**Linear Algebra**
**BS-CS**
**Time: 9:00am – 3:00pm**
**Date: 25/06/2020**

## Instructions:

- Allowed time is 6 hours (9:00am to 3:00pm)
- Mark all the answers sheets with page numbers and ID on every sheet number
- Answers copied will both be marked zero
- Late Submission will not be accepted
- Submit in PDF format

1. Consider the following vectors $R^3$:

$$V_1 = \begin{pmatrix} ID1 \\ ID2 \\ ID3 \end{pmatrix}, \; V_2 = \begin{pmatrix} ID2 \\ ID3 \\ ID4 \end{pmatrix}, \; V_3 = \begin{pmatrix} ID3 \\ ID4 \\ ID5 \end{pmatrix}$$

Solve the system and find if these vectors are linearly independent?

Solution

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**Example 118** Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

Are they linearly independent?

We need to see whether the system

$$c^1 v_1 + c^2 v_2 + c^3 v_3 = 0$$

has any solutions for $c^1, c^2, c^3$. We can rewrite this as a homogeneous system:

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0.$$

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This system has solutions if and only if the matrix $M = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is singular, so we should find the determinant of $M$:

$$\det M = \det \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix} = 2 \det \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} = 12.$$

Since the matrix $M$ has non-zero determinant, the only solution to the system of equations

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0$$

is $c_1 = c_2 = c_3 = 0$. So the vectors $v_1, v_2, v_3$ are linearly independent.

2. Suppose that a company produces two products X and Y. For each unit of each product produced, money must be spent on materials, labor and overhead.

| Cost per unit | Product X | Product Y |
|---|---|---|
| Materials | Rs. 450 | Rs. 400 |
| Labor | Rs. 250 | Rs. 350 |
| Overhead | Rs. 150 | Rs. 150 |

The production vector x = $\begin{bmatrix} x1 \\ x2 \end{bmatrix}$ whereas, x1 and x2 are total number of units for product A and B i.e. 1000 and 500 respectively.

a. Find the total cost.

Video Link (Linear Transformation Lecture uploaded on SIC

https://www.youtube.com/watch?v=xuulVLujY1w&list=PLNr8B4XHL5kGDHOrU4IeI6QNuZHur4F86&index=21)

Find. AX where A is the 3x2 matrix of product X and Y, you will have cost of each unit i.e.

A = $\begin{bmatrix} 450 & 400 \\ 250 & 350 \\ 150 & 150 \end{bmatrix}$, X=$\begin{bmatrix} 1000 \\ 500 \end{bmatrix}$

A.X = matrix multiplication

b. Explain these linear transformation properties with the help of above problem as an example.

   $T(u+v) = T(u) + T(v)$

   The cost of combined product is the sum of costs (https://www.youtube.com/watch?v=xuulVLujY1w&list=PLNr8B4XHL5kGDHOrU4IeI6QNuZHur4F86&index=21)

   $T(cu) = cT(u)$

   Scaling production scales costs by the same factor

3. What are the four main things we need to define for a vector space? Which of the following is a vector space over R? For those that are not vector spaces, modify one part of the definition to make it into a vector space.
   a. V = { 2 x 2 matrices with entries in R}, usual matrix addition, and

   $$k.\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & b \\ kc & d \end{pmatrix} for\ k \in R$$

   b. V = {Polynomials with complex coefficients of degrees ≤3}, with usual addition and scalar multiplication of polynomials.

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8. The four main ingredients are (i) a set $V$ of vectors, (ii) a number field $K$ (usually $K = \mathbb{R}$), (iii) a rule for adding vectors (vector addition) and (iv) a way to multiply vectors by a number to produce a new vector (scalar multiplication). There are, of course, ten rules that these four ingredients must obey.

   (a) This is not a vector space. Notice that distributivity of scalar multiplication requires $2u = (1+1)u = u + u$ for any vector $u$ but

$$2 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b \\ 2c & d \end{pmatrix}$$

   which does *not* equal

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}.$$

   This could be repaired by taking

$$k \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

   (b) This is a vector space. *Although, the question does not ask you to, it is a useful exercise to verify that all ten vector space rules are satisfied.*

4. Determinants: Let M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2x2 matrix.
   a. For which values of det M does M have an inverse?
   b. Write down all 2x2 bit matrices with determinant 1. (Remember bits are either 0 or 1 and 1+1 = 0 in bits)
   c. Write down 2x2 bit matrices with determinant 0
   d. Compute det A for below 3x3 matrix.

$$A = \begin{pmatrix} ID1 & ID1 & ID1 \\ ID2 & ID3 & ID2 \\ ID4 & ID1 & ID5 \end{pmatrix}$$

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1. (a) Whenever $\det M = ad - bc \neq 0$.

(b) Unit determinant bit matrices:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}.$$

(c) Bit matrices with vanishing determinant:

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$

$$\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}.$$

*As a check, count that the total number of* $2\times2$ *bit matrices is* $2^{(\text{number of entries})} = 2^4 = 16$.