Response of underdamped systems to free vibrations

Underdamped systems

The solution for the under-damped system have the form

$$u(t) = e^{-\zeta\omega_n t}\left[A.\mathrm{Cos}(\omega_D t) + B.\,\mathrm{Sin}(\omega_D t)\right]$$

$$\text{where } \omega_D = \omega_n\sqrt{1-\zeta^2}$$

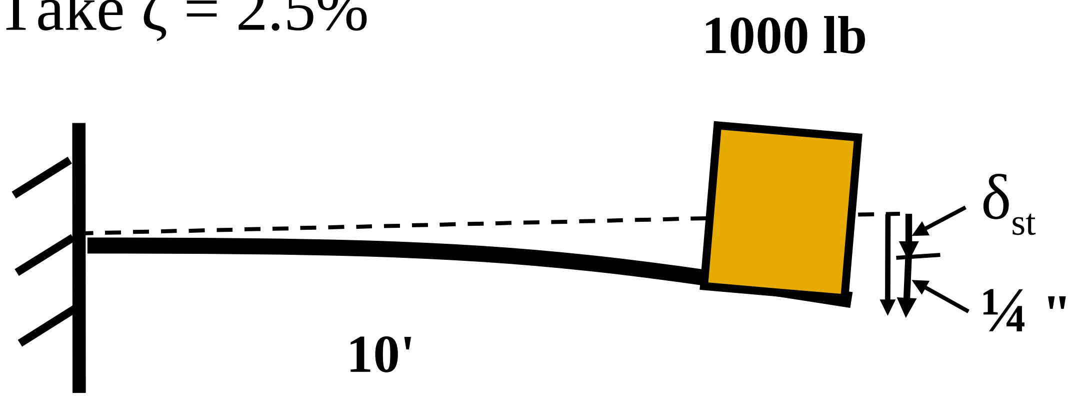
This is similar to the case for undamped free vibration except that the frequency is slightly smaller and there is a decay of the response with time. Again the constants **A** and **B** may be found from the solution at two different times or from the initial conditions at time t = 0

$$u(t) = e^{-\zeta\omega_n t}\left[u(0)\mathrm{Cos}(\omega_D t) + \frac{\dot{u}(0) + u(0)\zeta\omega_n}{\omega_D}\mathrm{Sin}(\omega_D t)\right]$$

Problem M4.3

For the beam's data given in problem M 4.1, develop and solve the equation of motion for vibrations resulting at free end.

Also develop an equation showing variation in the *Equivalent static forces* with time. Take $\zeta = 2.5\%$



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Solution (M4.3)

E.O.M for damped free vibrations is:

$$ku + c\dot{u} + m\ddot{u} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

It is known from problem M4.1 that:

$k = 90625$ lb/ft and $m = 31.06$ lb.sec^2/ft

$$c = \zeta * 2m\omega_n = 2*31.06*54.9*0.025$$

$c = 85.26$ lb. sec /ft

By substituting vlaues of k,c and m in (1) we get

$$90625u + 85.26\dot{u} + 31.06\ddot{u} = 0$$

Solution to the E.O.M for damped free vibration is:

$$u(t) = e^{-\zeta\omega_n t} \left[u(0) \cos(\omega_D t) + \frac{1}{\omega_D} [\dot{u}(0) + u(0)\, \zeta\, \omega_n] . \sin(\omega_D t) \right.$$

$\omega_D = 54.9$ rad/sec

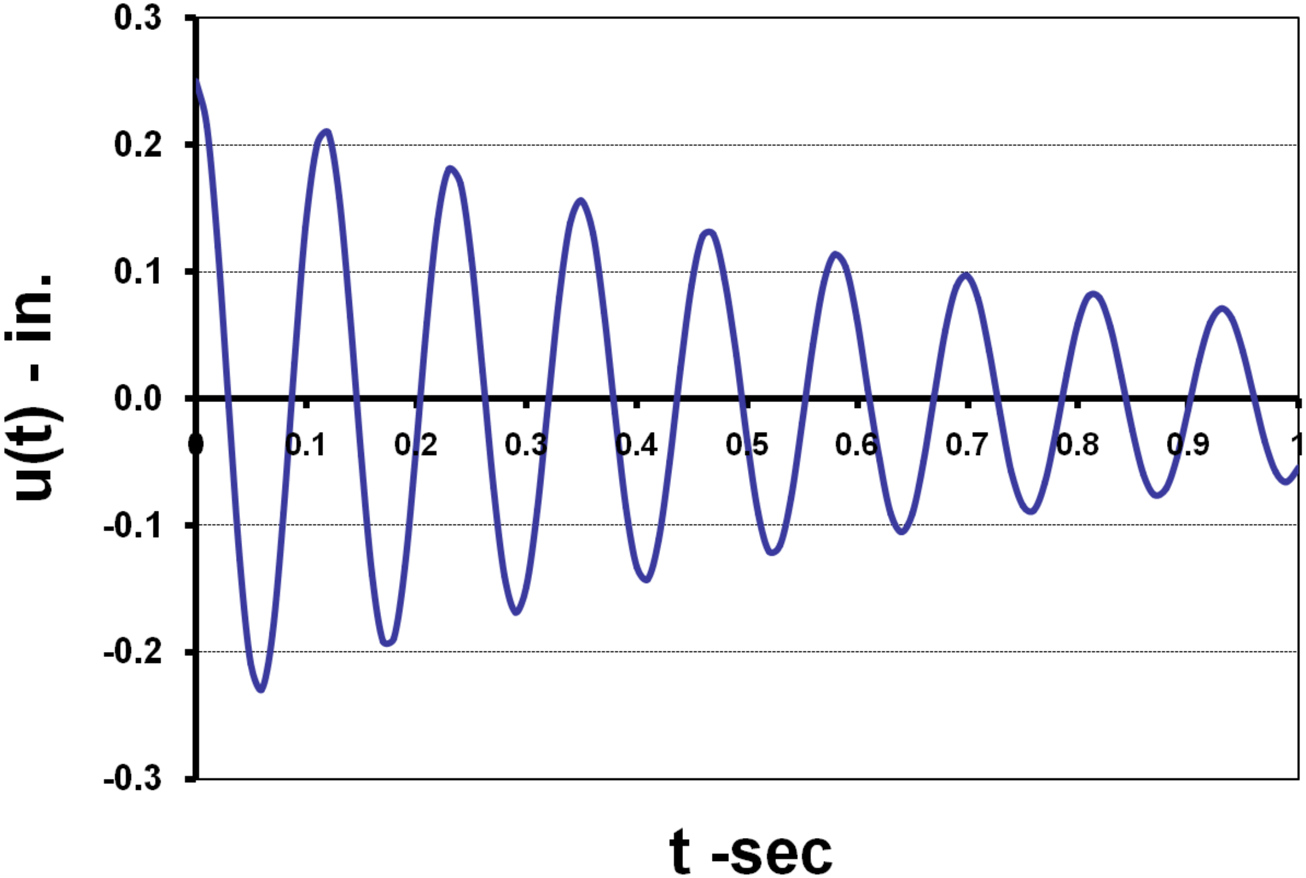
$$u(t) = = e^{-0.025*54.9t} \left[\frac{1}{48} * \cos(54.9t) + \frac{1}{54.9} * [\, 0 + \frac{1}{48} * 0.025 * 54.9 * \sin(54.9t)\right]$$

$$u(t) = = e^{-1.373t} [0.0208 * \cos(54.9t) + 0.00052 * \sin(54.9t)]$$

$$f_s(t) = k \,.\, u(t) = 90625 * u(t)$$

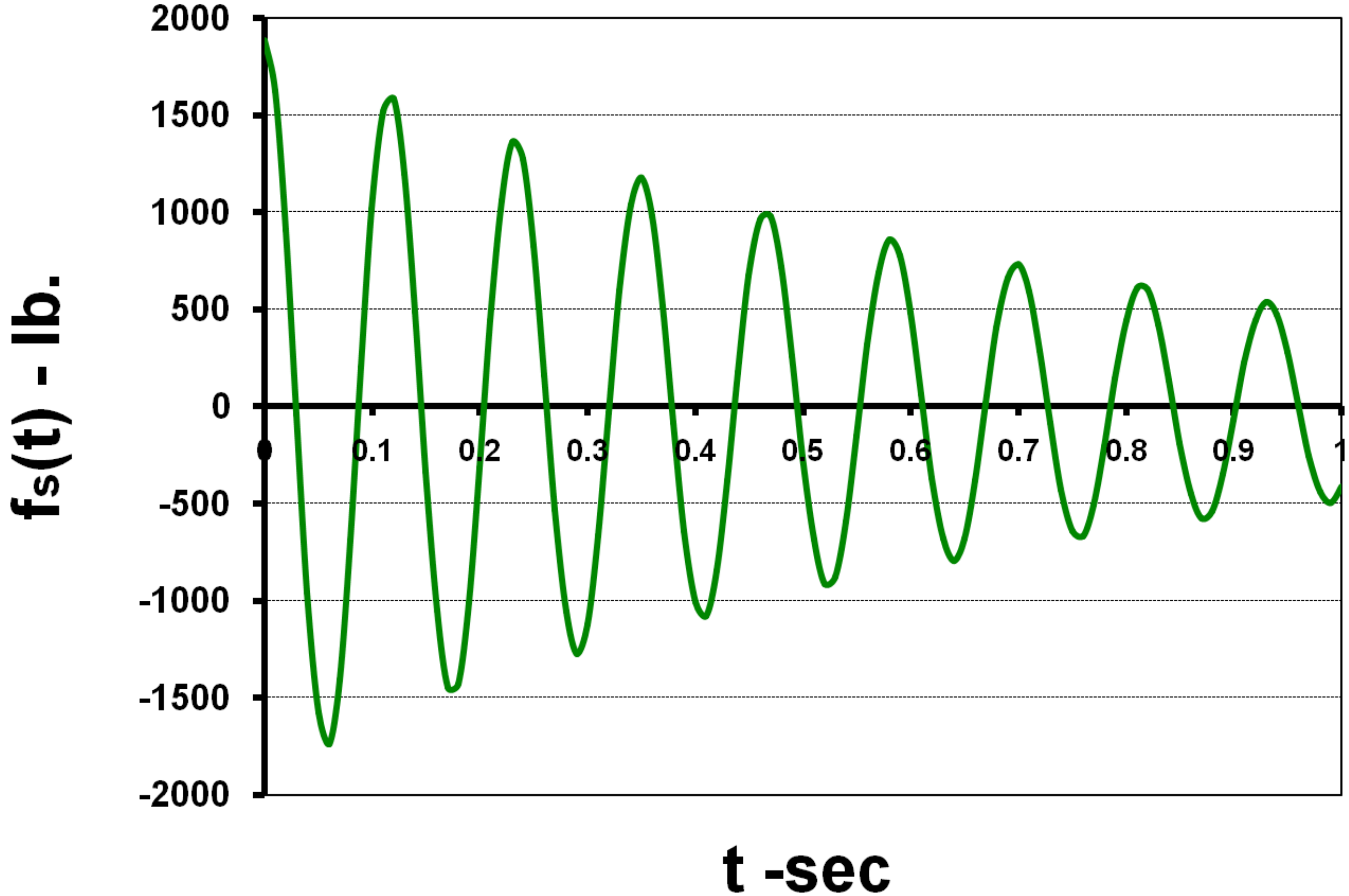
$$f_s(t) = e^{-1.373t} [1885 \cos(54.9t) + 47.13 * \sin(54.9t)]$$

Damped Free Vibration



Variation of displacement with time (Problem M4.3)

Damped Free Vibration

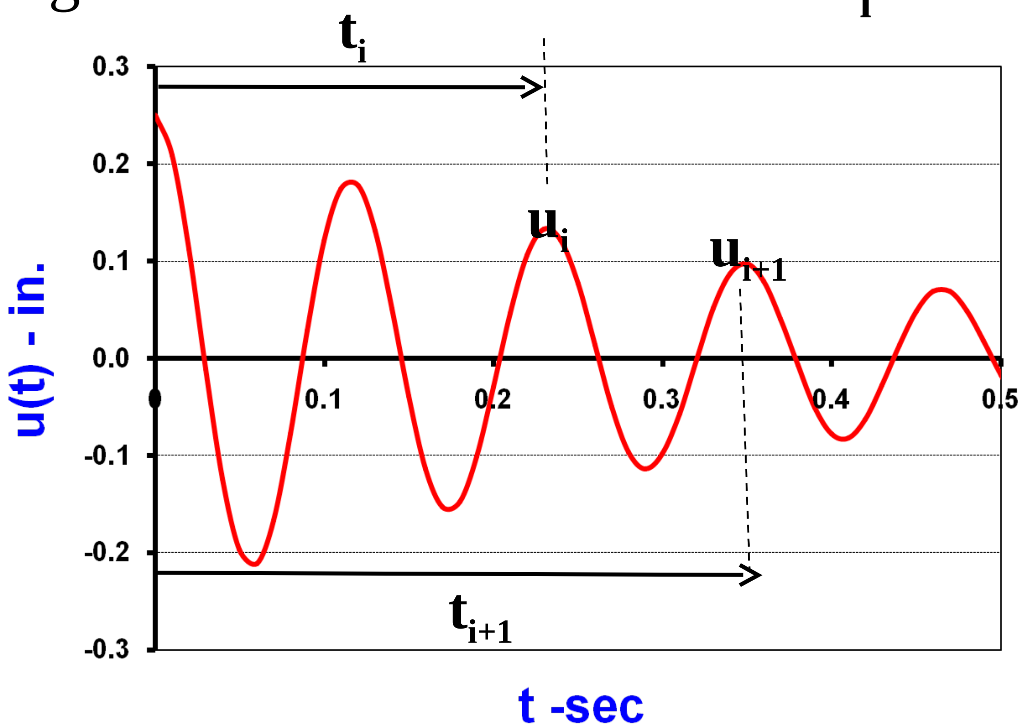


Variation of Equivalent static Forces with time (Problem M4.3)

Decay of Response Due To Damping

The decay observed in the response of a structure to some initial disturbance can be used to obtain a measure of the amount of viscous damping ,**c**, present in the structure.

Consider two successive positive peaks $\mathbf{u_i}$ and $\mathbf{u_{i+1}}$ in the response shown in the figure and which occur at times $\mathbf{t_i}$ and $\mathbf{t_{i+1}}$.



Decay of Response Due To Damping

It can be derived that:

$$\frac{u_i}{u_{i+1}} = e^{\left(\frac{2\pi\zeta\omega_n}{\omega_D}\right)}$$

Taking the natural logarithm of both sides we get the so-called *logarithmic decrement of damping*, δ, defined by the following equation.

$$\delta = \ln\left(\frac{u_i}{u_{i+1}}\right) = \frac{2\pi\zeta\omega_n}{\omega_D}$$

Decay of Response Due To Damping

$$\text{Since } \omega_D = \omega_n \sqrt{1-\zeta^2}$$

$$\Rightarrow \delta = \frac{2\pi\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} \quad \text{and;}$$

$$\Rightarrow \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

For Civil engineering systems, ζ is usually less than 0.1 and $\sqrt{1-\zeta^2} \approx 1$ and $\delta = 2\pi\zeta$ for lightly damped systems

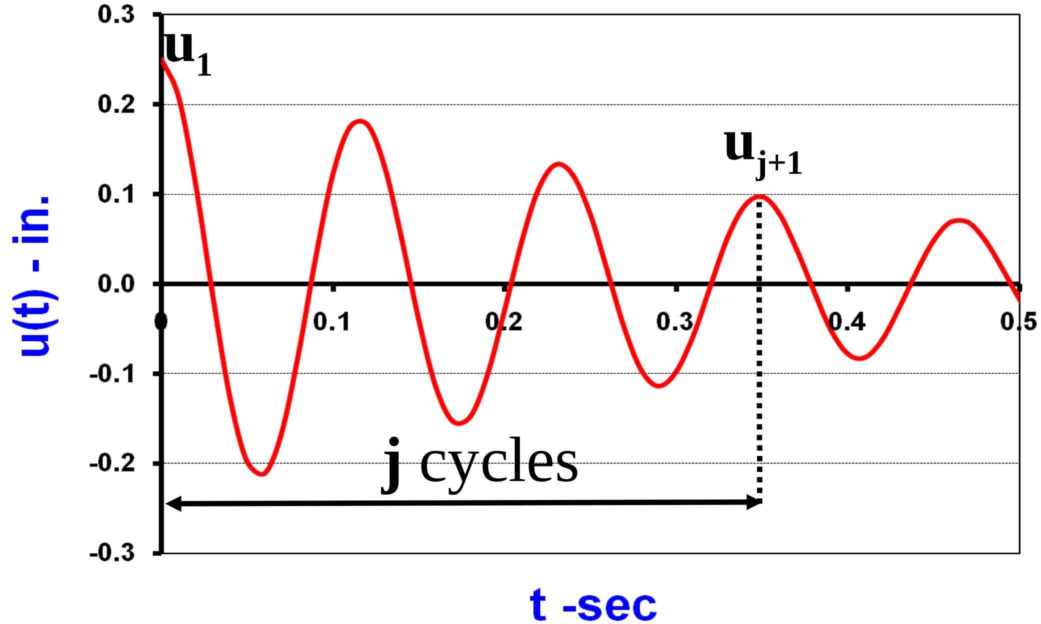
Decay of Response Due To Damping

These values may be improved by taking the differences in the peak response values not at two successive peaks but over a range of ***j*** peaks.

$$j\delta = \ln\left(\frac{u_1}{u_{j+1}}\right)$$

$$\Rightarrow \delta = \frac{1}{j}\ln\left(\frac{u_1}{u_{j+1}}\right) = 2\pi\zeta$$

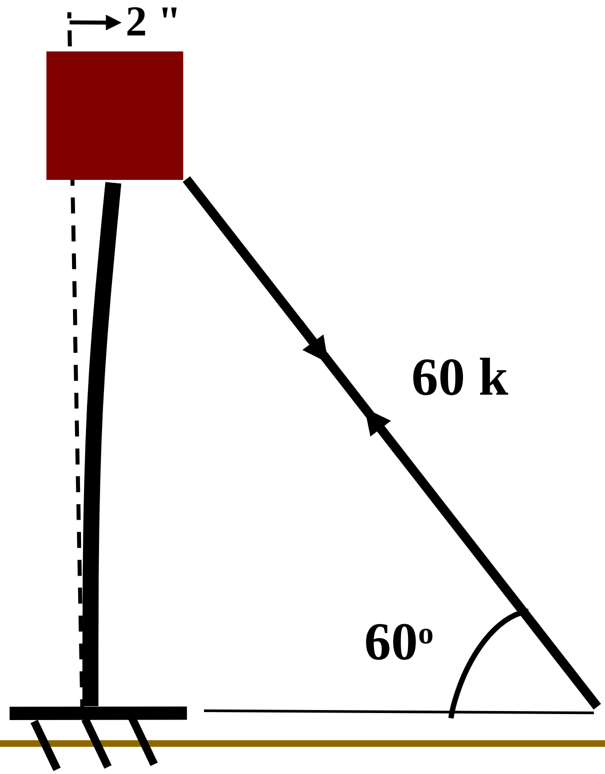
$$\Rightarrow \quad j = \frac{1}{2\pi\zeta}\ln\left(\frac{u_1}{u_{j+1}}\right)$$



Problem M4.4 :A free vibration test was conducted on an empty water tank shown in figure. A force of 60 kips, applied through a cable attached to the tank, displace the tank by 2 " in Horizontal direction. The cable is suddenly cut and the resulting vibration is recorded.

At the end of 5 cycles, which complete in 2.55 sec., the amplitude of displacement is 0.9 ". Ignore the vertical vibration of tank and compute the following:

a) Damping ratios
b) Natural period of undamped vibration
c) Stiffness of structures
d) Weight of tank
e) Damping coefficient
f) Number of cycles to reduce the displacement amplitude to 0.5 ″



Solution: Refer to class notes.

Solution (M4.4)

$u_1 = 2$ in.

After $j=5$, $u_{j+1}=u_6=0.9''$

a) ζ = Damping ratio=?

$$j=\frac{1}{2\Pi\zeta}\ln\left[\frac{u_1}{u_{j+1}}\right]$$

$$\Rightarrow 5 = \frac{1}{2\Pi\zeta}\ln(2/0.9)$$

$$\Rightarrow \zeta = 0.0254 = 2.54\%$$

b) T_n=?

5 cycles of vibrations are completed in 2.55 seconds

$\Rightarrow$Time required to complete one cycle = $2.55/5=T_D$

$\Rightarrow T_D = 0.51$Sec

Now

$$\omega_D = \omega_n \sqrt{(1 - \zeta^2)}$$

$$2\pi/\omega_D = 2\pi/(\omega_n\sqrt{1-\zeta^2})$$

$$\Rightarrow T_D = \frac{T_n}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow T_n = T_D * \sqrt{1-\zeta^2}$$

$$\Rightarrow T_n = 0.51*\sqrt{1-(0.0254)^2}$$

$$\Rightarrow T_n = 0.5098 = 0.51\text{sec}$$

c) k=?

$$k=\frac{60*\cos 60^{o}}{2} = 15 \text{ k/in} = 18000\text{lb/ft}$$

d) Weight of the tank, W=?

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{\left(\frac{W}{g}\right)}} = \sqrt{\frac{k*g}{W}}$$

$$\Rightarrow \omega_n^2 = k*g/W$$

$$\Rightarrow W = k*g \,/\, \omega_n^2$$

Also $\omega_n = 2\pi/T_n$

$$\Rightarrow W = kg/\left(\frac{4\pi^2}{T_n{}^2}\right) = kg*\frac{T_n{}^2}{4\pi^2}$$

$$W=\frac{\frac{18000\text{lb}}{\text{ft}}*\frac{32.2\text{ft}}{\text{sec}^2}.(0.51\text{ sec.})^2}{4\pi^2}$$

W=74875lb = 74.9 k

e) c = ?

It is known that $\zeta = \dfrac{c}{2m\omega_n}$

$\Rightarrow c = \zeta * 2m\omega_n = \zeta * 2m*(2\pi/T_n)$

$\Rightarrow c = \dfrac{0.0254*4*\pi*(\frac{74875}{32.2})}{0.51}$

$\Rightarrow c = 1455.3$ lb.sec/ft

f) No. of cycles to reduce displacement amplitude from 2 in. to 0.5 in., j = ?

$$j = \frac{1}{2\pi\zeta} \ln\left[\frac{u_1}{u_{j+1}}\right]$$

$$\Rightarrow j = \frac{1}{2*\pi*0.0254} \ln\left[\frac{2}{0.5}\right]$$

$\Rightarrow j = 8.69$ or 9 cycles

Problems

1. A spring- mass system has a natural frequency of 10 Hz. When the stiffness of spring is reduced by 800 N/m, the frequency is reduced by 45%. Find the mass and stiffness of actual system.

2. The natural frequency of a beam supporting a lumped mass, m, is 2 cycles per second (cps). When an additional mass of 50 kg is added, the natural frequency is reduced to 1.75 cps. Determine the stiffness and lumped mass, m.

3. The lateral stiffness of a cantilever beam, supporting a lumped mass of 25kg at the free end, is 5 kN/m. The beam, when hit with a hammer, start vibrating with an initial velocity of 0.1 m/s. Determine the displacement at free end from equilibrium position at the end of first second. Take viscous damping coefficient as 150 N.s/m.

4. A body of mass 1.25 kg is suspended from a spring with stiffness of 2kN/m. A dashpot attached to the spring-mass system require a force of 0.5 N to move with a velocity of 50 mm/s. Determine :
(a) Time required by the spring-mass system to complete one cycle during free vibration. (b) Amplitude of displacement of the body after 10 cycles when the body was released after an initial displacement of 20 mm. and
(c) Displacement after 1.25s from the start of free vibration.

5. An underdamped shock absorber is to be designed for a motor cycle of mass 200 kg. The vibration starts occurring in motor cycle in vertical direction when it reaches a road bump. Find necessary stiffness and damping ratio of the absorber if the damped period of vibration is to be 2 s as well that the ratio of amplitudes of displacement in successive peaks is to be 16.

Home Assignment M4H1

Solve problems 1,4 and 5