Question #01:- ①

Solution.

$$x^3y''' + 2x^2y' + 2y = 10x + \frac{10}{x} \quad — ①$$

Put

$x = e^t$ Then

$$\frac{dx}{dt} = e^t \Rightarrow \frac{dt}{dx} = e^{-t}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot e^{t}$$

or.

$$y' = \frac{dy}{dx} = e^{-t} Dy \quad \because \frac{d}{dt} \to D.$$

Similarly

$$y'' = e^{-2t} [D(D-1)]y.$$

$$y''' = e^{-2t}(D(D-1)(D-2)]y.$$

Using these values in eq. ①.

$$\Rightarrow e^{3t} \cdot e^{-3t}[D(D-1)(D-2)]y + 2e^{2t} \cdot e^{-2t}[D(D-1)]y + 2y = 10e^t + 10e^{-t}.$$

$$\Rightarrow (D^3 - 3D^2 + 2D)y + (2D^2 - 2D)y + 2y = 10e^t + 10e^{-t}.$$

$$\Rightarrow D^3y - D^2y + 2y = 10e^t + 10e^{-t}.$$

$$\Rightarrow \frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 10e^t + 10e^{-t} \to ②.$$

The assosiated homogenious eq. of ② is.

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 0.$$

Say, $\frac{d}{dt^2} = x^2$, $\frac{d^3}{dy^3} = K^3$.

$\Rightarrow (K^3y - K^2y + 2y) = 0.$

$(K^3 - K^2 + 2)y = 0.$

$\Rightarrow$ For non-trivial Sol., $y \neq 0$

$K3 - K^2 + 2 = 0.$

$\Rightarrow$ Root are, $K = -1, 1 \pm i$

$\Rightarrow y_c(t) = Ae^{-t} + (B\cos t + C\sin t)e^t.$

which is complementry Sol.

Question #03:-

(P-1)

Let $x^2 y'' + 2x y' - 6y = 10x^2$; $y(1) = 1$ & $y'(1) = -6$.

Now $x = e^t$ ie $t = \log x$

$xy' = \Delta y \Rightarrow x^2 y'' = \Delta(\Delta - 1)y$.

where $\Delta = \frac{d}{dt}$.

Then

eq. ① $\Rightarrow [\Delta(\Delta-1) + 2\Delta - 6]y = 10e^{2t}$.

$[\Delta^2 - \Delta + 2\Delta - 6]y = 10e^{2t}$.

$(\Delta^2 + \Delta - 6]y = 10e^{2t}$.

Char form eq.

$\Delta^2 + \Delta - 6 = 0$.

$\Delta + 3\Delta - 2\Delta - 6 = 0$.

$\Delta = -3, \Delta = 2$.

(P-2):- Compementry Function

$C.F = c_1 e^{-3t} + c_2 e^{2t}$

Also P. Integral.

$P.I = \frac{1}{\Delta^2 + \Delta - 6} 10e^{2t}$

$= 10 \frac{1}{(2)^2 + 2 - 6} e^{2t}$ replace Δ by in case of fialiure.

$P.I = 10t \frac{1}{2\Delta + 1} e^{2t} = 10t \cdot \frac{1}{2(2)+1} e^{2t}$.

$= 10t \frac{1}{5} e^{2t} = 2t e^{2t}$.

Hence general: Sol $y = C.F + PI$.

$y = c_1 e^{-3t} + C_2 e^{2t} + 2t e^{2t}$

$= C_1 x^{-3} + C_2 x^2 + 2(\log x) x^2 \rightarrow (*)$.

P-3:-

Applying intial Cond.

$y(1) = 1$ we get,

$1 = C_1 + C_2 + 0 \longrightarrow (A)$.

$1 \cdot y'(1) = -6$

$y' = -3c_1 x^{-4} + 8c_2 x + 2x + 4x \log x$

$-6 = -3c_1 + 2C_2 + 2 + 0$

$-3c_1 + 2c_2 = -8 \longrightarrow (B)$.

eq. (A) $\times$ 3 & add with eq (B)

$3 = \cancel{3c_1} + 3c_2$

$-8 = \cancel{-3c_1} + 2c_2$

$5c_2 = -5$.

$\boxed{c_2 = -1}$

eq. (A) $\Rightarrow 1 = c_1 - 1$

$\boxed{c_1 = 2}$

Thus (*) $\Rightarrow \boxed{y = 3x^{-3} - x^2 + 2x^2 \log x.}$

Question #04:-

$$x^2y'' + 7xy' + 5y = x^5 \quad , \quad y(0) = 2 \;\&\; y'(1) = 2.$$

Let,
$$x = e^t \Rightarrow t = \log x \quad , \quad \Delta = \frac{d}{dt}.$$

Now
$$xy' = \Delta y \Rightarrow x^2 y = \Delta(\Delta - 1)y.$$

Then
$$(\Delta(\Delta-1) + 7\Delta + 5)y = e^{5t}.$$
$$(\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}.$$
$$(\Delta^2 - 6\Delta + 5)y = e^{5t}$$

Char. eq. is;
$$\Delta^2 + 6\Delta + 5 = 0.$$
$$\Delta^2 + 5\Delta + \Delta + 5 = 0.$$
$$\Delta = -5, -1\Delta.$$

Complerafore eq. is
$$C.F. = C_1 e^{-5t} + C_2 e^{-t}.$$

P. Integral

$$P.I. = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}$$
$$= \frac{1}{5^2 + 6(5)} e^{5t} \quad \text{replacing } \Delta \text{ by } 5.$$
$$= \frac{1}{60} e^{5t}$$

Thus G. solution.
$$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$$

$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5.$

$y' = -5 c_1 x^{-6} - c_2 x^{-2} + \frac{1}{12} x^4.$

$y(0) = 2 \qquad x = 0, y = 2.$

$2 = c_1 + c_2 + \frac{1}{60}.$

$c_1 + c_2 = \frac{119}{60} \rightarrow A.$

$y'(1) = 2 \qquad x = 1, y' = 2.$

$2 = -5c_1 - c_2 + \frac{1}{12}.$

$-5c_1 - c_2 = \frac{23}{12} \rightarrow B$

A+B $\quad -4c_1 = \frac{234}{60} \Rightarrow 4 = \frac{-117}{120}.$

Now

$y = \frac{-117}{120} x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5$

$c_1 = \frac{-117}{120}$ Put in eq. A $\quad \frac{-117}{120} + c_2 = \frac{119}{60}.$

$c_2 = \frac{119}{60} + \frac{117}{120}$

$= \frac{238 + 117}{120}. \quad = \frac{355}{120}.$

Question #05:-

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2 \quad \rightarrow (1)$$

Let, $x+1 = e^t \Rightarrow x = e^t - 1$

Diff. $\log(x+1) = t$

Also, $(x+1)y' = \Delta y$. $\left[\frac{d}{dt} = \Delta \text{ \& } D = \frac{d}{dx}\right]$

$(x+1)^2 y'' = \Delta(\Delta - 1)y$.

Then eq. (1) $\Rightarrow (\Delta(\Delta-1) - 3\Delta + 4)y = (e^t - 1)^2$.

$$(\Delta^2 - 4\Delta + 4)y = e^{2t} - 2e^t + 1$$

Chare eq. is $\Delta^2 - 4\Delta + 4 = 0$

$$(\Delta - 2)^2 = 0$$

$\Delta = 2, 2$.

Thus Complemuitry Function is

C.F. $= (c_1 + c_2 t)e^{2t}$.

Also Particular Integral is

P.I $= \frac{1}{(\Delta-2)^2}(e^{2t} - 2e^t + 1)$

$= \frac{1}{(\Delta-2)^2}e^{2t} - 2\frac{1}{(\Delta-2)^2}e^t + \frac{1}{(\Delta-2)^2} \Rightarrow (2)$.

Now, $\frac{1}{(\Delta-2)^2}e^{2t} = \frac{1}{(2-2)^2}e^{2t} = \frac{1}{0}e^{2t}$.

Case of failure

$\frac{1}{(\Delta-2)^2}e^{2t} = t\frac{1}{2(1-2)^2}e^t = \frac{t^2 e^t}{2}$

& $2\frac{1}{(\Delta-2)^2}e^t = 2\frac{1}{(1-2)^2}e^t = 2e^t$.

& $\frac{1}{(\Delta-2)^2}(1) = \frac{1}{(\Delta-2)^2} e^{0t} = \frac{1}{4}.$

eq (5) ⟹ PI $= \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}.$

Hence complete solution is

$y = C.F + P.I$

$y = (c_1 + c_2 t) e^{2t} + \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}.$

repeat the value of e^t

$y = (c_1 + c_2 \log(x+1)) (x+1)^2 + \frac{1}{2}\left[(\log(x+1)^2)(x+1)^2\right] - 2(x+1) + \frac{1}{4}$

OR:-

$y = c_1 + c_2 \log(x+1)(x+1)^2 + \frac{1}{2}\left[\log(x+1)^2 (x+1^2)\right] - 2x - 7/4$

Which is the required.

& $\frac{1}{(\Delta-2)^2}(1) = \frac{1}{(\Delta-2)^2} e^{2t} = \frac{1}{4}.$

eq. 2 $\Rightarrow$ $P.I. = \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}.$

Hence complete solution is.

$y = C.F. + P.I.$

$y = (c_1 + c_2 t) e^{2t} + \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}.$

repeat the value of e^t.

$y = (c_1 + c_2 \log(x+1)(x+1)^2 + \frac{1}{2}\left[(\log(x+1)^2)(x+1)^2\right] - 2(x+1) + 1/4$

<u>OR:-</u>

$y = c_1 + c_2 \log(x+1)(x+1)^2 + \frac{1}{2}\left[\log(x+1)^2(x+1^2\right] - 2x - 7/4$

which is the required.