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I-D # "7209"

Subject # Differential Equation.

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Question 01:- The wave Equation

by one-dimensional ~~E~~ wave Equation.

Show the following functions are all solutions of the wave eq. by determining relevent Partial derivaties;

i):- $w = \sin(x+ct) + \cos(2x+2ct)$

Given,

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \longrightarrow (1)$$

Now,

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right].$$

$$\frac{\partial}{\partial t}(\sin(x+ct)) + \frac{\partial}{\partial t}(\cos(2x+2ct))$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Now,

$$\boxed{\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 4c^2\cos(2x+2ct)\right.}$$

Now,

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right].$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right].$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$\Rightarrow$ ①

$$-c^2 \sin(x+ct) - 4c^2\cos(2x+2ct) = c^2$$

$$c^2[-\sin(x+ct) - 4\cos(2x+2ct).$$

$$-c^2 \cancel{\sin(x+ct)} - 4c^2 \cancel{\cos(2x+2ct)} = -c\cancel{\sin(x+ct)}$$

$$-4c^2 \cancel{\cos(2x+2ct)}$$

$$\boxed{0 = 0} \text{ (Satisfied).}$$

ii)= $w = \tan(2x+ct)$

Now, $\dfrac{\partial w}{\partial t} = c\sec^2(2x+ct)$

& $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial t}(c\sec^2(2x+ct)$

$$= c^2 \cdot 2\sec(2x+ct)\tan(2x+ct)$$

Now, $\dfrac{\partial w}{\partial x} = 2\sec^2(2x+ct)$

$$\frac{\partial^2 w}{\partial t^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

$\Rightarrow$ ① $4c^2 \cancel{\sec^2(2x+ct)\tan(2x+ct)} = 4c^2 \cancel{\sec^2(2x+ct)}$
$\tan(2x+ct)$

$$\boxed{0 = 0} \text{ (Satisfied).}$$

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Question #02:- Expand the following functions in a fourier Series;

$$F(x) = x, \quad -\pi < x \leq 0$$
$$= 2x, \quad 0 \leq x \leq \pi$$

Solution;

We have to find the fourier co-efficient, a_0, a_n & b_n,

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \cancel{\frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \left\{\frac{\pi^2}{2} - 0\right\}}$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left(\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{\cancel{2}} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \pi/2} \quad \text{— (1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x\cos nx)dx\}$$

$$= \frac{1}{\pi}\left\{x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} +$$

$$+\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos nx}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos nx}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So,

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2} ; & \text{if } n \text{ is odd.} \\ 0 ; & \text{if } n \text{ is even} \end{cases}} \rightarrow (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right]$$

$$\boxed{b_n = -\frac{3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}} \rightarrow (3)$$

So the requried fouries Series is;

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2x-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

again use the another Initial Condition

$$y'(0) = 2$$

So,

$\Rightarrow$ $y' = C_1 2e^{2x} \cos 3x + C_1 e^{2x}(-3 \sin 3x)$

$+ C_2 2e^{2x} \sin 3x + C_2 e^{2x}(3 \cos 3x)$

$+ \frac{2}{10}(\cos 3x - 3 \sin 3x)$.

$\Rightarrow$ $y'(0) = C_1 2e^{(0)} \cos(0) + C_1 e^{(0)}(-3 \sin(0))$.

$+ C_2 2e^{(0)} \sin(0) + C_2 e^{(0)}(3 \cos(0))$

$+ \frac{2}{10}(\cos(0) - 3(\sin(0)))$.

$2 = 2C_1 + 0 + 0 + C_2 3(1) + \frac{2}{10}(1 - 3(0))$

$2 = 2C_1 + 3C_2 + \frac{2}{10}$.

$2 = 2(2/5) + 3C_2 + \frac{2}{10}$. $\boxed{\text{use } C_1 = 2/5}$

$1/3\left(2 - \frac{4}{5} - \frac{2}{10}\right) = \boxed{C_2 \Rightarrow C_2 = 1/3\left(\frac{20 - 8 - 2}{10}\right) = \frac{1}{3}}$.

So the gernal Solution is ;

$y = 2/5\, e^{2x} \cos 3x + 1/3\, e^{2x} \sin 3x + \frac{2}{10}[\sin 3x + 2\ 3\cos 3x]$.

is the requried Solution

Question #03:- Solve the initial Value Problem

$$y'' - 4y' + 13y = 8\sin 3x ,$$

$$y(0) = 1 \quad \& \quad y'(0) = 2.$$

Solution:-

we have to find $y = y_c + y_p$

For y_c the characteristics (Auxilery Eq.) Eq. is;

$$m^2 - 4m + 13 = 0.$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-52}}{2} \Rightarrow m = \frac{4 \pm 6i}{2}.$$

So,

$$y_c = e^{2x}\{C_1 \cos 3x + C_2 \sin 3x\}.$$

For y_p Let;

$$y_p = \text{Imag}\left(\frac{1}{m^2 - 4m + 13} 8e^{3ix}\right)$$

$$= 8\,\text{Imag} \frac{e^{3ix}}{(3i)^2 - 4(3i) + 13}$$

$$= 8\,\text{Imag} \frac{e^{3ix}}{-9 - 12i + 13}$$

$$= 8\,\text{Imag} \frac{e^{3ix}}{4 - 12i}$$

$$y_p = 2\, \text{Imag.}\ \frac{e^{3ix}}{(1-3i)} \times \frac{(1+3i)}{(1+3i)}$$

$$y_p = 2\, \text{Imag.}\ \frac{(1+3i)\,(e^{3ix})}{(1)^2 - (3i)^2}$$

$$y_p = 2\, \text{Imag}\ \frac{(1+3i)\,(e^{3ix})}{10}$$

$$y_p = \frac{2}{10}\left(\text{Imag}\,(1+3i)\,(\cos 3x + i \sin 3x)\right)$$

$$y_p = \frac{2}{10}\left\{1 \sin 3x + 3\cos 3x\right]$$

So the general solution is;

$$y = y_c + y_p$$

$$y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x + \frac{2}{10}\left\{\sin 3x + 3\cos 3x\right]$$

Now use the initial condition $y(0) = 1$.

$$y(0) = C_1 e^{(0)} \cos(0) + C_2 e^{(0)} \sin(0) + \frac{2}{10}\left[\sin(0) + 3\cos(0)\right]$$

$$1 = C_1(1) + 0 + 0 + 0 + \frac{2}{10}\,(3(1))$$

$$1 = C_1 + \frac{6}{10} \Rightarrow \boxed{C_1 = \frac{1-6}{10} = \frac{\cancel{4}}{\cancel{10}} = 2/5.}$$

Question #04: Solve;

$$(D^2 - DD')z = \cos x \cos 2y.$$

Auxiliary Equation is;

$$m^2 - m = 0 \Rightarrow m = 0 \; , \; m = 1.$$

Hence the complemently function is given by.

$$Z_c = f_1(y) + f_2(y + x)$$

For the Particular Integrial, we have,

$$Z_p = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$Z_p = \frac{1}{2} \cdot \frac{1}{D^2 - DD'} \left[\cos(x + 2y) + \frac{1}{D^2 - DD'} \cos(x - 2y) \right]$$

$$Z_p = \frac{1}{2} \left[\frac{1}{D^2 - DD'} \cos(x + 2y) + \frac{1}{D^2 - DD'} \cos(x - 2y) \right].$$

$$Z_p = \frac{1}{2} \left[\frac{1}{-1+2} \cos(x + 2y) + \frac{1}{-1-2} \cos(x - 2y) \right].$$

$$Z_p = \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$$

Hence the Complete Solution is given by.

$$= f_1(y) + f_2(y + x) + \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$$

Answer