Final paper

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Subject Differential Equations

Section A

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Q no 1
(part - 1)

(1)
$w = \sin(x+ct) + \cos(2x+2ct)$

Solution :

$w = \sin(x+ct) + \cos(2x+2ct)$

Now
$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}(\sin(x+ct)) + \frac{\partial}{\partial t}(\cos(2x+2ct))$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Now
$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)}$$

Now
$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

(1) $\rightarrow$

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2\left[-\sin(x+ct) - 4\cos(x+2ct)\right] - c^2\sin(x-ct) - 4c^2\cos(2x+2ct)$$

$$= -c\sin(x+ct) - 4c^2\cos(x+2ct)$$

$0 = 0$ (satisfied)

Q ① **part - 2**

Give Data

(i:) $W = \tan(2x + ct)$

Solution

$$W = \tan(2x + ct)$$

Now $\dfrac{\partial w}{\partial t} = c\sec^2(2x + ct)$

& $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial t}\left(c\sec^2(2x + xt)\right)$

$= c_2 \cdot 2\sec^2(2x + ct)\tan(2x + ct)$

Now

$$\frac{\partial w}{\partial x} = 2\sec^2(2x + 2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x + ct)\tan(2x + ct)$$

① $\Rightarrow$ $4c^2\cancel{\sec^2(2x+ct)}\tan(2x + ct) = 4c^2\cancel{\sec^2(2x+ct)}\tan(2x + ct)$

$(0 = 0)$ Satisfied.

Q2

Expand the following function in a fourier series.

$P(x) = x, \; -\pi < x \le 0$

$= 2x, \; 0 \le x \le \pi$

Solution:

$$F(x) \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

we have to find the fourier's co-efficient a_0, a_n $f b_n$.

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \pi^2/2\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \pi/2} \leftarrow (i)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}$

$\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$

$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}$

$\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$

$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$

So

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2} ; \text{if } n \text{ is add} \\ 0 ; \text{if } n \text{ is even} \end{cases}} \quad -(2)$$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$

$+ \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$

$= \frac{1}{\pi}\left[x\left(\frac{-\cos x}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$

$+ \frac{2}{\pi}\left[x\left(-\frac{\cos x}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi\cos\pi}{n} + \frac{2}{\pi}\left\{-\frac{\pi\cos\pi}{n}\right\}\right] = \frac{-3\cos\pi}{n}$$

$$= \frac{3(-1)^{n+1}}{n}$$

So the required fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$\boxed{= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}}$$

DKS

Q(5)

Solve the initial value problem
$y'' - 4y' + 13y = 18\sin 3x$ $y(0) = 1$ and
$y'(0) = 2$

Solutions

$y'' - 4y' + 13y = 8\sin 3x$ $y(0) = 1$
$y'(0) = 2$

Associated Homogenous eq of (1) is
$y'' - 4y' + 13y = 0 \rightarrow (2)$

change (2) into auxiliary equations

Put $y = m$ in (1)

$m^2 - 4m + 13 = 0$

Use Quadratic formula

$a = 1, b = -4, c = 13.$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36 i}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$$m_1 = 2 + 3i$$

$$m_2 = 2 - 3i$$

$$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \rightarrow (A)$$

let

$$y_p = A\cos 3x + B\sin 3x \rightarrow x)$$

Diff, w.r.t "x"

$$y'_p = -3A\sin 3x + 3B\cos 3x$$

Again Diff w.r.t "x"

$$y''_p = -9A\cos 3x - 9B\sin 3x$$

Put in (1)

$$\Rightarrow (-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x + 3B\cos^3 x) + 13(A\cos 3x + B\sin 3x) - B\sin 3x$$

$$\Rightarrow -9A\cos - 12B\cos 3x + 13A\cos 3x - 9B\sin 3x + 12A\sin 3x + 13B\sin 3x = 8\sin 3x$$

$$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)\sin 3x = 8\sin 3x.$$

$$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)\sin 3x = 8\sin 3x$$

Comparing co-effinits

$\sin 3x \Rightarrow \quad 4B + 12A = 8 \rightarrow$ (a)

$\cos 3x \Rightarrow \quad 4A - 12B = 0 \Rightarrow 4A = 12B$

$$\Rightarrow \boxed{A = 3B} \rightarrow \text{(b)}$$

put (b) in (a)

$$4B + 12(3B) = 8$$

$$4B + 36B = 8$$

$$40B = 8$$

$$B = 1/5 \rightarrow \text{(c)}$$

put (c) in (b)

$$\Rightarrow \boxed{A = 3/5} \longrightarrow \text{(d)}$$

Put (E) & (d) in x

$$y_p = \frac{13}{5}\cos 3x + \frac{1}{5}\sin 3x \rightarrow (B)$$

Then G. sol is

$$y = y_c + y_p$$

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{3}{5}\cos 3x$$
$$+ \frac{3}{5}\sin 3x \rightarrow (C)$$

Now we need to find the values of c_1 & c_2 for this.

Put $x=0$ & $y=1$ in (C)

$$1 = e^{x(0)}(c_1 \cos 3(0) + c_2 \sin 3(0)) + \frac{3}{5}\cos 3(0)$$
$$+ \frac{1}{5}\sin 3(0)$$

$$1 = (c_1(1) + c_2(0)) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = c_1 + \frac{3}{5}$$

$$c_1 = 1 - \frac{3}{5}$$

$$c_1 = \frac{2}{5}$$

(C) w.r.t.

$$c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x +$$
$$3e^{2x}\cos 3x) - \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x \rightarrow (D)$$

Put $y' = 2$, $x = 0$ in (D)

$y' = c_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right)$
$+ c_2\left(2e^{2x}\sin 3x + 3e^{2x}\cos 3x\right)$
$- \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x$

put $y' = 2,\ x = 0$

$2 = c_1\left(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)\right)$
$+ c_2\left(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0)\right)$
$- \frac{6}{5}\sin 3(0) + \frac{3}{5}\cos 3(0)$

$= c_1(2) + c_2(3) - 0 + \frac{3}{5}$

$2 = 2c_1 + 3c_2 + \frac{3}{5}$

put $c_1 = \frac{2}{5}$

$2 = \frac{4}{5} + 3c_2 + \frac{3}{5}$

$2 = \frac{7}{5} + 3c_2$

$3c_2 = 2 - \frac{7}{5}$

$3c_2 = \frac{3}{5}$

$\boxed{c_2 = \frac{3}{15}} \rightarrow \boxed{(iii)}$

put (ii) & (iii) in (iC)

$y = e^{2x}\left(\frac{2}{5}\cos 3x + \frac{2}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin \ldots$

$\boxed{y = \frac{2}{5}e^{2x}\cos 3x + \frac{1}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\ldots}$

$\rightarrow$ Required General solution

(i) (4)

Solve

$(D^2 - DD')z = \cos x \cos 2y$

Solution

It is already in symbolic form

$(D^2 - DD')z = \cos x \cos 2y$ — (a)

Put A.E $D^2 - DD' = 0$

As we know

$D/D' = m$ i.e $D = m, D' = 1$

$\Rightarrow m^2 - m = 0$

$m = 0, 1$

Therefore $C.F = f_1(y) + f_2(y+x)$

from eq (a)

$P.I = \frac{1}{D^2 - DD'} \cos x \cos y$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} 2\cos x \cos 2y$$

As

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$CF = f_1(y-x) + x f_2(y-x)$$

$$PI = \frac{1}{Dx + 2DD' + D'^2}\left[2(y-x) + \sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2}\left[2(y-x) + \sin(x-y)\right]$$

By General method

$$m = -1 \; ; y - x = c$$

$$= \frac{1}{D+D'}\left[\int (2c + \sin(-c)\right] dx$$

$$= \frac{1}{D+D'}\left[2cx - (\sin c)\, x\right]$$

Replacing by $y-x$

$$= \frac{1}{D+D'}\left[2x(y-x) - x\sin(y-x)\right]$$

Again put $y - x = c$

$$= \int (2xc - x\sin c)\, dx \Rightarrow cx^2 - \frac{x^2}{2}\sin c$$

Replacing by $y-x$

$$= x^2(y-x) - \frac{x^2}{2}\sin(y-x) = x^2y - x^3 + \frac{x^2}{2}\left(\sin(x-y)\right)$$

Hence the required solution is

$$Z = C.F + P.I = f_1(y-x) + x f_2(y-x) + xy - x^3 + \tfrac{1}{2}x\left(\sin(x-y)\right)$$