University of Engineering and Technology Peshawar, Pakistan

CE-409: Introduction to Structural Dynamics and Earthquake Engineering

MODULE 4:

UNDAMPED & DAMPED FREE VIBRATIONS IN S.D.O.F SYSTEMS

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Undamped Free Vibration

We will first consider the case where there is no load acting on the Structure i.e. p(t)=0. This case is known as free vibration.
The trivial solution for this case is where there is no displacement of the structure at t=0 i.e. u(t) = 0

$$m\ddot{u} + ku = 0$$

There are several ways of obtaining a solution to this second order differential equation. The simplest is to assume that the solution of the equation is of following form

$$u(t) = Ge^{st}$$

Undamped Free Vibration

Substituting this solution into the equation of motion (given on previous slide) result in:

$$m(s^2Ge^{st}) + kGe^{st} = 0$$

By rearranging: $$(s^2m + k)\,(Ge^{st}) = 0$$

Since $Ge^{st} \neq 0$, only possibility is that $s^2m + k = 0$

By solving one get $$s = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_n$$

The variable ω_n is known as the *natural circular frequency* and the units are radians/second.

Undamped Free Vibration

Inserting $s = \pm i\omega_n$ in $s^2 m + k = 0$

and solving we get:

$$u(t) = G_1 e^{i\omega_n t} + G_2 e^{-i\omega_n t}$$

The above after further simplification results in:

$$u(t) = A\text{Cos}(\omega_n t) + B\text{Sin}(\omega_n t)$$

This is the solution of equation of motion of undamped free vibration in the form of Simple Harmonic Motion with an angular velocity ω_n

Initial Conditions

The constants **A** and **B** can be found by evaluating the solution at two different times or more commonly from the velocity and displacement at time t = 0

$$u(t) = ACos(\omega_n t) + BSin(\omega_n t)$$

$$\text{At } \mathbf{t} = 0,\ u(t) = u(0) = ACos0 + BSin0$$

$$\Longrightarrow A = u(0)$$

Initial Conditions

$$\text{Similarly}, \dot{u}(t) = -A\omega_n \text{Sin}(\omega_n t) + B\omega_n \text{Cos}(\omega_n t)$$

$$\text{At } \mathbf{t} = 0, \dot{u}(t) = \dot{u}(0) = -A\omega_n \text{Sin}0 + B\omega_n \text{Cos}0$$

$$\Rightarrow B = \dot{u}(0) / \omega_n$$

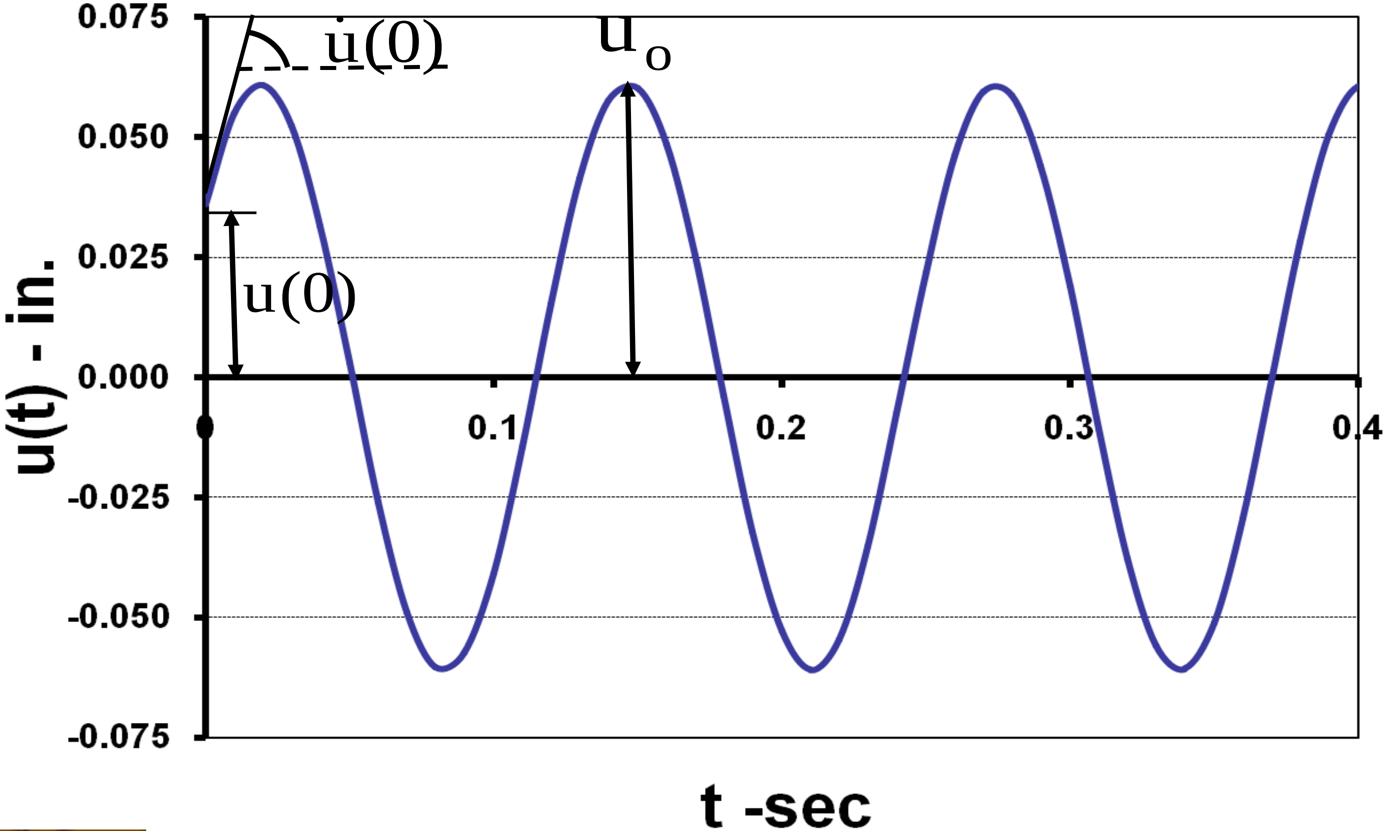
Using the value of **A** & **B** in equation given in slide 4 results in:

$$u(t) = u(0)\text{Cos}(\omega_n t) + \frac{\dot{u}(0)}{\omega_n}\text{Sin}(\omega_n t)$$

Amplitude of displacement during undamped free vibration

It can be determined that the amplitude (i.e. peak value) of displacement during undamped free vibration is:

$$u_o = \sqrt{[u(0)]^2 + \left[\frac{\dot{u}(0)}{\omega_n}\right]^2}$$



Element forces $f_s(t)$ from displacements $u(t)$

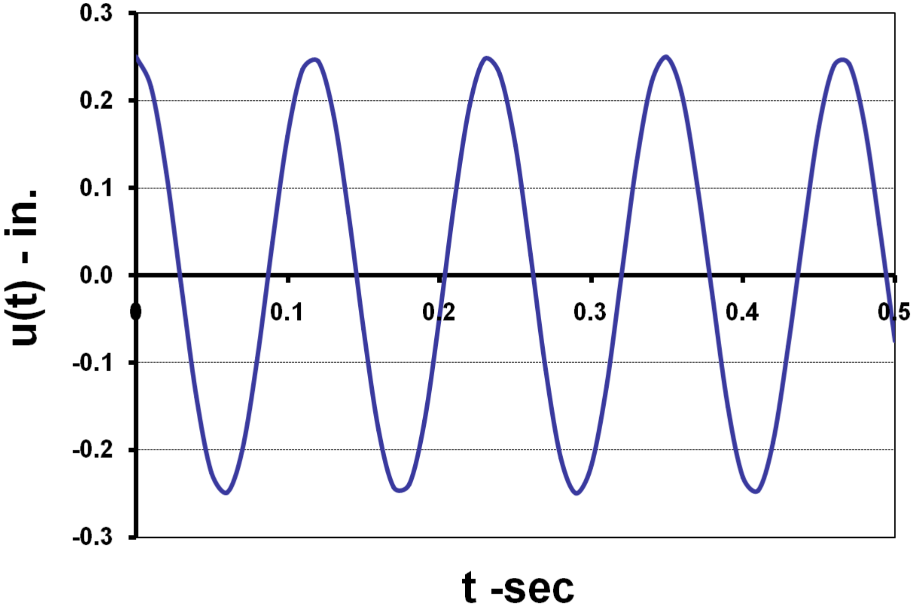
- One of the most appealing approach for calculating element forces in a structural system is to use ***Equivalent Static force*** approach.

- According to this approach, at any instant of time **t** the equivalent static force $\mathbf{f}_s$ is the external force that will produce the deformation **u** at the same **t** in the stiffness component of structure (i.e., the system without mass and damping). Thus

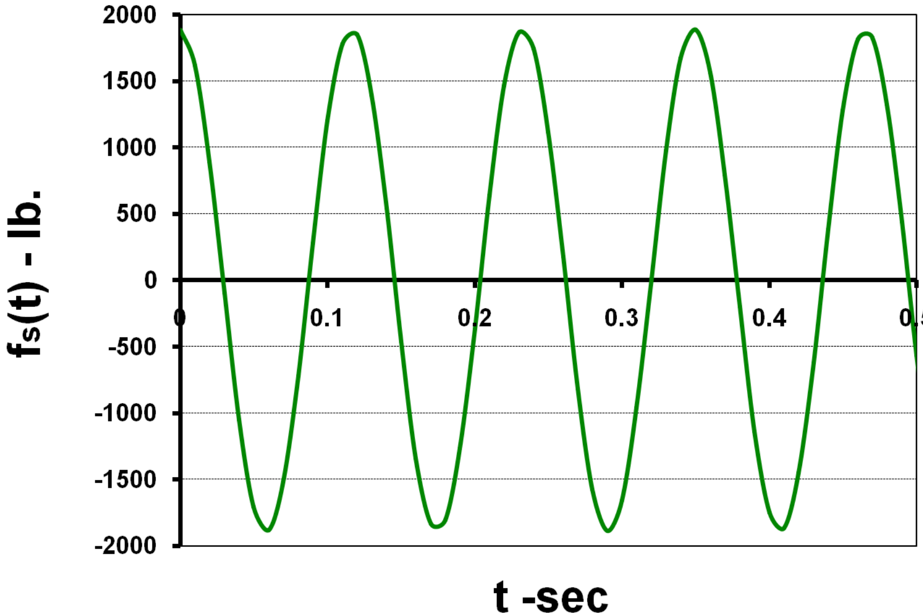
$$f_s(t) = k.u(t)$$

Where **k** is the lateral stiffness of the structure. Element forces or stresses can be determined at each instant of time by the static analysis of the structure subjected to the force $\mathbf{f}_s$.

Element forces $f_s(t)$ from displacements $u(t)$



Variation of displacement with time in an undamped system



Variation of corresponding equivalent static forces with time, $f_s(t)$.

$$f_s(t) = k.u(t)$$

Natural frequencies and Periods of free vibration

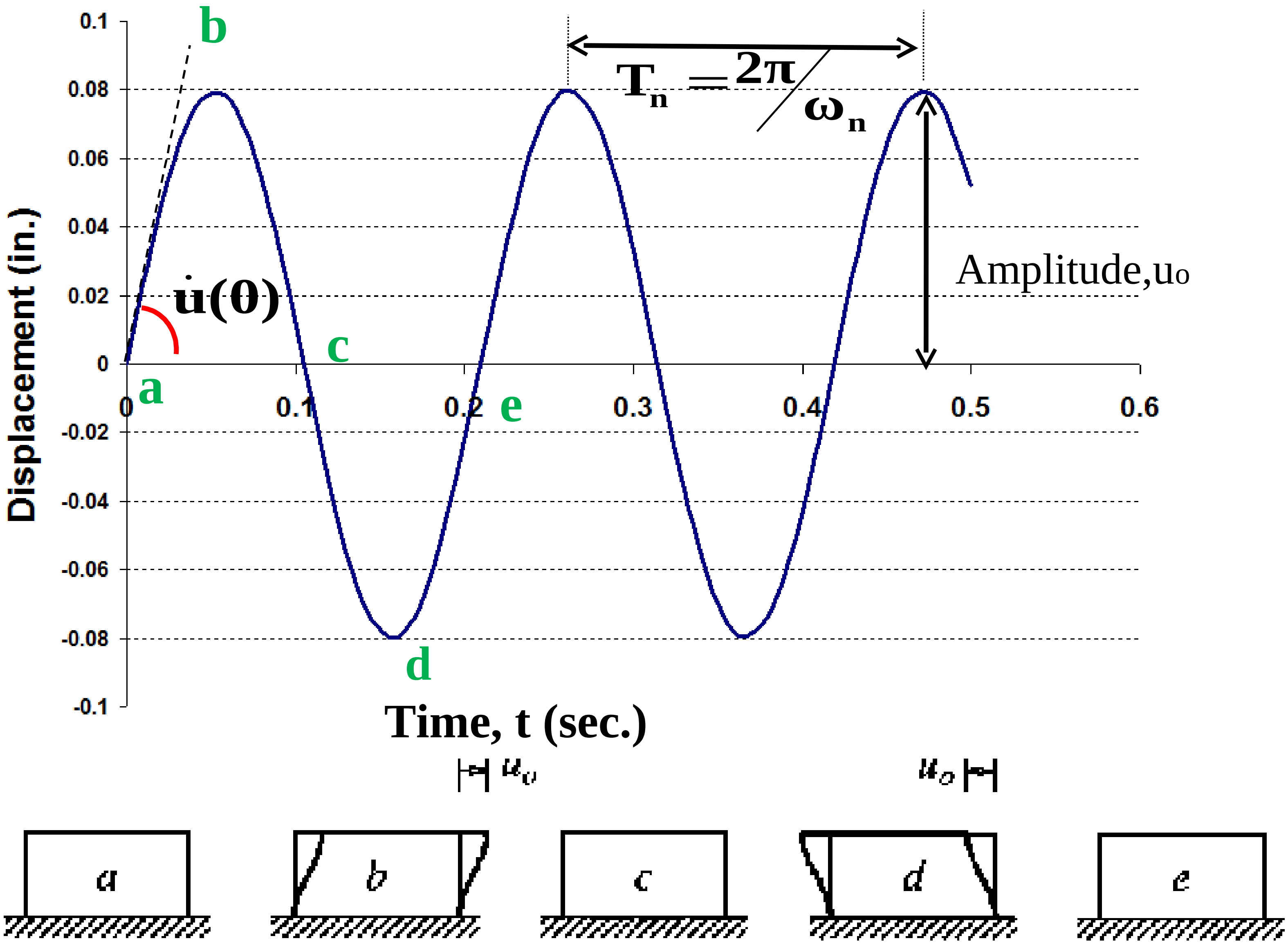
The Natural Circular Frequency ω_n is not a convenient measure for most engineers. The preferred usage is the Natural Frequency f_n which is usually measured in cycles/second (cps) or Hertz

$$f_n = \frac{\omega_n}{2\pi}$$

However, in earthquake engineering the preferred measure of the dynamic characteristic of structures is the Natural Period of free vibration $\mathbf{T_n}$ measured in units of seconds.

$$T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

Natural Period of free vibration



Natural period of free vibration

Alcoa Building, San Francisco

- Steel structure, 26 stories.

- **Periods of vibration**:

  - **Transverse** (east-west): 2.21 sec
  - **Longitudinal** (north-south): 1.67 sec
  - **Torsional**: 1.12 sec

Natural period of free vibration

Golden Gate Bridge, San Francisco

- Steel structure, center span of 4200 feet.

- **Periods of vibration**:

  - **Transverse**: 18.2 sec
  - **Vertical** : 10.9 sec
  - **Longitudinal**: 3.81 sec
  - **Torsional**: 4.43 sec

Natural period of free vibration

Rule of thumb:

Natural Time Period (sec) ≈ Number of Stories / 10

e.g. a 15 story building has a fundamental period of approximately 1.5 seconds. This applies primarily to moment frame buildings. This rule is very rough, but is good for understanding the general ballpark.

Undamped Free Vibration

Problem M4.1

A beam shown in Figure is pulled for ¼ inch in the downward direction and then suddenly released to vibrate freely. Determine natural time Period of the system and develop and solve the equation of motion for vibrations resulting at free end.

Also develop the equation showing variation in the *Equivalent static forces* with time. What will be the amplitude of equivalent static force? Ignore the self weight of beam as well as damping effect. Take **E** = 29,000 ksi and **I** = 150 in^4. $\boldsymbol{\delta_{st}}$ **=** Deflection due to 1000 lb static load

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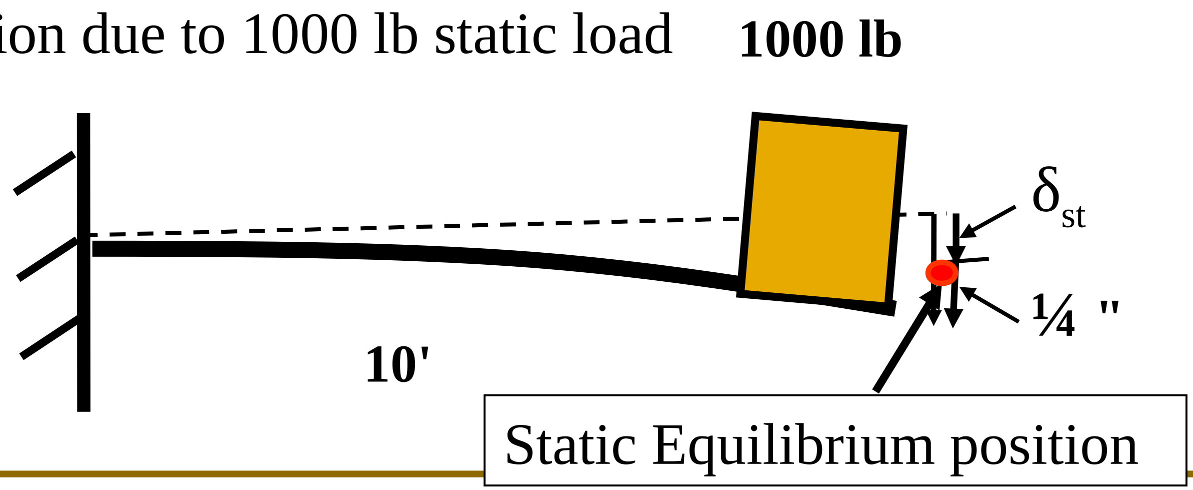


Solution (M4.1)

The general E.O.M for SDOF system is

$$Ku + c\dot{u} + m\ddot{u} = p(t)$$

In our case system is undamped (c=0) undergoing free vibration (P(t)=0)

Hence general EOM becomes $ku + m\ddot{u} = 0$ ……………………………(1)

$$k = 3EI/L^3$$

$$= \frac{3 * 29000\frac{k}{in^2} * 150in^4}{(10*12in)^3}$$

$$= 7.55 \text{ k/in}$$

In order to eliminate the chances of mistake during calculation, it is more appropriate to use fundamental units like lb, ft sec or kg, m, sec.

$$k = 7.55 \text{k/in} = 90625 \text{ lb/ft}$$

$$m = \frac{1000lbsec^2}{32.2ft} = 31.06 \text{ slug}$$

$$\omega_n = \sqrt{k/m} = \sqrt{96025/31.06} = 54.91 \text{ rad/sec}$$

$$T_n = 2\pi/\omega_n = 2\pi/54.91 = 0.114 \text{ sec}$$

Substituting the corresponding values in eq-1

$$90625u + 31.06\,\ddot{u} = 0$$

Where 'k' is in lb/ft and 'm' is in lb sec/ft^2

General solution to the EOM for undamped free vibration is,

$$u(t) = u(0)\cos(\omega_n t) + \frac{\dot{u}(0)}{\omega_n}\sin(\omega_n t)$$

$u(0) = 1/4'' = 1/48$ ft and $\dot{u}(0) = 0$

u(t) = (1/48) * cos(54.9t) + 0 = (1/48) * cos(54.9t)

Equivalent static force at any time "t" is

$f_s(t) = k.u(t) = \frac{90625*\cos(54.9t)}{48}$

$f_s(t) = 1888\cos(54.9t)$

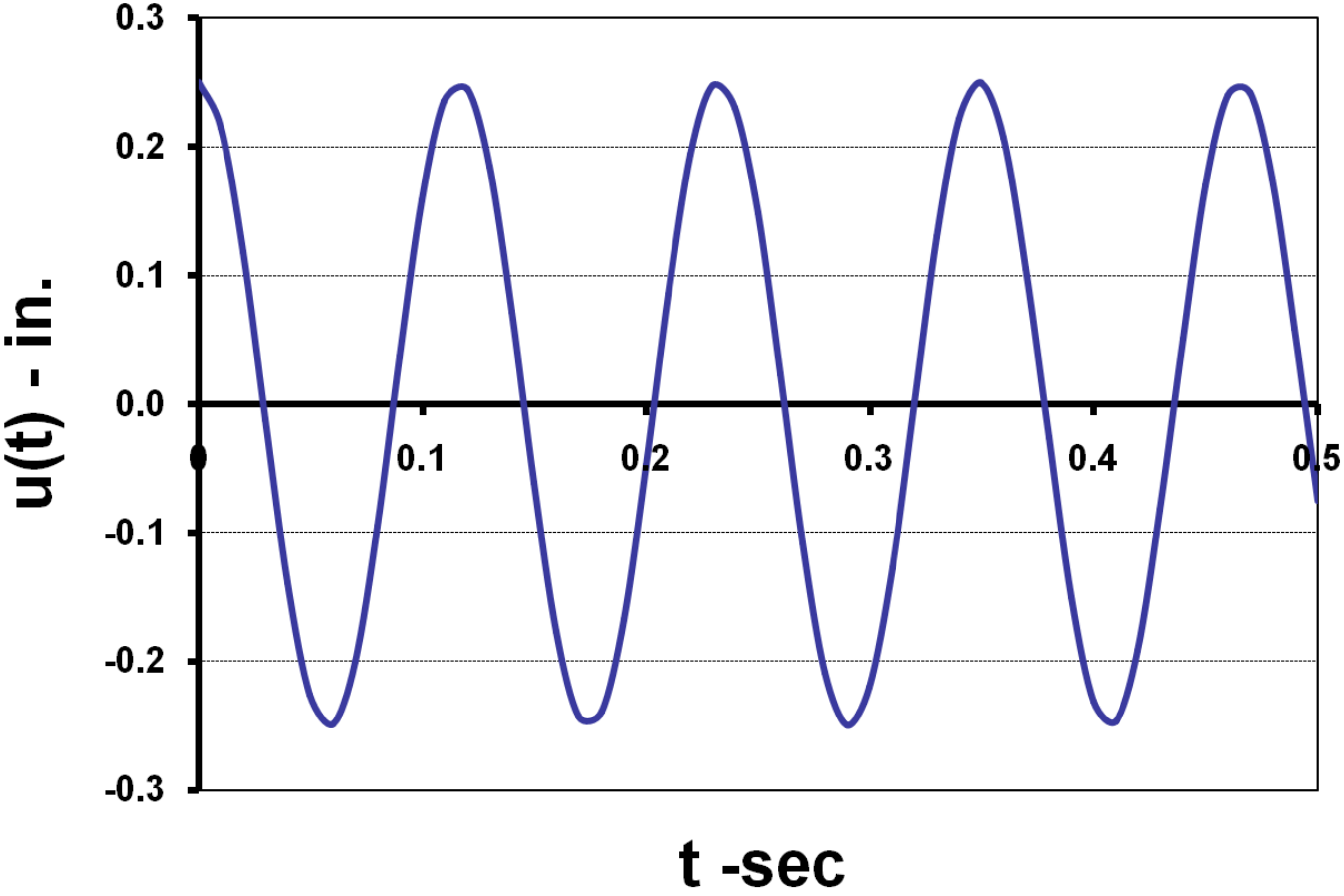
Amplitude of dynamic displacement, u_0 for undamped free vibration is

$$u_o = \sqrt{\left[(u(0))^2 + \left(\dot{u}(0)/\omega_n\right)^2\right]} = \sqrt{\left(\left(1/48\right)^2 + 0\right)} = 1/48 \text{ ft}$$

Amplitude of equivalent static force, f_{so}

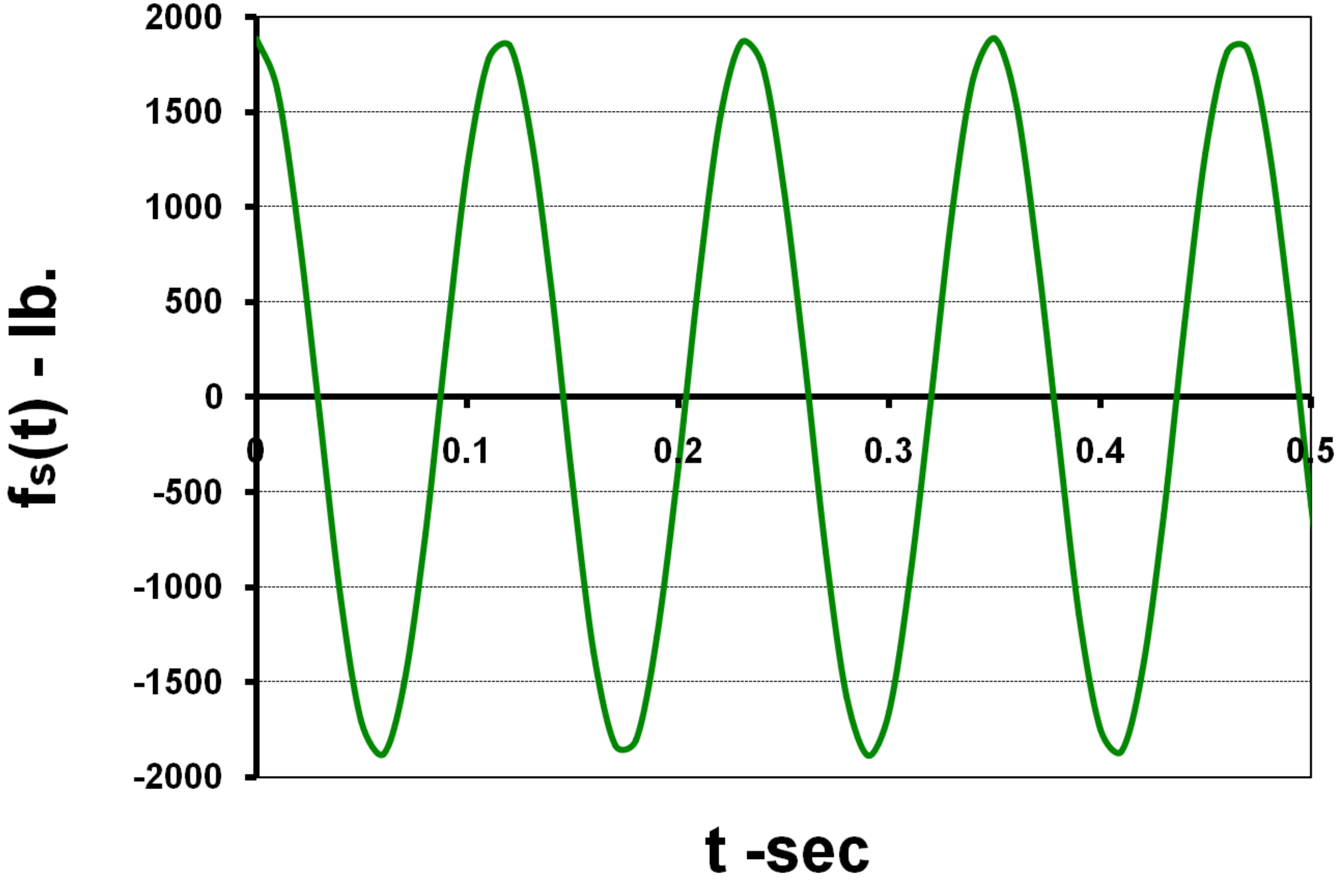
$ku_0 = 90625 * 1/48 = 1888\text{lb}$

Undamped Free Vibration



Variation of displacement with time (Problem M4.1)

Undamped Free Vibration



Variation of Equivalent static Forces with time (Problem M4.1)

Problem M4.2:

Considering free vibration, solve the equation of motion developed for the frame given in Problem **M3.3**. Also develop an equation showing variation with the *Equivalent static forces*.

What will be the amplitude of equivalent static force if the Displacement and velocity (which occur in the same direction ?) at the start of free vibration are 0.003 ft and 0.2 ft/sec., respectively

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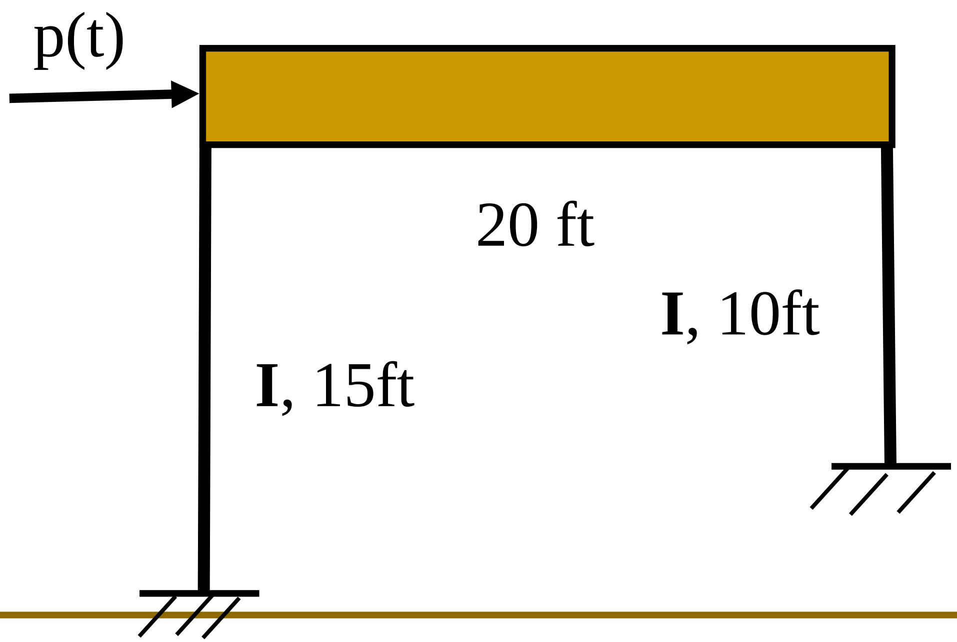


Solution (M4.2)

$3106\ddot{u} + 3.76*10^6\, u = p(t)$

Where u and p(t) are in ft and lb respectively

For undamped free vibrations

$$u(t) = u(0)\cos(\omega_n t) + \frac{\dot{u}(0)}{\omega_n}\sin(\omega_n t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{3.76*10^6\ lb/ft}{3106\ lb.sec^2/ft}}$$

ω_n=34.79 rad/sec

$T_n = 2\pi/\omega_n = 0.18$sec

It is also given that:

$u(0) = 0.003$ ft and $\dot{u}(0) = 0.2$ ft/sec

By substituting the corresponding values in eq-1, we get:

$$u(t) = 0.003\cos(34.8t) + 0.2/34.8 * \sin(34.8\,t)$$

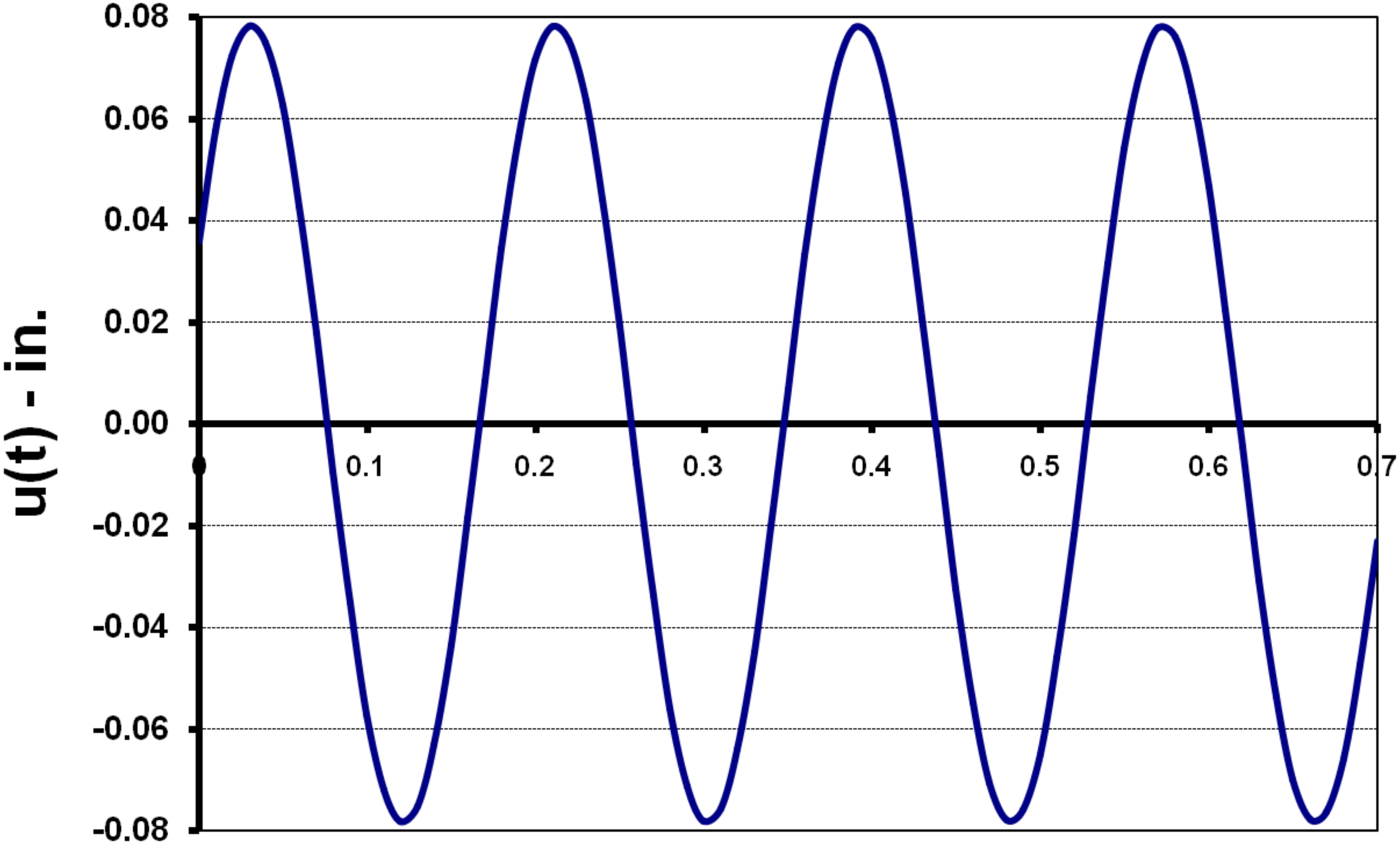
$$f_s(t) = k.u(t) = 3.76 * 10^6\,[u(t)]$$

$$= 11280\cos(34.8t) + 21808\sin(34.8t)$$

$$f_{so} = k.\,u_o = k * \sqrt{\{u(0)\}^2 + \{\dot{u}(0)/\omega_n\}^2}$$

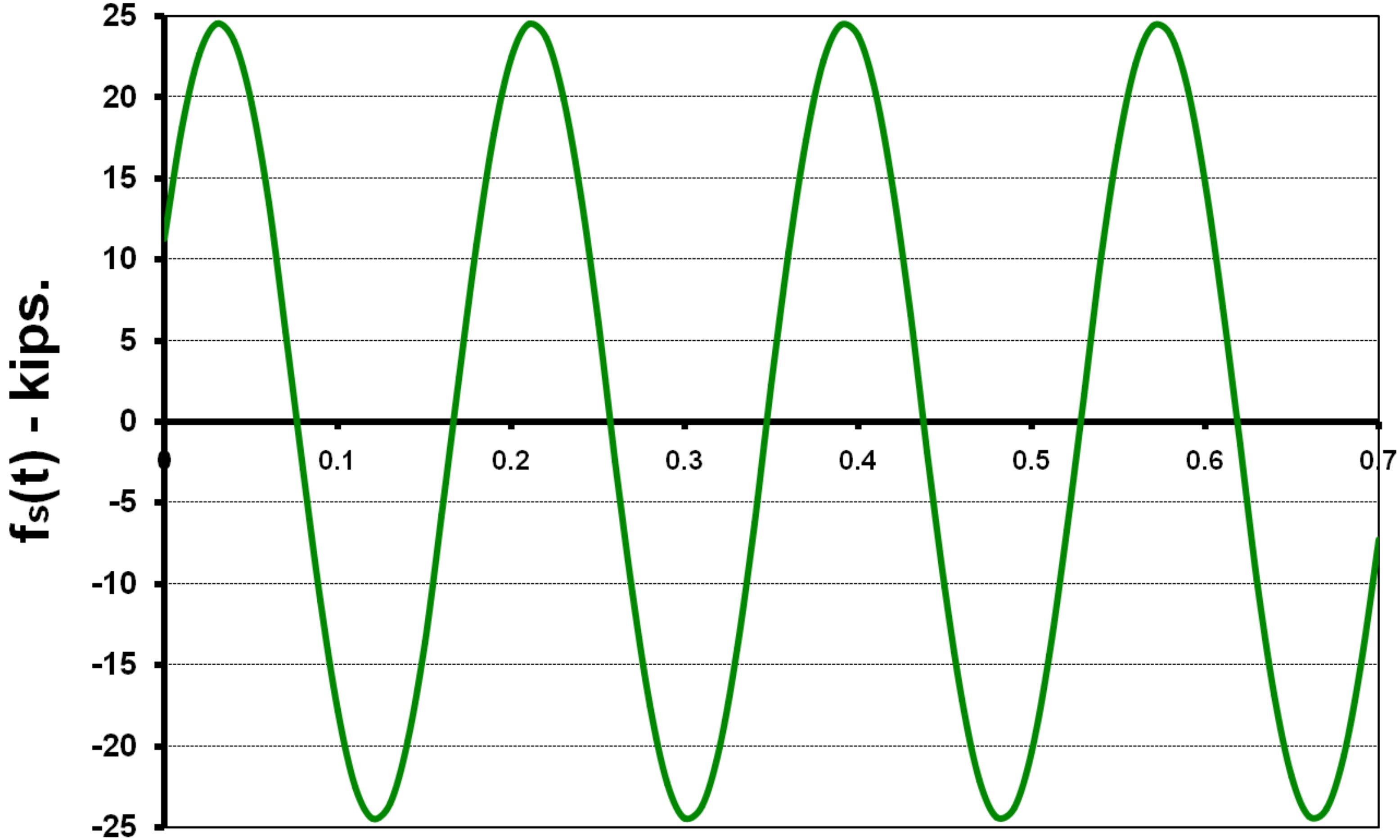
$$f_{so} = 3.76 * 10^6 * \sqrt{(0.003)^2 + \left({}^{0.2}/_{34.8}\right)^2} = 24376\text{ lb} = 24.38\text{ kips}$$

Undamped Free Vibration



Variation of displacement with time (Problem M4.2)

Undamped Free Vibration



Variation of Equivalent static forces with time (Problem M4.2)

Response of damped systems to free vibrations

Viscously Damped Free Vibration

The equation of free vibration for damped free vibration has the form

$$m\ddot{u} + c\dot{u} + ku = 0$$

The solution to this equation will be taken in the same form as for the undamped form i.e

$$u(t) = Ge^{st}$$

Substituting this value in equation of motion result in:

$$m(s^2Ge^{st}) + c(sGe^{st}) + kGe^{st} = 0$$

Viscously Damped Free Vibration

By rearranging we get: $(s^2 m + sc + k)(Ge^{st}) = 0$

Since $Ge^{st} \neq 0 \quad \Longrightarrow s^2 m + sc + k = 0$

or $$s = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{4km}{4m^2}}$$

Using the relation $k = m\omega_n^2$ $$s = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \omega_n^2}$$

By rearranging: $$s = -\frac{c}{2m\omega_n}\omega_n \pm \omega_n \sqrt{1 - \left[\frac{c}{2m\omega_n}\right]^2}$$

Viscously Damped Free Vibration

$2m\omega_n = c_{cr}$ is known as Critical damping coefficient

where $\frac{c}{2m\omega_n} = \zeta$ (Greek alphabet for ***Zeta***) is known as damping ratio or fraction of critical damping, which when substituted in the equation mentioned on previous slide gives:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$$

$$\text{or } s = -\zeta\omega_n \pm \omega_D$$

Where $\omega_n\sqrt{1-\zeta^2}$ is known as damped natural frequency and represented by ω_D

Viscously Damped Free Vibration

There are 3 forms of solution available depending on the magnitude of the damping coefficient **c**

1 $c = 2m\omega_n$ → **Critically damped system**

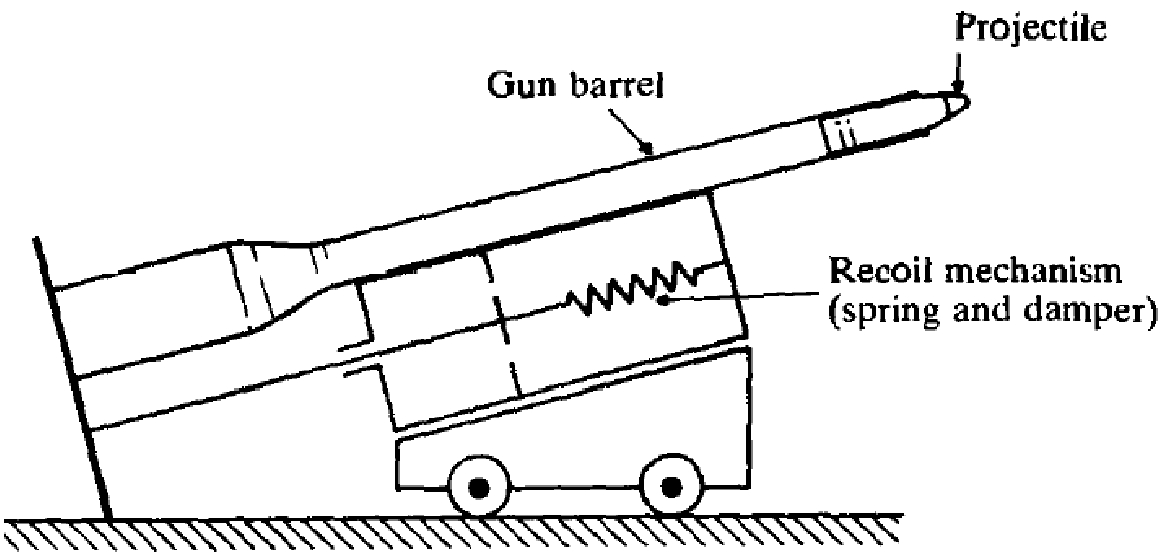
2 $c > 2m\omega_n$ → **Over damped system**

3 $c < 2m\omega_n$ → **Under damped system**

Viscously Damped Free Vibration

Critically damped system $c = 2m\omega_n$

- The structure is said to be ***Critically damped.***
- There is no vibration in the response. The structure returns to its initial position without vibrating about the zero position but in the shortest time



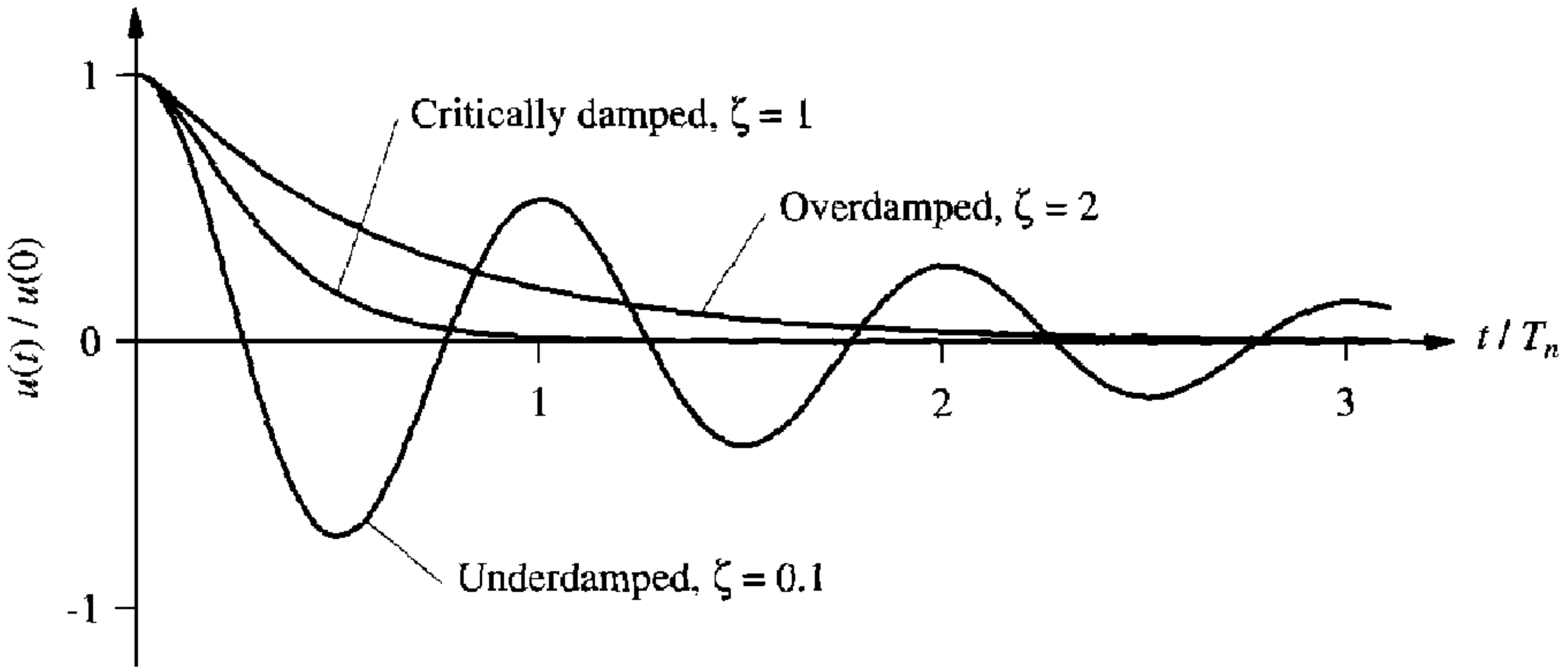
An example of critically damped system

Viscously Damped Free Vibration

Over damped system $$c > 2m\omega_n$$

- The structure is said to be ***Over damped.***
- Like critically damped systems, there is no vibration. However, the structure returns to its initial position slowly as compared to critically damped system.
- ***Critically damped and over damped systems are of no interest to the civil or structural engineer*.**

An automatic door close is an example of an over damped system



Free vibration of under damped, critically damped , and over damped systems

Viscously Damped Free Vibration

Underdamped system $c < 2m\omega_n$

- The structure is said to be under-damped
- The structure again returns to its origin but now vibrates.
- This is the only case that is of interest to civil or structural engineers as in all our structures the level of damping is very small, usually less than 5% of critical damping.

Approximate Damping Ratios

Working Stress Level (1/2 yield point)	**ζ**
Welded steel, Prestressed concrete, Well reinforced concrete (slight cracking)	2-3%
Reinforced concrete with considerable cracking	3-5%
Bolted or riveted steel, Timber	5-7%
At or just below yield point	
Welded steel, Prestressed concrete (without loss of prestress)	5-7%
Reinforced concrete	7-10%
Bolted or riveted steel, Bolted timber	10-15%
Nailed timber	15-20%