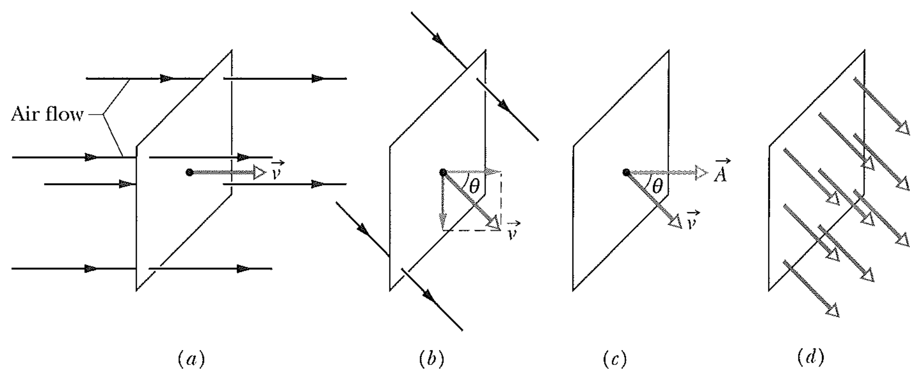


Fig. 23-2 (*a*) A uniform airstream of velocity $\vec{v}$ is perpendicular to the plane of a square loop of area A. (*b*) The component of $\vec{v}$ perpendicular to the plane of the loop is $v \cos \theta$, where θ is the angle between $\vec{v}$ and a normal to the plane. (*c*) The area vector $\vec{A}$ is perpendicular to the plane of the loop and makes an angle θ with $\vec{v}$. (*d*) The velocity field intercepted by the area of the loop.

Before we discuss a flux involved in electrostatics, we need to rewrite Eq. 23-1 in terms of vectors. To do this, we first define an *area vector* $\vec{A}$ as being a vector whose magnitude is equal to an area (here the area of the loop) and whose direction is normal to the plane of the area (Fig. 23-2*c*). We then rewrite Eq. 23-1 as the scalar (or dot) product of the velocity vector $\vec{v}$ of the airstream and the area vector $\vec{A}$ of the loop:

$$\Phi = vA \cos \theta = \vec{v} \cdot \vec{A}, \tag{23-2}$$

where θ is the angle between $\vec{v}$ and $\vec{A}$.

The word "flux" comes from the Latin word meaning "to flow." That meaning makes sense if we talk about the flow of air volume through the loop. However, Eq. 23-2 can be regarded in a more abstract way. To see this different way, note that we can assign a velocity vector to each point in the airstream passing through the loop (Fig. 23-2*d*). Because the composite of all those vectors is a *velocity field*, we can interpret Eq. 23-2 as giving the *flux of the velocity field through the loop*. With this interpretation, flux no longer means the actual flow of something through an area—rather it means the product of an area and the field across that area.

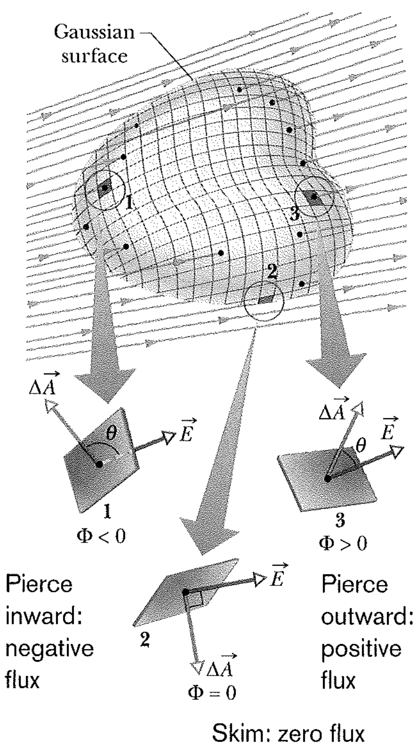


Fig. 23-3 A Gaussian surface of arbitrary shape immersed in an electric field. The surface is divided into small squares of area ΔA. The electric field vectors $\vec{E}$ and the area vectors $\Delta\vec{A}$ for three representative squares, marked 1, 2, and 3, are shown.

23-3 Flux of an Electric Field

To define the flux of an electric field, consider Fig. 23-3, which shows an arbitrary (asymmetric) Gaussian surface immersed in a nonuniform electric field. Let us divide the surface into small squares of area ΔA, each square being small enough to permit us to neglect any curvature and to consider the individual square to be flat. We represent each such element of area with an area vector $\Delta\vec{A}$, whose magnitude is the area ΔA. Each vector $\Delta\vec{A}$ is perpendicular to the Gaussian surface and directed away from the interior of the surface.

Because the squares have been taken to be arbitrarily small, the electric field $\vec{E}$ may be taken as constant over any given square. The vectors $\Delta\vec{A}$ and $\vec{E}$ for each square then make some angle θ with each other. Figure 23-3 shows an enlarged view of three squares on the Gaussian surface and the angle θ for each.

A provisional definition for the flux of the electric field for the Gaussian surface of Fig. 23-3 is

$$\Phi = \sum \vec{E} \cdot \Delta\vec{A}. \tag{23-3}$$

This equation instructs us to visit each square on the Gaussian surface, evaluate the scalar product $\vec{E} \cdot \Delta\vec{A}$ for the two vectors $\vec{E}$ and $\Delta\vec{A}$ we find there, and sum the results algebraically (that is, with signs included) for all the squares that make up the surface. The value of each scalar product (positive, negative, or zero) determines whether the flux through its square is positive, negative, or zero. Squares like square 1 in Fig. 23-3, in which $\vec{E}$ points inward, make a negative contribution to the sum of Eq. 23-3. Squares like 2, in which $\vec{E}$ lies in the surface, make zero contribution. Squares like 3, in which $\vec{E}$ points outward, make a positive contribution.

The exact definition of the flux of the electric field through a closed surface is found by allowing the area of the squares shown in Fig. 23-3 to become smaller and smaller, approaching a differential limit dA. The area vectors then approach a differential limit $d\vec{A}$. The sum of Eq. 23-3 then becomes an integral:

$$\Phi = \oint \vec{E} \cdot d\vec{A} \quad \text{(electric flux through a Gaussian surface).} \tag{23-4}$$

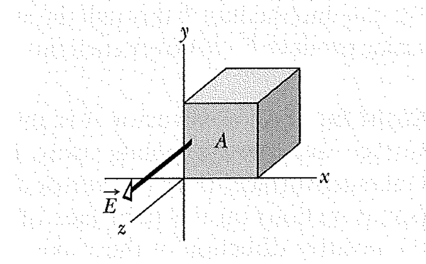
The loop on the integral sign indicates that the integration is to be taken over the entire (closed) surface. The flux of the electric field is a scalar, and its SI unit is the newton–square-meter per coulomb ($N \cdot m^2/C$).

We can interpret Eq. 23-4 in the following way: First recall that we can use the density of electric field lines passing through an area as a proportional measure of the magnitude of the electric field $\vec{E}$ there. Specifically, the magnitude E is proportional to the number of electric field lines per unit area. Thus, the scalar product $\vec{E} \cdot d\vec{A}$ in Eq. 23-4 is proportional to the number of electric field lines passing through area $d\vec{A}$. Then, because the integration in Eq. 23-4 is carried out over a Gaussian surface, which is closed, we see that

The electric flux Φ through a Gaussian surface is proportional to the net number of electric field lines passing through that surface.

CHECKPOINT 1

The figure here shows a Gaussian cube of face area A immersed in a uniform electric field $\vec{E}$ that has the positive direction of the z axis. In terms of E and A, what is the flux through (a) the front face (which is in the xy plane), (b) the rear face, (c) the top face, and (d) the whole cube?



Sample Problem

Flux through a closed cylinder, uniform field

Figure 23-4 shows a Gaussian surface in the form of a cylinder of radius R immersed in a uniform electric field $\vec{E}$, with the cylinder axis parallel to the field. What is the flux Φ of the electric field through this closed surface?

KEY IDEA

We can find the flux Φ through the Gaussian surface by integrating the scalar product $\vec{E} \cdot d\vec{A}$ over that surface.

Calculations: We can do the integration by writing the flux as the sum of three terms: integrals over the left cylinder cap a, the cylindrical surface b, and the right cap c. Thus, from Eq. 23-4,

$$\Phi = \oint \vec{E} \cdot d\vec{A}$$

$$= \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}. \tag{23-5}$$

For all points on the left cap, the angle θ between $\vec{E}$ and $d\vec{A}$ is 180° and the magnitude E of the field is uniform. Thus,

$$\int_a \vec{E} \cdot d\vec{A} = \int E(\cos 180^\circ)\, dA = -E \int dA = -EA,$$

where $\int dA$ gives the cap's area A $(= \pi R^2)$. Similarly, for the right cap, where $\theta = 0$ for all points,

$$\int_c \vec{E} \cdot d\vec{A} = \int E(\cos 0)\, dA = EA.$$

Finally, for the cylindrical surface, where the angle θ is 90° at all points,

$$\int_b \vec{E} \cdot d\vec{A} = \int E(\cos 90^\circ)\, dA = 0.$$

Substituting these results into Eq. 23-5 leads us to

$$\Phi = -EA + 0 + EA = 0. \quad \text{(Answer)}$$

The net flux is zero because the field lines that represent the electric field all pass entirely through the Gaussian surface, from the left to the right.

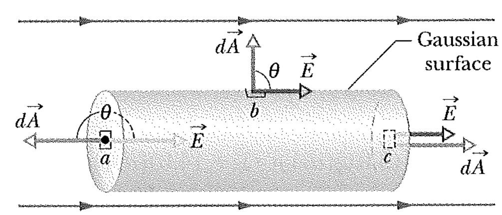


Fig. 23-4 A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.