velocity $\vec{v}$ at a small square loop of area A. Let Φ represent the *volume flow rate* (volume per unit time) at which air flows through the loop. This rate depends on the angle between $\vec{v}$ and the plane of the loop. If $\vec{v}$ is perpendicular to the plane, the rate Φ is equal to vA.

If $\vec{v}$ is parallel to the plane of the loop, no air moves through the loop, so Φ is zero. For an intermediate angle θ, the rate Φ depends on the component of $\vec{v}$ normal to the plane (Fig. 23-2*b*). Since that component is $v\cos\theta$, the rate of volume flow through the loop is

$$\Phi = (v\cos\theta)A. \tag{23-1}$$

This rate of flow through an area is an example of a **flux**—a *volume flux* in this situation.

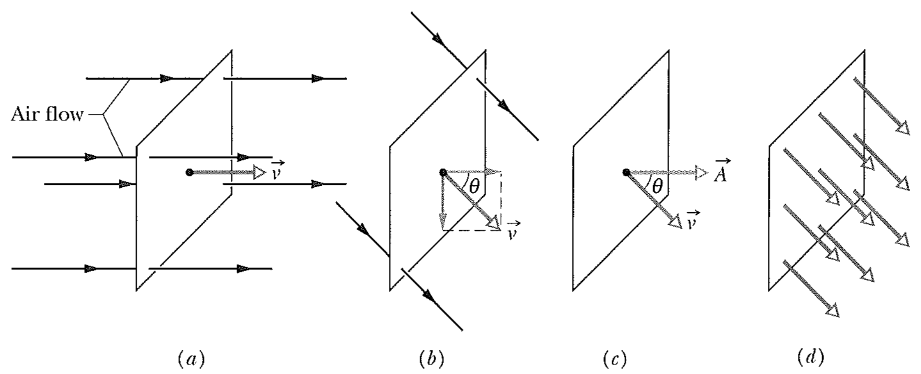

Fig. 23-2 (*a*) A uniform airstream of velocity $\vec{v}$ is perpendicular to the plane of a square loop of area A. (*b*) The component of $\vec{v}$ perpendicular to the plane of the loop is $v \cos \theta$, where θ is the angle between $\vec{v}$ and a normal to the plane. (*c*) The area vector $\vec{A}$ is perpendicular to the plane of the loop and makes an angle θ with $\vec{v}$. (*d*) The velocity field intercepted by the area of the loop.

Before we discuss a flux involved in electrostatics, we need to rewrite Eq. 23-1 in terms of vectors. To do this, we first define an *area vector* $\vec{A}$ as being a vector whose magnitude is equal to an area (here the area of the loop) and whose direction is normal to the plane of the area (Fig. 23-2*c*). We then rewrite Eq. 23-1 as the scalar (or dot) product of the velocity vector $\vec{v}$ of the airstream and the area vector $\vec{A}$ of the loop:

$$\Phi = vA \cos \theta = \vec{v} \cdot \vec{A}, \tag{23-2}$$

where θ is the angle between $\vec{v}$ and $\vec{A}$.

The word "flux" comes from the Latin word meaning "to flow." That meaning makes sense if we talk about the flow of air volume through the loop. However, Eq. 23-2 can be regarded in a more abstract way. To see this different way, note that we can assign a velocity vector to each point in the airstream passing through the loop (Fig. 23-2*d*). Because the composite of all those vectors is a *velocity field*, we can interpret Eq. 23-2 as giving the *flux of the velocity field through the loop.* With this interpretation, flux no longer means the actual flow of something through an area—rather it means the product of an area and the field across that area.

23-3 Flux of an Electric Field

To define the flux of an electric field, consider Fig. 23-3, which shows an arbitrary (asymmetric) Gaussian surface immersed in a nonuniform electric field. Let us divide the surface into small squares of area ΔA, each square being small enough to permit us to neglect any curvature and to consider the individual square to be flat. We represent each such element of area with an area vector $\Delta\vec{A}$, whose magnitude is the area ΔA. Each vector $\Delta\vec{A}$ is perpendicular to the Gaussian surface and directed away from the interior of the surface.

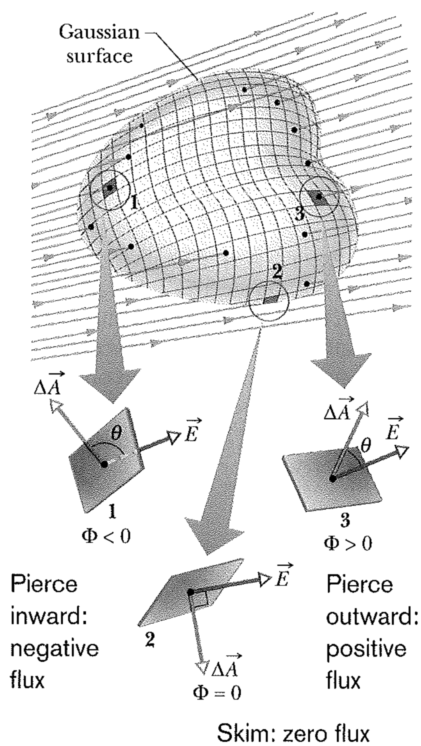

Fig. 23-3 A Gaussian surface of arbitrary shape immersed in an electric field. The surface is divided into small squares of area ΔA. The electric field vectors $\vec{E}$ and the area vectors $\Delta\vec{A}$ for three representative squares, marked 1, 2, and 3, are shown.

Because the squares have been taken to be arbitrarily small, the electric field $\vec{E}$ may be taken as constant over any given square. The vectors $\Delta\vec{A}$ and $\vec{E}$ for each square then make some angle θ with each other. Figure 23-3 shows an enlarged view of three squares on the Gaussian surface and the angle θ for each.

A provisional definition for the flux of the electric field for the Gaussian surface of Fig. 23-3 is

$$\Phi = \sum \vec{E} \cdot \Delta\vec{A}. \tag{23-3}$$

This equation instructs us to visit each square on the Gaussian surface, evaluate the scalar product $\vec{E} \cdot \Delta\vec{A}$ for the two vectors $\vec{E}$ and $\Delta\vec{A}$ we find there, and sum the results algebraically (that is, with signs included) for all the squares that make up the surface. The value of each scalar product (positive, negative, or zero) determines whether the flux through its square is positive, negative, or zero. Squares like square 1 in Fig. 23-3, in which $\vec{E}$ points inward, make a negative contribution to the sum of Eq. 23-3. Squares like 2, in which $\vec{E}$ lies in the surface, make zero contribution. Squares like 3, in which $\vec{E}$ points outward, make a positive contribution.