Name
Muhammad Asif
ID
16950
Paper
Differential Equation
Teacher
Sir. Latif Jan

ID# 16950
Name = Muhammad Asif

Q. Define differential equation along with two example?

Definition:

An ordinary differential equation or just differential equation is another type of equation where the unknown equation is not a number, but a function.

Example:

1) Show that $y(x) = x^{-3/2}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$

we'll need to the first and second derivative to do this

$\frac{dy}{dx} = \frac{d}{dx}(x^{-3/2})$

$y'(x) = \frac{-3}{2} x^{-5/2}$ $\quad y''(x) = \frac{15}{2} x^{-7/2}$

Plug these as well as function into the differential equation

$4x^2\left(\frac{15}{4} x^{-7/2}\right) + 12x\left(\frac{-3}{2} x^{-5/2}\right) + 3(x$

$15x^{-3/2} - 18x^{-3/2} + 3x^{-3/2} = 0$

$0 = 0$

So $y(x) = x^{-3/2}$ does satisfy the differential equation.

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2) $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = \frac{-3}{64}$?

Solution:-

As well saw in previous example the function is a Solution and we can then note that

$$y(4) = 4^{-\frac{3}{2}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{8}$$

$$y'(4) = \frac{-3}{2} 4^{-\frac{5}{2}} = \frac{-3}{2}\frac{1}{(\sqrt{4})^5} = -\frac{3}{64}$$

and so this solution also meets the intial condition of $y(4) = \frac{1}{8}$ and $y'(4) = \frac{-3}{64}$

in fact, $y(x) = x^{-\frac{3}{2}}$ is the only Solution to this (function) differential equation.

b) Define Separable Differential Equation ?

$-3/2) = 0$

Definition:

A Separable differential Equation is any D.E that we can write in the following

$$N(y)\frac{dy}{dx} = M(x)$$

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Solve the following using separable D.E and the find the interval of validity of the solution:

a) $y' = \frac{xy^3}{\sqrt{1+x^2}}$ $y(0) = -1$

Sol

$$y' = \frac{xy^3}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

$$\frac{dy}{y^3} = \frac{x}{\sqrt{1+x^2}}\, dx$$

$$y^{-3}\, dy = x(1+x^2)^{-1/2}\, dx$$

integrating on b/s

$$\int y^{-3}\, dy = \int x(1+x^2)^{-1/2}\, dx$$

$$\int y^{-3}\, dy = \frac{1}{2}\int 2x(1+x^2)^{-1/2}\, dx$$

$$\frac{y^{-3+1}}{-3+1} = \frac{1}{2}\, \frac{(1+x^2)^{-1/2+1}}{-1/2+1} + C$$

$$\frac{y^{-2}}{-2} = \frac{1}{2} \times 2\, (1+x^2)^{1/2} + C$$

$$\frac{-1}{2y^2} = (1+x^2)^{1/2} + C$$

$\because$ $y(0) = -1$

$$-\frac{1}{y^2} = 2(1+x^2)^{1/2} + C \quad \text{---} \; (*)$$

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$\Rightarrow -y^2 = \frac{1}{2(1+x^2)^{1/2}} + C$

$\because x = 0, \quad y = -1$

$-(-1)^2 = \frac{1}{2(1+0)^{1/2}} + C$

$-1 = \frac{1}{2} + C$

$-1 - \frac{1}{2} = C \Rightarrow -3/2$

$\boxed{C = -3/2}$

putting the value of C in (*)

eq (*) $\Rightarrow \frac{-1}{2y^2} = (1+x^2)^{1/2} - \frac{3}{2}$

(b) $y' = e^{-y}(2x-4) \qquad y(5) = 0.$

Solution:-

$\frac{dy}{dx} = e^{-y}(2x-4)$

$\frac{dy}{e^{-y}} = 2x-4)\,dx$

$e^{y}dy = (2x)dx - 4dx$

integrating on b/s

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$\int e^y dy = 2\int x dx - 4\int dx$

$e^y = \frac{2x^2}{2} - 4x + c$

$e^y = x^2 - 4x + c$ —— (4)

$\because y(5) = 0$

$e^0 = (5)^2 - 4(5) + c$

$1 = 25 - 20 + c$

$1 = 5 + c$

$1 - 5 = c$

$-4 = c$

$\boxed{c = -4}$

putting the c value in (4)

eq (4) $\Rightarrow$

$e^y = x^2 - 4x + c$

$e^y = x^2 - 4x - 4$

Q2 Solve the following using linear Differential method.

1) Explain the steps of solving Differential Equation.

1) Identify the variables
2) Then separate the given equation.
3) Then arrange them & integrating according to given variable.

4) After integrating add "integrating constant" at the end of solution.

(ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \quad y[\pi/4] = 3\sqrt{2},$

$0 \leq x \leq \pi/2$.

Solution:

$$\cos(x)\frac{dy}{dx} + \sin(x)\,y = 2\cos^3(x)\sin x - 1$$

$$\{\cos(x)\,dy + \sin(x)\,y\}dx = (2\cos^3 x\sin x - 1)\,dx$$

$$\int \cos x\,dy + \int \sin x\,y\,dx = 2\int \cos^3 x\sin x - \int dx$$

$$\cos\cdot y + \cos y = -2\int \cos^3 x(-\sin x)\,dx - x + c$$

$$0 = -2\frac{\cos^4 x}{4_2} - x + c$$

$$\Rightarrow -\frac{\cos^4 x}{2} - x + c = 0$$

$$\Rightarrow -\cos^4 x - 2x + c' = 0$$

$$-\cos^4(\pi/4) - 2(\pi/4) + c' = 0$$

$$-\left(\frac{1}{\sqrt{2}}\right)^4 - \pi/2 + c' = 0$$

$$-\frac{1}{4} - \pi/2 + c' = 0$$

$$c' = \pi/2 + \frac{1}{4}$$

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$C' = \frac{2\pi+1}{4}$

eq(7) $\Rightarrow -\cos^4 x - 2x + \frac{2\pi+1}{4} = 0$

(iii) $x' + 2x = \sin t$

$x' + 2x = \sin t$

Solution:

$$\frac{dx}{dt} + 2x = \sin t$$

$$dx + 2x\,dt = \sin t\,dt$$

integration on b/s

$$\int dx + 2\int x\,dt = \int \sin t\,dt$$

$$x + 2xt = -\cos t + c$$

$$x + 2xt + \cos t - c = 0$$

Ans

Q3 Solve the following by exact equation.

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 . \; y(0) = -3$

Solution:

$M = 2xy - 9x^2 \quad M_y = 2x$

$N = 2y + x^2 + 1 \quad N_x = 2x$

Now, how do we actually find $\Psi(x, y)$?

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well react that

$$\Psi x = M$$

$$\Psi y = N$$

Integrating $\Psi(x, y)$.

$$\Psi = \int M dx \quad \text{OR} \quad \Psi = \int N dy$$

so

$$\Psi(x,y) = \int 2xy - 9x^2 dx = x^2y - 3x^3 + h(y)$$

$\Psi x = M$ to find most of $\Psi(x,y)$
$\Psi y = N$ to find $h(y)$ Differentiate our $\Psi(x,y)$ with respect to y and set equal to N

$$\Psi = x^2 + h'(y) = 2y + x^2 + 1 = N$$

from this we can see that

$$h'(y) = 2y + 1$$

we can now find $h(y)$ by integrating

$$h(y) = \int 2y + 1 \, dy = y^2 + y + K$$

constant K.

so we can now write down $\Psi(x,y)$

$$\Psi(x,y) = x^2y - 3x^3 + y^2 + y + K = y^2 + (x^2+1)y - 3x^3 + K$$

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we can now go straight to implicit solution using

$$y^2+(x^2+1)y-3x^3+K=c$$

Combining

$$y^2+(x^2+1)y-3x^3=c-K$$

$$y^2+(x^2+1)\ y-3x^3=c$$

Let find c to apply initial condition

$$(-3)^2+(0+1)(-3)-3(0)^3=c \Rightarrow c=6$$

The implicit solution is then

$$y^2+(x^2+1)\ y-3x^3-6=0$$

for solving y(x) using quadratic formula:

$$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2-4(1)(-3x^3-6)}}{2(1)}$$

$$= \frac{-(x^2+1) \pm \sqrt{x^4+12x^3+2x^2+35}}{2}$$

reapply intial condition

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = -3, 2$$

The explicit solution is then

$$y(x) = -(x^2+1) - \sqrt{x^4+12x^3+2x^2+25}$$

Now, for the interval of validity it looks like we might well have problems with square root of negative, so, we need to solve:

$$x^4+12x^3+2x^2+25=0$$

upon solving this equation is zero at $x = -11.81557624$ and $x = -1.3969$.

(ii) $\frac{2ty}{t^2+1} - 2t - (2-\ln(t^2+1))y' = 0 \quad y(5)=0$

Separating two terms

$$\frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0$$

Now, find M and N and check that its exact

$M = \frac{2ty}{t^2+1} - 2t \qquad M_y = \frac{2t}{t^2+1}$

$N = \ln(t^2+1) - 2 \qquad N_t = \frac{2t}{t^2+1}$

So, it's exact. we'll integrate the first one in this case

$$\Psi(t,y) = \int \frac{2ty}{t^2+1} - 2t\,dt = y\ln(t^2+1) - t^2 - h(y)$$

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$\Psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N$

So, it looks like we've got

$h'(y) = 2 \Rightarrow h(y) = -2y$

This gives us

$\Psi(t,y) = y(\ln(t^2+1) - t^2 - 2y$

implicit solution

$y(\ln(t^2+1) - t^2 - 2y = c$

initial condition gives

$-25 = c$

Now implicit solution

$y(\ln(t^2+1) - 2) - t^2 = -25$

The term in the logarithm is always positive so we don't need to worry about negative number.

$\ln(t^2+1) - 2 = 0$

$\ln(t^2+1) = 2$

$t^2+1 = e^2$

$t = \pm\sqrt{e^2-1}$

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we now have three possible intervals of validity

$$-\infty < t < -\sqrt{e^2-1}$$

$$-\sqrt{e^2-1} < t < \sqrt{e^2-1}$$

$$\sqrt{e^2-1} < t < \infty$$

The last one contain $t=5$

So interval of validity for this problem is

$$\sqrt{e^2-1} < t < \infty.$$