Iqra National University,
Peshawar Department of
Computer Science Spring
Semester, Date: 25th June 2020
Final term – Semester
Examination

Course Code: **Course Title: Differential Equations Instructor: Engr. Latif Jan**

Program: BS (CS-SE & EE) **Total Marks: 50** **Time Allowed: 120 minutes Note: Attempt all Questions:**

Q 1: a) Define 2^{nd} order linear homogenous/non-homogenous differential equation along with examples? **(1+1 Marks)**

b) Solve the following 2^{nd} order Linear homogeneous /non-homogenous differential equation? **(5+5 Marks)**

i. 4y"-6y'+7y=0
ii. y"-4y'-12y=3e^(5x)

Q 2: Solve the following IVP for the 2^{nd} order linear equations. **(5+5+5+5 Marks)**

(i) $16y''-40y'+25y=0 \quad y(0)=3\ y'(0)=-9/4$
(ii) $y''+14y'+49y=0 \quad y(-4)=-1 y'(-4)=5$
(iii) $y''-4y'+9y=0 \quad y(0)=0 y'(0)=-8$
(iv) $y''-8y'+17y=0 \quad y(0)=-4 y'(0)=-1$

Q 3: Define Laplace transform along with example? **(2 Marks)**

A. Find the Laplace transforms of the given functions. **(2+2+2 Marks)**

1. f(t) = 6(e^−5t)+e^3t+5(t^3)−9
2. g(t) = 4cos(4t)−9sin(4t)+2cos(10t)
3. h(t) = e^3t+cos(6t)−e^(3t)cos(6t)

Q4: Solve the following IVP using Laplace Transform. **(5+5 Marks)**

(i) $y''-10y'+9y=5t, \quad y(0)=-1,\ y'(0)=2$
(ii) $y''-6y'+15y=2\sin(3t), \quad y(0)=-1\ \ y'(0)=-4$

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Differential Equations

Question # 01

Part (a)

⇒ **Homogenous:-** The differential Equation of any order is homogenous if once all the terms involving the unknown function are collected together on one side of the equation & the other side is Identically zero.

Example:- $y'' - 2y' + y = 0$

⇒ **Non Homogenous:-** The non homogenous equation has terms on both sides. This type of equation has the form of

Example:- $y'' + Py' + qy = f(x)$

where P, q are real no & can be real & complex.

Part (b)

(i) $4y'' - 6y' + 7y = 0$

Solution:-

For an equation $ay' + by' + cy = 0$ assume solution of form e^{xt} recurring equation with $y = e^{xt}$

$$4[(e^{xt})''] - 6[(e^{xt})'] + 7[e^{xt}] = 0$$

$$e^{xt}[4x^2 - 6x + 7] = 0 \quad \text{——} \ (i)$$

by solving (i)

$$\lambda = 3/4 + i\sqrt{19}/4, \quad \lambda = \frac{3}{4} - i\frac{\sqrt{19}}{4}$$

2 complex roots

$r_1 \neq r_2$

where $r_1 = a + i\beta$, $r_2 = a - i\beta$ The general solution is

$$\boxed{y = e^{at}(C_1 \cos(\beta t) + C_2 \sin \beta t)}$$

(ii) $y'' - 4y' - 12y = 3e^{(5x)}$

Solution:-

$$-12y(x) - 4\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 3e^{5x}$$

The differential equation has the form

$$y'' + Py' + qy = s$$

where,

$$P = -4$$
$$q = -12$$
$$s = -3e^{5x}$$

$$y'' + Py' + qy = 0$$

Find the root of the characteristics equation

$$q + (k^2 + kP) = 0$$

$$\Rightarrow k^2 - 4k - 12 = 0$$

The roots of this equation

$$k_1 = -2$$
$$k_2 = 6$$

$$\cancel{y(x) = c_1 e^{k}}$$

$$y(x) = C_1^{k_1 x} + C_2^{k_2 x}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{6x}$$

The general solution is:

$$y(x) = C_1(x)e^{-2x} + C_2(x)e^{6x}$$

$$y_1(x)\frac{d}{dx}C_1(x) + y_2(x)\frac{d}{dx}C_2(x) = 0$$

$$\frac{d}{dx}C_1(x)\frac{d}{dx}y_1(x) + \frac{d}{dx}C_2(x)\frac{d}{dx}y_2(x) = 0$$

$$\Rightarrow y_1(x) = \exp(-2x) \quad (C_1 = 1, C_2 = 0)$$

$$\Rightarrow y_2(x) = \exp(6x) \quad (C_1 = 0, C_2 = 1)$$

The free term $f = -8$ or

$$f(x) = 3e^{5x}$$

So the system has the form:

$$e^{6x}\frac{d}{dx}C_2(x) + e^{-2x}\frac{d}{dx}C_1(x) = 0$$

$$\frac{d}{dx}C_1(x)\frac{d}{dx}e^{-2x} + \frac{d}{dx}C_2(x)\frac{d}{dx}e^{6x} = 3e^{5x}$$

or

$$e^{6x}\frac{d}{dx}C_2(x) + e^{-2x}\frac{d}{dx}C_1(x) = 0$$

$$6e^{6x}\frac{d}{dx}C_2(x) + e^{-2x}\frac{d}{dx}C_1(x) = 3e^{5x}$$

Solve the System

$$\frac{d}{dx} C_1(x) = -\frac{3e^{7x}}{8}$$

$$\frac{d}{dx} C_2(x) = \frac{3e^{-x}}{8}$$

$$C_1(x) = C_3 + \int\left(-\frac{3e^{7x}}{8}\right) dx$$

$$C_2(x) = C_4 + \int\left(\frac{3e^{-x}}{8}\right) dx$$

or

$$C_1(x) = C_3 - \frac{3e^{7x}}{56}$$

$$C_2(x) = C_4 - \frac{3e^{-x}}{8}$$

Substitute found $C_1(x)$ & $C_2(x)$ to

$$y(x) = C_1(x)e^{-2x} + C_2(x)e^{6x}$$

$$\boxed{\Rightarrow y(x) = C_3e^{-2x} + C_4e^{6x} - \frac{3e^{5x}}{7}}$$

Question # 02

(i) $16y'' - 40y' + 25y = 0$ $\quad y(0) = 3, \; y'(0) = 9/4$

Solution:-

$$25y(x) - 40\frac{d}{dx}y(x) + 16\frac{d^2}{dx^2}y(x) = 0$$

divide both sides of the equation by the multiplier of the derivative of y''

$$\Rightarrow 16$$

we get the equation

$$\frac{25y(x)}{16} - \frac{5\frac{d}{dx}y(x)}{2} + \frac{d^2}{dx^2}y(x) = 0$$

This differential equation has the form

$$y'' + Py' + qy = 0$$

where, $P = -\frac{5}{2}$ & $q = \frac{25}{16}$

roots of the characteristic equation

$$q + (k^2 + kP) = 0$$

$$k^2 - \frac{5k}{2} + \frac{25}{16} = 0$$

root of this equation is

$$k_1 = 5/4$$

$$\boxed{y(x) = C_1 e^{5x/4} + C_2 x e^{5x/4}}$$

(ii) $y'' + 14y' + 49y = 0$ $\quad y(-4) = -1$, $y'(-4) = 5$ (8)

=> Solution :-

$$49y(x) + 14\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 0$$

=> The differential equation has the form

$$y'' + Py + qy = 0$$

where $P = 14$ & $q = 49$

root of the characteristic equation

$$q + (k^2 + kP) = 0$$

$$k^2 + 14k + 49 = 0$$

The root of this equation is :

$$k_1 = -7$$

$$y(x) = e^{k_1 x} C_1 + e^{k_1 x} C_2 x$$

Substitute :

$$k_1 = -7$$

$$y(x) = C_1 e^{-7x} + C_2 x e^{-7x}$$

$$C_1 = -\frac{9}{e^{28}}$$

$$C_2 = -\frac{2}{e^{28}}$$

$$y(x) = \left(-\frac{2x}{e^{28}} - \frac{9}{e^{28}}\right) e^{-7x}$$

$$y(x) = (C_1 + C_2 x)\, e^{-7x}$$

(9)

$$y(-4) = -1$$

$$\left(\begin{cases} -4 & \text{for } 0 = 1 \\ 1 & \text{for } 1 = 1 \\ 0 & \text{otherwise} \end{cases}\right) \frac{d}{dx} y(x) \Big|_{x=-4} = 5$$

$$\frac{d}{dx} y(x) = C_2 e^{-7x} - 7 (C_1 + C_2 x) e^{-7x}$$

$$y(x) = (C_1 + C_2 x) e^{-7x}$$

$$5 = C_2 e^{--28} - 7 (C_1 + (-4) C_2) e^{--28}$$

$$-1 = (C_1 + (-4) C_2) e^{--28}$$

$$\boxed{C_1 = \frac{-9}{e^{28}}}$$

(iii) $y'' - 4y' + 9y = 0$ $\quad\quad y(0) = 0, \; y'(0) = -8$

Solution:-

$$9y(x) - 4\frac{d}{dx} y(x) + \frac{d^2}{dx^2} y(x) = 0$$

This differential equation has the form:

$$y'' + Py + qy = 0$$

where $P = -4$ & $q = 9$

roots of the characteristic equation are:

$$q + (k^2 + kP) = 0$$

$$k^2 - 4k + 9 = 0$$

⑨

$$k_1 = 2-\sqrt{5}i$$

$$k_2 = 2+\sqrt{5}i$$

$$y(x) = e^{k_1 x}C_1 + e^{k_2 x}C_2$$

$$y(x) = C_1 e^{x(2-\sqrt{5}i)} + C_2 e^{x(2+\sqrt{5}i)}$$

$$y(0) = 0$$

$$\left(\begin{cases} 0 & \text{for } 0=1 \\ 1 & \text{for } 1=1 \\ 0 & \text{otherwise} \end{cases}\right) \frac{d}{dx} y(x)\bigg|_{x=0} = -8$$

$$\frac{d}{dx} y(x) = 2(C_1 \sin(\sqrt{5}x) + C_2 \cos(\sqrt{5}x))\, e^{2x}$$

$$y(x) = (C_1 \sin(\sqrt{5}x) + C_2 (\sqrt{5}x))\, e^{2x}$$

$$-8 = 2(C_1 \sin(0\sqrt{5}) + C_2 (0\sqrt{5}))\, e^{0.2}$$

$$0 = (C_1 \sin(0\sqrt{5}) + C_2 \cos(0\sqrt{5}))\, e^{0.2}$$

$$C_2 = 0$$

$$C_1 = -\frac{8\sqrt{5}}{5}$$

$$\boxed{y(x) = -\frac{8\sqrt{5}e^{2x}\sin(\sqrt{5}x)}{5}}$$

Question # 03

Q:- Define Laplace Transform along with Example:-

=> **Laplace Transform:-**

Laplace transform is the Integral Transform of the given derivative function with real variable t to convert into complex function with variable s.

$$F(s) = \int_0^\infty f(t) \cdot e^{-st} \, dt$$

Question # 03

(i) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

Solve:-

$$F(s) = 6\,\frac{1}{s-(-5)} + 1\,\frac{1}{s-3} + 5\,\frac{3!}{s^{3+1}} - 9\,\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

(ii) $g(t) = 4\cos^{4t} - 9\sin(4t) + 2\cos(10t)$

Solution:-

$$g(s) = 4\,\frac{s}{s^2+4^2} - 9\,\frac{4}{s^2+4^2} + 2\,\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+10} + \frac{2s}{s^2+100}$$

(iii) $h(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

Solution:-

$$g(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Question # 04

(i) $y'' - 10y' + 9y = 5t \qquad y(0) = -1 \quad y'(0) = 2$

Solution:-

Applying the Laplace transform to both sides:

$$(s^2 - 10s + 9)Y + s - 2 - 10 = \frac{5}{s^2}$$

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

To find the Inverse Laplace Transform we will first simplify the expression for $Y(s)$ using the partial-fraction decomposition

$$\frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

we find,

$$B = \frac{5}{9}, \quad D = -2, \quad C = \frac{31}{81}, \quad A = \frac{50}{81}$$

Therefore using the linearity of the Inverse Laplace Transform

$$\boxed{y(t) = \frac{50}{81} + \frac{5t}{9} + \frac{31}{81}e^{9t} - 2e^{t}}$$

$y'' - 6y' + 15y = 2\sin(3t)$ $\quad y(0) = -1 \quad y'(0) = -4$

Solution:- we have:

$$(s^2 - 6s + 15)y + s - 2 = \frac{6}{s^2+9}$$

$$\Rightarrow y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

To find the constants we need to simplify the expression on the right & equate the coefficients at the equal powers

$$s^3 = A + C = -1$$
$$s^2 = -6A + B + D = 2$$
$$s^1 = 15A - 6B + 9C = -9$$
$$s^0 = 15B + 9D = 24$$

$$A = \frac{1}{10}, \quad B = \frac{1}{10}, \quad C = \frac{11}{10}, \quad D = \frac{5}{2}$$

$$y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right)$$

Now we need to find the Inverse Laplace Transform

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}\left\{\frac{3}{s^2+9}\right\}$$

$$= \cos 3t + \frac{1}{3}\sin 3t$$

$$\Rightarrow \frac{11s+25}{s^2-6s+15} = \frac{-11s+25}{(s-3)^2+6}$$

$$= \frac{-11(s-3)-8}{(s-3)^2+6}$$

$$= -11\,\frac{(s-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}}\,\frac{\sqrt{6}}{(s-3)^2+6}$$

$$\mathcal{L}^{-1}\left\{\frac{-11s+25}{s^2-6s+15}\right\} = -11\,e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{10}\left(\cos 3t + \frac{1}{3}\sin 3t - 11\,e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t\right)}$$

"**End of Paper**"