ID

# 7964

Sec

# B

Paper

# Differential Equation

Instructor

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Question NO 07

Part (ii)

(i) $w = \sin(x+ct) + \cos(2x+2ct)$

$$w = \sin(x+ct) + \cos(2x+2ct)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct)+c - \sin(2x+2ct)+2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct)+c^2 - \cos(2x+2ct)+4c^2$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct)+2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$= \{-\sin(x+ct) - 4\cos(x+2ct)\}$$

$$\frac{\partial^2 w}{\partial t^2} = +c^2\{-\sin(x+ct) - 4\cos(2x+2ct)\}$$

$$c^2 = \frac{\partial^2 w}{\partial x^2}$$

Hence

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

Question No $\cancel{Q}$ 09
Part (ii)

Given :

$$w = \tan(2x + ct)$$

Sol :-

we know that

$$w = \tan(2x+ct)$$

taking derivative w-r-t 't'

$$\frac{\delta w}{\delta t} = \frac{\delta}{\delta t} \tan(2x+ct) \cdot c$$

$$\frac{\delta w}{\delta t} = \sec^2(2x+ct) \cdot c$$

$$\frac{\delta w}{\delta t} = c \sec^2(2x+ct)$$

Again taking derivative

$$\frac{\delta^2 w}{\delta t^2} = c \frac{\delta}{\delta t} \sec^2(2x+ct)$$

$$\frac{\delta^2 w}{\delta t^2} = c \cdot 2\sec(2x+ct) \cdot \sec(2x+ct) \cdot \tan(2x+ct) \cdot c$$

$$\frac{\delta^2 w}{\delta t^2} = 2c^2 \sec^2(2x+ct) \cdot \tan(2x+ct)$$

Now $w = \tan(2x+ct)$

taking derivative w.r.t '$x$'

$$\frac{\delta w}{\delta x} = \frac{\delta}{\delta x} \tan 2x+ct$$

$$\frac{\delta w}{\delta x} = \sec^2(2x+ct) \cdot 2$$

$$\frac{\delta w}{\delta x} = 2\sec^2(2x+ct)$$

Again taking derivative

$$\frac{\delta^2 w}{\delta x^2} = 2\frac{\delta}{\delta x} \sec^2(2x+ct)$$

$$\frac{\delta^2 w}{\delta x^2} = 2 \cdot 2\sec(2x+ct)\sec(2x+ct)\tan(2x+ct) \cdot 2$$

$$\frac{\delta^2 w}{\delta x^2} = 6\sec^2(2x+ct)\tan(2x+ct)$$

we know the wave equation

$$\frac{\delta^2 w}{\delta t^2} = c^2 \frac{\delta^2 w}{\delta x^2}$$

Put values in wave equation

$$\cancel{2}^{1} \cancel{c^2} \cancel{\sec^2(2x+ct)} \cancel{\tan(2x+ct)} = \cancel{c^2}\,\cancel{6}^{3}\,\cancel{\sec^2(2x+ct)}\,\cancel{\tan(2x+ct)}$$

$$1 \neq 3$$

So

L.H.S $\neq$ R.H.S

Question No 2

Given :-

$$f(x)\begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

we have to find Fourier co-efficients $a_0$, $a_n$ & $b_n$

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left\{\frac{x^2}{2}\right\}_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2}} \quad \text{---(i)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x\cos nx \, dx) + \frac{1}{\pi}\int_{0}^{\pi} (2x\cos nx)\, dx$$

$$= \frac{1}{\pi}\left[ x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left( x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right)_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos nx}{n^2} + \frac{2}{\pi}\left[\frac{\cos nx}{n^2} - \frac{\cos(0)}{n^2}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{1 - (-1)^n + 2(-1)^n - 2}{n^2}\right]$$

$$= \frac{(-1)^n - 1}{\pi n^2}$$

So $\begin{cases} \frac{-2}{\pi n^2} ; \text{ if } n \text{ is odd} \\ 0 ; \text{ if } n \text{ is even} \end{cases}$

$$b_x = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$+\frac{2}{\pi}\int_{0}^{x} x\sin nx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} +$$

$$\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right]+\frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right]$$

$$= -3\frac{\cos n\pi}{n} = 3\frac{(-1)^{n+1}}{n}$$

So the required Fourier is

$$f(x) = \frac{A_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2x-1)}{(2x-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Question No 3

Given :-

$$y'' - 4y' + 13y = 8\sin 3x$$
$$y(0) = 1 \, , \, y'(0) = 2$$

Sol:-

$$y'' - 4y' + 13y = 8\sin 3x \longrightarrow (1)$$

Associated homogenous eq (1) is

$$y'' - 4y' + 13y = 0 \longrightarrow (2)$$

put $y = m$ in (2)

$$m^2 - 4m + 13 = 0$$

by quadratic formula

$$a = 1 \, , \, b = -4 \, , \, c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2}$$

$$m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$m = \frac{4 \pm \sqrt{-36}}{2}$$

$$m = \frac{4 \pm \sqrt{36i}}{2} = \frac{4 + 6i}{2}$$

$$m = 2 \pm 3i$$

$$m_1 = 2 + 3i, \; m_2 = 2 - 3i$$

$$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \rightarrow A$$

Let $y = A \cos 3x + B \sin 3x \rightarrow A_{ii}$

Diff w.r.t '$x$'

$$y_p = -3A \sin 3x + 3B \cos 3x$$

put in (1)

$$\Rightarrow -9A \cos 3x - 9B \sin 3x)$$

Again Diff w-r-t 'x'

$y_p'' = -9A\cos 3x - 9B\sin 3x$

put in (i)

$(-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x + 3B\cos 3x)$

$+ 13(A\cos 3x + B\sin 3x) - 8\sin 3x$

$\Rightarrow -9A\cos 3x - 12B\cos 3x + A13\cos 3x$

$-9Bx + 12A\sin 3x + 13B\sin 3x - 8\sin 3x$

$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)$
$\sin 3x = 8\sin 3x$

Comparing Co-efficient

$\sin 3x \Rightarrow \quad 4B + 12A = 8 \rightarrow (a)$

$4A - 12B = 0 \Rightarrow 4A - 12B$

$\boxed{A = 3B} \rightarrow (b)$

Put (b) in a

$4B + 12(3B) = 8$

$40B = 8$

$\boxed{B = 1/5}$ -(i)

put (i) in (ii)

$\boxed{A = 3/5} \rightarrow d$

Put These all in eq $y_P$

$$y_P = 3/5 \cos 3x + 1/5 \sin 3x \rightarrow (B)$$

$$y = y_c + y_P$$

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + 3/5 \cos 3x + 1/5 \sin 3x \rightarrow (C)$$

Now we need to find the value of $c_1$ & $c_2$

Put $x=0$ & $y=1$ in (C)

$$1 = e^{x(0)}(c_1 \cos 3(0) + c_2 \sin 3(0) + 3/5 \cos 3(0) + 1/5 \sin 3(0))$$

$$1 = c_1(1) + c_2(0) + 3/5(1) + 1/5(0)$$

$$1 = c_1 + 3/5$$

$$\boxed{c_1 = 2/5} \rightarrow ★★$$

Diff w-r-t $x$

$$y' = c_1(2e^{2x}\cos3x - 3e^{2x}\sin3x) + c_2(2e^{2x}\sin3x + 3e^{2x}\cos3x) - 6/5 \sin3x + 3/5\cos3x \rightarrow (D)$$

Put

$y' = 2$, $x = 0$ in (D)

$$y' = c_1(2e^{2x}\cos3x - 3e^{2x}\sin3x) + c_2(2e^{2x}\sin3x + 3e^{2x}\cos3x) - 6/5 \sin3x + 3/5\cos3x$$

Put $y' = 2$ ; $x = 0$

$$2 = c_1(2e^{2(0)}\sin3(0) + 3e^{2(0)}\sin(0) + c_2(2e^{2(0)}\sin3(0) + 3e^{2(0)}\cos3(0))$$

$$2 = c_1(2) + c_2(3) - 0 + 3/5$$

$$2 = 2c_1 + 3c_2 + 3/5$$

Put $c_1 = 2/5$

$$2 = 4/5 + 3c_2 + 3/5$$

$2 = 7/5 + 3C_2$

$3C_2 = 3/5$

$\boxed{C_2 = 3/15}$ → ★★★

Put ★★ & ★★★ in (c)

$y = e^{3x}(2/5 \cos 3x + 3/15 \sin 3x) + 3/5 \cos 3x$
$+ 1/5 \sin 3x$

$$\boxed{y = 2/5 e^{2x} \cos 3x + 3/15 e^{2x} \sin 3x + 3/5 \cos 3x + 1/5 \sin 3x}$$

So this is the required equation.

Question No 04

Given :-

$$(D^2 - DD')z , \cos x \cos 2y$$

Sol:-

$$(D^2 - DD')z = \cos x \cos 2y$$

$$CF = \phi_1(y) + \phi_2(y+x)$$

while its PI is given by

$$PI = \frac{1}{(D^2 - DD')} + \frac{1}{2}\{\cos(x+2y) + \cos(x-2y)\}$$

$$= \frac{1}{2}\left\{\frac{1}{(-1+2)} \cos(x+2y) + \frac{1}{(-1-2)} \cos(x+2y)\right\}$$

$$= \frac{1}{2}\cos(x-2y) - \frac{1}{6}\cos(x-2y)$$

Solution of the give PDE's

$$z = \phi_1(y) + \phi_2(y+x) + \tfrac{1}{2}\cos(x+2y) - \tfrac{1}{6}\cos(x-2y)$$

Ans