Zarak Khan ID: 13794

Q1: Use method of separation of variables to find the general solution to the following differential equations.

(a): $x' = \sqrt{x}$

Sol: Step 1:

Given: $x' = \sqrt{x}$

Here x is the dependent variable. let us take t as the independent variable.

then differential equation becomes

$$\frac{dx}{dt} = \sqrt{x}$$

According to the method of separation of variables, all terms involving dependent variable brought to one side and all independent variable to other side. then integrate for the solution

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Step 2: Consider

$$\frac{dx}{dt} = \sqrt{x}$$

$$\frac{dx}{\sqrt{x}} = dt$$

On integration on both sides, we get

$$\int \frac{dx}{\sqrt{x}} = \int dt$$

$$\int x^{-\frac{1}{2}} dx = \int dt$$

$$-\frac{1}{2} x^{-\frac{1}{2}-1} = t + C$$

$$\boxed{-\frac{1}{2x^{\frac{3}{2}}} = t + C}$$ (C is the constant of integration)

This is the required solution

$$\Rightarrow \tan^{-1}(y) = x + C$$

$$\boxed{\Rightarrow y = \tan(x + C)}$$ where C is albitary constant

Step 3:

Hence the general solution of the differential equation

$$y' = 1 + y^2 \text{ is}$$

$$\boxed{y = \tan(x + C)}$$

On Further simplification

$$u^2 - 5u = C - V$$

$$\left(u^2 - 5u + \frac{25}{4} - \frac{25}{4}\right) = C - V$$

$$\left(u - \frac{5}{2}\right)^2 = C - V$$

$$\left(u - \frac{5}{2}\right) = \sqrt{C - V}$$

$$\boxed{u = \frac{5}{2} + \sqrt{C - V}}$$

Thus, the general solution is

$u = \frac{5}{2} + \sqrt{C - V}$, where C is constant.

(3)

(b): $x' = e^{-2x}$

Sol: **Step 1:**

Consider the given differential equation

$$x' = e^{-2x}$$

it can be written as,

$$\frac{dx}{dt} = e^{-2x}$$

Multiply both sides by dt

$$dx = e^{-2x}\,dt$$

Divide both sides by e^{-2x}

$$\frac{dx}{e^{-2x}} = dt$$

it can be written as

$$e^{2x}\,dx = dt$$

(f) :- $Q' = \dfrac{Q}{4+Q^2}$

Sol:- Step 1:

Given differential equation

$$Q' = \frac{Q}{4+Q^2}$$

let the independent variable be t and thus we have

$$Q' = \frac{dQ}{dt}$$

So given differential equation becomes

$$\frac{dQ}{dt} = \frac{Q}{4+Q^2}$$

Rewrite the above differential equation as

$$\frac{(4+Q^2)}{Q}\,dQ = dt$$

(e): $x' = au + b, \quad a, b > 0$

Sol: Step 1:

given

$$x' = au + b \quad \dots (1) \quad a>0, \; b>0$$

Step 2:

Equation (1) can be written as

$$\frac{dx}{du} = au + b$$

by variable of separation

$$dx = (au+b)\,du$$

integration on both sides

$$\int dx = \int (au+b)\,du + C$$

$$\int dx = \int au\,du + \int b\,du + C$$

$$\int dx = a\int u\,du + b\int du + C$$

$$\boxed{x = a\frac{u^2}{2} + bu + C}$$

(4)

Step 2: The equation $e^{2x}dx = dt$ in variable separable form

Integrated both sides

$$\Rightarrow \int e^{2x}dx = \int dt$$

$$\Rightarrow \frac{e^{2x}}{2} = t + C$$

Multiply both sides by 2

$$e^{2x} = 2(t+C)$$

Taking natural logarithm function from both sides

$$\Rightarrow \ln(e^{2x}) = \ln(2t + 2C)$$

$$\Rightarrow 2x = \ln(2t + C_1)$$

$$\boxed{\Rightarrow x = \frac{\ln(2t + C_1)}{2}}$$

The solution of the given differential equation is $x = \frac{\ln(2t+C_1)}{2}$

(d) :- $u' = \frac{1}{5-2u}$

Sol:- Step 1: Given $u' = \frac{1}{5-2u}$

Compute general equation as follows.

Rewrite the given differential equation as

$$\frac{du}{dv} = \frac{1}{5-2u}$$

$$(5-2u)\,du = dv$$

Step 2:

Now integrate on both sides

$$\int (5-2u)\,du = \int dv$$

$$5u - \frac{2u^2}{2} + C = v + C$$

$$5u - u^2 = v + C$$

(c): $y' = 1 + y^2$

Sol: Step 1: Here we use the method of separation of variable to find the general solution of the differential equation

$$y' = 1 + y^2$$

Step 2: Here

$$y' = 1 + y^2$$

$$\Rightarrow \frac{dy}{dx} = 1 + y^2 \qquad \left[\because y' \frac{dy}{dx}\right]$$

$$\Rightarrow dy = (1 + y^2)\,dx$$

$$\Rightarrow \frac{dy}{1+y^2} = dx$$

integrating both sides, we get

$$\int \frac{dy}{1+y^2} = \int dx + C \qquad \left[\because \int \frac{dx}{1+y^2} = \tan^{-1}(y)\right]$$

The left hand side consist only one variable Q and right hand side consist only of one variable t.

Hence, differential Equation $\left(\frac{4+Q^2}{Q}\right) dQ = dt$ is a variable separable equation

Step 2:

Proceed using the method of separation of variables to obtain the solution of the differential equation $\frac{(4+Q^2)}{Q} dQ = dt$ as follows

$$\int \frac{(4+Q^2)}{Q} dQ = \int dt$$

$$\int \left(\frac{4}{Q} + \frac{Q^2}{Q}\right) dQ = \int dt$$

$$\int \left(\frac{4}{Q} + Q\right) dQ = \int dt$$

$$\int \frac{4}{Q} dQ + \int Q\,dQ = \int dt$$

$$4\int \frac{1}{Q} dQ + \int Q\,dQ = \int dt$$

$$\boxed{4 \ln |Q| + \frac{Q^2}{2} = t + C}$$ Answer

The left hand side consist only one variable Q and right hand side consist only of one variable t.

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$$4\int \frac{1}{Q} dQ + \int Q\,dQ = \int dt$$

$$\boxed{4 \ln|Q| + \frac{Q^2}{2} = t + C}$$ Answer

Q: $x' = e^{x^2}$

Sol: Step 1:

Given Equation $x' = e^{x^2}$

Rewrite equation as $\frac{1}{e^{x^2}} \frac{dx}{dt} = 1$

$$\frac{1}{e^{x^2}} dx = dt$$

Integrating both sides

$$\int \frac{1}{e^{x^2}} dx = \int 1 dt$$

$$\int e^{-x^2} dx = \int 1 dt$$

Step 2: $\int \frac{1}{e^{x^2}} dx = \int 1 dt$

$$\int e^{-x^2} dx = \int 1 dt$$

Multiply & divide by $\sqrt{\pi}$

$$\int \frac{1}{e^{x^2}}\,dx = \int 1\,dt$$

$$\sqrt{\pi}\int \frac{1}{\sqrt{\pi}} e^{-x^2}\,dx = \int 1\,dt$$

Multiply & divide by 2

$$\frac{\sqrt{\pi}}{2}\int 2\frac{1}{\sqrt{x}} e^{-x^2}\,dx = \int 1\,dt \quad \text{.... (1)}$$

the integral $\int \frac{2}{\sqrt{x}} e^{-x^2}\,dx$ is the $\operatorname{erf}(x)$

Therefore, (1) reduces to $\frac{\sqrt{x}}{2}\operatorname{erf}(x) = t + C$

where C is the constant of integration

The answer is: $\boxed{\frac{\sqrt{x}}{2}\operatorname{erf}(x) = t + C}$

(h) :- $y' = r(a-y)$

Sol :- Step 1 :

Given $y' = r(a-y)$

Step 2 :

Given differential equation $y' = r(a-y)$

$$y' = r(a-y)$$

$$\frac{dy}{dt} = r(a-y)$$

$$\frac{dy}{(a-y)} = r\,dt$$

Integrating we get

$$\int \frac{dy}{(a-y)} = \int r\,dt$$

$$-\ln(a-y) = rt + C$$

$$\ln(a-y) = -rt - C$$

Step 3:

Solve the equation $\ln(a-y) = -rt - C$ for y as shown below

$$\ln(a-y) = -rt - C$$

$$e^{\ln(a-y)} = e^{-rt-C}$$

$$a - y = e^{-rt-C}$$

$$\boxed{y = -e^{-rt-C} + a}$$

Therefore

The general solution of the differential equation

$y' = r(a-y)$ is $\boxed{y = -e^{-rt-C} + a}$

ID # 13794

SubJ # Differential Equations

Teacher # Latif Jan