HASSAN ZAIB KHATTAK

ID 7958

Deptt BE (EC)

Semester 4^{Th}

Subject Differential Equation

SUBMITTED TO MISS SHUMAILA MAZHAR

IQRA NATIONAL UNIVERSITY.

Question # 01

The Cahuchy euler Equation.

$$① \; x^3 y''' + 2x^2 y' + 2y - 10x + 10/x$$

Solution:-

$$x^3 \frac{d^3 y}{dx^3} + 2x^2 \, dy/dx + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$$

$$(x^3 + D^3 + 2x^2 D + 2) y = 10x + 10x^{-1} \text{ — } ①$$

let $x = et \Rightarrow t = \ln x$

$$xD = D$$

$$x^2 D^2 = D(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2)$$

Subtituting into eq # ①.

$$(\Delta - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2) y = 10^x + 10x^{-1}.$$

$$(\Delta^2 - \Delta^2 + 2) y = 10x - 10x^t$$

$$(m^3 - m^2 + 2) y = 10e^t + 10/et$$

Using synthetic division.

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$$\Delta^2 - 2\Delta + 2 = 0$$

Now using Quadratic formula.

$$a = 1, \; b = -2, \; c = 2$$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a.}$$

$$D = \frac{2 \pm \sqrt{4-8}}{2}$$ ②

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$D = \frac{2 + 2i}{2}$$

$$D = \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$D = 1 \pm i$$

Since Roots are complex

$$y_c = e^{-x}(c_1 \cos t + c_2 \sin t)$$

Now particular integration.

$$y_P = \frac{1}{D^3 - D^2 + 2} 10e^t + \frac{1}{D^3 + D^2 + 2} \cdot 10/e^t$$

$$= \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{10e^{-t}}{(1)^3 - (1)^2 + 2}$$

$$= \frac{\overset{5}{\cancel{10}}e^t}{\cancel{2}} + \frac{\overset{5}{\cancel{10}}e^{-t}}{\cancel{2}}$$

$$= 5e^t + 5e^{-t}$$

$$y_P = 5e^t + 5e^{-t}$$

General solution.

$$y = y_c + y_P$$

$$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^t + 5e^{-t} \text{ (put } e^t = x \text{ and } t = \ln x)$$

$$y = e^{-x}(c_1 \ln x + c_2 \sin \ln x) + 5e^x + 5e^{-x} \text{ Ans.}$$

Scanned with CamScanner

Question No:2

③

$$2\text{-}\ x^3\frac{d^3y}{dx} + 4x^2\frac{d^2y}{dx} - 5x\frac{dy}{dx} - 15y = x^4$$

Solution:

Let $d/dx = \Delta$

$$x^3\Delta^3 y + 4x^2D^2y - 5xDy - 15y = x^4$$

$$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$$

Let $x = e^t \Rightarrow t = \ln x$

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$$

$$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Now subtituting

$$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$$

$$(\Delta^3 + 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5(\Delta) - 15)y = e^{4t}$$

$$(\Delta^3 + \Delta^2 - 7\Delta - 15)y = e^{4t}$$

Synthetic division

5	1	+1	-7	-15
		3	12	15
	1	4	5	0

$$\Delta^2 + 4\Delta + 5 = 0$$

Quadratic formula.

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= -4 \pm 2i/2$$

$\Delta = \frac{2(-2 \pm i)}{2}$

$y_c = e^{5x}(c_1 \cos t + c_2 \sin t)$

For $y_p = ?$

$y_p = \frac{1}{\Delta^3 + \Delta^2 - 7D - 15} \cdot e^{4t}$

$= \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} \cdot e^{4t}$

$= \frac{1}{64 + 16 - 28 - 15} \cdot e^{4t}$

$= \frac{1}{80 - 43} e^{4t}$

$y_p = \frac{1}{37} e^{4t}$

Hence

$y = y_c + y_p$

$y = (c_1 \cos t + c_2 \sin t) + \frac{1}{37} \cdot e^{4t}$

Again put $t = \ln x$ and $x = \ln x$

$y = e^{3x}(c_1 \cos \ln x + c_2 \sin \log x) + \frac{1}{37} e^{4x}$ $\boxed{\text{Ans}}$

Question No: 3

$x^2 y'' + 2xy' - 6y = 10x^2$

Solution:

$y(1) = 1$ and $y'(1) = -6$

$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2$

$\Rightarrow (x^2 d^2/dx^2 + 2x\, d/dx - 6)y = 10x^2$

Put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$, $x = e^t$ and $\log x = t$

$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$

The characteristic equation: (5)

$\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$\Rightarrow \Delta(\Delta+3) - 2(\Delta+3) = 0$

$\Delta+3 = 0, \quad \Delta-2 = 0$

$\Delta = 2, \quad \Delta = -3$

Since roots are real and distinct

For $y_c = ?$

$y_c = C_1 e^{-3t} + C_2 e^{2t}$

For $y_P = ?$

$$y_P = \frac{1}{\Delta^2 - \Delta - 6} \, 10^{2t}$$

$$= \frac{10}{\Delta^2 - \Delta - 6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t}$$

Now

$$10 \frac{1}{d/d\Delta (\Delta^2 + \Delta - 6)} e^{2t}$$

$$\Rightarrow 10 \, t/2\Delta+1 \; e^{2t}$$

$$= 10 \frac{1-t}{4+1} e^{2t}$$

$y_P = 2te^{2t}$

General Solution:

$y = y_c + y_P$

$= C_1 e^{-3t} + C_2 e^{2t} + 2te^{2t}$

$y = C_1 x^{-3} + C_2 x^2 + 2(\log x) x^2$ —— (B)

Put $y(1) = 1$ i-e $x = 1, y = 1$ in (B)

$1 = C_1(1)^{-3} + C_2(1)^2 + 2\log(1)$

$1 = c_1 + c_2$ —— (C) (6)

Now differentiate between eq (B) cost $\cdot x$

$y' = -3c_1 x^{-4} + 2(2x + 2/x(x^2) + 4x \log x)$

Now put $y'(1) = -6$ i-e $y' = 6$ and $x = -6$

$-6 = -3(1 + 2c_2 + 2 + 0$

$\Rightarrow -6 = -3c_1 + 2c_2 + 2$

$\Rightarrow -6 - 2 = -3c_1 + 2c_2 + 2$

$-8 = -3c_1 + 2c_2$ —— (D)

Mutiplying eq (C) with (2) and substracting from (D)

$$\begin{array}{l} 2c_1 + \cancel{2c_2} = 2 \\ \pm 3c_1 \pm \cancel{2}c_2 = \mp 8 \\ \hline 5c_1 = 10 \end{array}$$

$c_1 = \cancel{10}^{2}/\cancel{5}^{1}$ $\boxed{c_1 = \boxed{2}}$

$-8 = -3(2) + 2c_2$

$-8 = -6 + 2c_2$

$2c_2 = -8 + 6$

$2c_2 = -2$

$c_2 = -\cancel{2}/\cancel{2}^{1}$

$\boxed{c_2 = -1}$

Now put the value of c_1 and c_2. in eq (B)

$y = 2x^{-3} - x^2 + 2 \ln x \, x(x^2)$

$y = 2/x^3 - x^2 + 2x^2 \log x$ $\boxed{\text{Ans}}$

Question No 4

$x^2 y'' + 7xy' + 5y + x^5$

$y(10) = 2$ and $y'(1) = 2$

Solution:

$$x^2 \frac{dy^2}{dx^2} + 7x \frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2 \frac{d^2}{dx^2} + 7x \frac{d}{dx} + 5y\right) = x^5 \quad \text{— (A)}$$

Put $xD = \Delta \Rightarrow x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t \Rightarrow \log x = t$ in eq^n (A)

$$\Rightarrow (\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$$

$$\Rightarrow (\Delta^2 + 6\Delta + 5)y = e^{5t}$$

By Quadratic formula.

$$\Delta = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{42}}{2}$$ 8

$$= \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$D = -3 \pm 2$

Since Root are Real and distinct

$y_c = C_1 e^{-5t} + C_2 e^{-t}$

For $y_P = ?$

$$y_P = \frac{1}{D^2 + 6D + 5} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now General solution is

$y = y_c + y_P$

$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$

$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^{5} \longrightarrow$ B

Scanned with CamScanner

$x = 0$ Put in this equation (9)

Now in eq (B) $e^0 = 1$

put $y(0) = 2$ i-e $y = 2$ and $x = 2$

$$2 = C_1(2)^{-5} + C_2(2)^{-1} + \frac{1}{60}(2)^5$$

$$2 = -32C_1 - 2C_2 + \frac{1}{\cancel{60}_{15}}(\cancel{32}^{8})$$

$$2 = -32C_1 - 2C_2 + \frac{8}{15}$$

$$2 - 8/15 = -32C_1 - 2C_2$$

$$\frac{22}{15} = -32C_1 - 2C_2 \quad \text{——} \quad (C)$$

Now differentiate eq (B) w.r.t (x)

$$y^1 = -5C_1 x^{-6} - C_2 x^{-2} + 1/12\, x^4$$

Put $y^1(1) = 2$ i-e $y' = 2$ and $x = 2$ in above equation

$$2 = -5C_1(2^{-6}) - C_2(2)^{-2} + 1/12\,(2)^4$$

$$2 = -5C_1(-64) - C_2(4) + \frac{1}{12}(16)$$

$$2 = 320C_1 + 4C_2 + 4/3$$

$$2 - 4/3 = 320C_1 + 4C_2$$

$\Rightarrow 2/3 = 320c_1 + 4c_2 \longrightarrow$ D

Multiplying eq C with 2 and then subtracting eq C from D.

$\frac{-44}{15} = 64c_1 + 4c_2$

$\frac{-44}{15} = 64c_1 + 4c_1$

$+2/3 = \pm 320c_1 \pm 4c_2$

$34/15 = -256c_1$

$c_1 = 34/15 \times 256$

$\boxed{c_1 = 580}$

Put the value of c_1 in eq C

$22/15 = -32(580) - 2c_2$

$\Rightarrow 22/15 = -18560 - 2c_2$

$\Rightarrow 22/15 + 18560 = -2c_2$

$\Rightarrow \frac{18561}{-2} = c_2$

$\Rightarrow \boxed{c_2 = -9280}$

Now put the value of c_1 and c_2 in eq B

$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$ (11)

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5 \quad \text{Ans}$$

Question No: 5

$$(x+1)^2 y'' - 3(x+1)' y' + 4y = x^2$$

Solution:

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right)y = x^2$$

$$\Rightarrow \left((x+1)^2 D^2 - 3(x+1)D + 4\right)y = x^2 \quad \text{— (A)}$$

Put $(x+1)D = \Delta \Rightarrow (x+1)^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t$ in eq (A)

$$\Rightarrow [\Delta^2 - \Delta - 3\Delta + 4)y = e^{2t}$$

$$\Rightarrow [\Delta^2 - 4\Delta + 4)y = e^{2t}$$

$$\Rightarrow (\Delta^2 - 4\Delta + 4)2 = e^{2t}$$

For y_c we find the roots

$$D^2 - 4D + 4 = 0$$

$$D^2 - 2D - 2D + 4 = 0$$

$$D(D-2) - 2(D-2) = 0$$

$D - 2 = 0$, $D = 2$

$D - 2 = 0$, $D = 2$

So the Roots are real and repeat

The General solution.

$$y = (c_1 + c_2 x)^{mx}$$

$$y = (c_1 + c_3 x)^{2x}$$

For $y_P = ?$

$$y_P = \frac{1}{D^2 + 4D + 4}$$

$(2)^2 - 4(2) + 4$

$\Rightarrow 0$

$$y_P = \frac{2}{2D - 4} e^{2t}$$

if we put 2

$2D - 4 \Rightarrow 2(2) - 4 = 0$

We take again derivative (13)

$$y_p = \cancel{1}/\cancel{1}\ e^{2t}$$

$$y = (c_1 + c_2 x)^{2t} + e^{2t} \quad \boxed{Ans}$$

General solution.