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Subject = Differential Equation

Program = BS (S.E)

Semester = 3rd

Q1 = Use any of the methods for solving the ordinary differential Equations as given below.

12. :- $x^2y'' - 4xy' + 6y = 0,$

$y(1) = 0.4, \quad y'(1) = 0$

Sol:-

Put $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

Put y'' in given D.E

$x^2m(m-1)x^{m-2} - 4xmx^{m-1} + 6x^m = 0$

$x^2m(m-1)x^m \cdot x^{-2} - 4xmx^m \cdot x^{-1} + 6x^m = 0$

$m(m-1) - 4m + 6 = 0$

$m^2 - 5m + 6 = 0$

Now finding roots

$m^2 - 5m + 6 = 0$

$m_{1/2} = \dfrac{5 \pm \sqrt{(-5)^2 + 4}}{2}$

$m_{1/2} = \dfrac{5 \pm 1}{2}$

$m = 3 \wedge m_2 = 2$

This provide two real Solution

$y_1 = x^m = x^3 \wedge x^2 = x^{m_2} = x^2$

Solution is :

$y = C_1 y_1 + C_2 y_2$

$= C_1 x^3 + C_2 x^2$

$y' = 3C_1 x^2 + 2C_2 x$

Now to determine C_1 and C_2

$$\begin{cases} 0.4 = 4(1) = C_1 . I^3 + C_2 . I^2 \\ 0 = y'(1) = 3C_1 . I^2 + 2C_3 . 7^2 \end{cases}$$

$$\begin{cases} 0.4 = C_1 + C_2 \\ 0 = 3C_1 + 2C_2 \end{cases}$$

$$\begin{cases} 0.4 - C_2 = C_1 \\ 0 = 3(0.4 - C_2) + 2C_2 \end{cases}$$

$$\begin{cases} 0.4 - C_2 = C_1 \\ 1.2 = C_2 \end{cases}$$

$$\begin{Bmatrix} 0.4 - 1.2 = C_1 \\ 1.2 = C_2 \end{Bmatrix}$$

$$\begin{Bmatrix} -0.8 = C_1 \\ 1.2 = C_2 \end{Bmatrix}$$

The particular Solution is

$$y = (-0.8x^3) + 1.2x^2$$

Ans

13:— $x^2y'' + 3xy' + 0.75y = 0, \quad y(1) = 1,$
$y'(1) = -1.5$

Sol:—

Put $y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

Putting in the given D.E

$x^2 m(m-1) x^{m-2} + 3x m x^{m-1} + 0.75 x^m = 0$

$\cancel{x^2} m(m-1) x^m . \cancel{x^{-2}} + 3\cancel{x} m x^m . \cancel{x^{-1}} + 0.75 x^m = 0$

Dropping Common factor x^m

$m(m-1) + 3m + 0.75 = 0$

$m^2 + 2m + 0.75 = 0$

finding roots

$m^2 + 2m + 0.75 = 0$

$m_{1/2} = \frac{-2 \pm \sqrt{2^2 - (4)(0.75)}}{2}$

$m_{1/2} = \frac{-2 \pm 1}{2}$

$m_1 = -1/2 \wedge m_2 = -3/2$

$y_1 = x^{m_1} = x^{-1/2} = x^{-0.75} \wedge y_2 = \cancel{x}^{m_2} = x^{-3/2}$

The general solution is

$y = c_1 y_1 + c_2 y_2$

$= c_1 x^{0.5} + c_2 x^{1.5}$

$y' = -0.5 c_1 x^{-1.5} - 1.5 c_2^{-2.5}$

To determine c_1 and c_2

$\begin{cases} 1 = y(1) = c_1 \cdot 1^{-0.5} + c_2 \cdot 1^{-1.5} \\ 1.5 = y'(1) = -0.5 c_1 \cdot 1^{-1.5} - 1.5 c_2 \cdot 1^{-2.5} \end{cases}$

$\begin{Bmatrix} 1 = c_1 + c_2 \\ 3 = c_1 + 3c_2 \end{Bmatrix}$

$$\begin{Bmatrix} 1 - c_2 = c_1 \\ 1 = c_2 \end{Bmatrix}$$

$\Rightarrow$

$$\begin{Bmatrix} 0 = c_1 \\ 1 = c_2 \end{Bmatrix}$$

Particular Solution is

$y = x^{1.5}$

Ans

14 :—

$x^2y'' + xy' + 9y = 0, \quad y(1) = 0,$
$y'(1) = 2.5$

Sol:— Put $y = x^m$ and $y'' = m(m-1)x^{m-2}$

$x^2 m(m-1)x^{m-2} + mx^m + 9x^m = 0$

$x^2 m(m-1)x^m \cdot x^{-2} + mx^m + 9x^m = 0$

Dropping common factor x^m

$m(m-1) + m + 9 = 0$

$m^2 - m + m + 9 = 0$

$m^2 + 9 = 0$

finding roots

$$m^2+9=0 \Rightarrow m^2-(3i)^2=0 \Rightarrow$$

$$\Rightarrow (m-3i)(m+3i)=0$$

$$m_1=3i \wedge m_2=3i$$

$$x^{m_2}=x^{3i}=(e^{\ln x})^{3i}=e^{3i}$$

$$x^{m_2}=x^{-3i}=e^{\ln x}=e^{-3i}$$

$$e^{z}=e^{a+ib}=e^{a}(\cos b+i\sin b)$$

$$x^{m_2}=\cos(3\ln x)+i\sin(3\ln x)$$

$$x^{m_2}=\cos(3\ln x)-i\sin(3\ln x)$$

$$x^{m_1}+x^{m_2}=\cos(3\ln x)+i\sin(3\ln x)+\cos(3\ln x)$$

$$-i\sin(3\ln x)=2\cos(3\ln x)$$

$$\frac{x^{m_1}+x^{m_2}}{2}=\cos(3\ln x)$$

(sub and divide)

$$\frac{x^{m_1}-x^{m_2}}{2i}=\sin(3\ln x$$

$$y_1=\cos(3\ln x \wedge y_2=\sin(3\ln x)$$

$$y=c_1y_1+c_2y_2$$

$$c_1\cos(3\ln x)+c_2\sin(3\ln x)$$

$$\begin{Bmatrix} 0=c_1 \\ 5/2=3c_1 \ :3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0=c_1 \\ 5/6=c_2 \end{Bmatrix} \Rightarrow \text{Solution is :-}$$

$$\boxed{y=\frac{5}{6}\sin(3\ln x)}$$

Q2-15

$x^2y'' + 3xy' + y = 0$ $\qquad y(1) = 3.6$

$y'(1) = 0.4$

Put $y = x^m$, $y'' = m(m-1)x^{m-2}$

$x^m(m-1)x^{m-2} + 3mx^m + x^m = 0$

$x^2 m(m-1)x^m \cdot x^{-2} + 3mx^m + x^m = 0$

Dropping Common factor

$m(m-1) + 3m + 1 = 0$

$m^2 - m + 3m + 1 = 0$

$m^2 + 2m + 1 = 0$

$m^2 + 2m + 1 = 0$, $(m+1)^2 = 0$

$m = 1$

$y_1 = x^m = x^{-1} = 1/x$

$y'' + 3/x \cdot y' + \frac{1}{x^2} \cdot y = 0$

$P(x) = 3 \cdot \frac{1}{x}$ $\Rightarrow \int P dx = 3\ln|x|$

$y_2 = Uy_1$

$$v = \int u\,dx \quad , \quad u = \frac{1}{y_1^2} e^{-\int p\,dx}$$

To find u

$$e^{-\int p\,dx} = e^{-3\ln|x|} = \left(e^{\ln|x|}\right)^{-3} = x^{-3}$$

$$u = x^{-3} \cdot \frac{1}{x^{2}} = x^{-3+2} = x^{-1} = \frac{1}{x}$$

$$v = \int \frac{dx}{x} = \ln|x|$$

$$y_2 = v y_1 = y_1 \ln x = \frac{1}{x} \ln x$$

$$y = c_1 y_1 + c_2 y_2$$

$$c_1 \frac{1}{x} + c_2 \frac{1}{x} \ln(x)$$

$$\frac{1}{x} \cdot c_1 + c_2 \ln x$$

$$y' = (x^{-1})' (c_1 + c_2 \ln(x)) + x^{-1} (c_1 + c_2 \ln x)'$$
$$= -x^{-2} (c_1 + c_2 \ln x) + \frac{1}{x} c_2 \cdot \frac{1}{x}$$
$$= \frac{1}{x^2} (-c_1 - c_2 \ln x + c_2)$$

Now finding c_1 and c_2

$$\begin{cases} 3.6 = y(1) = \frac{1}{1}(c_1 + c_2 \ln 1) \\ 0.4 = y'(1) = \frac{1}{2}(-c_1 - c_2 \ln 1 + c_2) \end{cases}$$

$$\begin{cases} 3.6 = c_1 \\ 0.4 = -c_1 + c_2 \end{cases}$$

$$\begin{cases} 3.6 = c_1 \\ 0.4 = -3.6 + c_2 \end{cases}$$

$$\begin{cases} 3.6 = c_1 \\ 4.0 = c_2 \end{cases}$$

$$y = (3.6 + 4.0 \ln x)\, \frac{1}{x}$$

Ans

Q1-11

$(x^2D^2 - 3xD + 4I) = 0 \qquad y(1) = \pi, \; y'(1) = 2\pi$

$x^2D^2y - 3xDy + 4Iy = x^2D(Dy) - 3xDy + 4y$

$= x^2y'' - 3xy + 4y$

$x^2y'' - 3xy' + 4y = 0$

Put $\quad y = x^m$ and $y'' = m(m-1)x^{m-2}, \; y' = mx^{m-1}$

$x^2m(m-1)x^{m-2} - 3xmx^{m-1} + 4x^m = 0$

$\cancel{x^2}m(m-1)x^m \cdot \cancel{x^{-2}} - 3\cancel{x}mx^m\cancel{x^{-1}} + 4x^m = 0$

Dropping x^m

$m(m-1) - 3m + 4 = 0, \quad m^2 - 4m + 4 = 0$

hence $y = x^n$ is a Solution

$m^2 - 4m + 4 = 0, \quad (m-2)^2 = 0$

$y_1 = x^m = x^2$

$y'' = \frac{-3}{x} \cdot y' + \frac{4}{x^2} \cdot y = 0$

$P(x) = -3 \cdot \frac{1}{x} \Rightarrow \int P dx = -3\ln(x)$

$y_2 = uy_1$

$u = \int v\,dx$ & $v = \dfrac{1}{y_1^2} e^{-\int p\,dx}$

To find v

$e^{-\int p\,dx} = e^{3\ln(x)} = \left(e^{\ln(x)}\right)^{3} = x^3$

$v = x^3 \cdot \dfrac{1}{(x^2)^2} = x^{3-4} = x^{-1} = 1/x$

$u = \int \dfrac{dx}{x} = \ln|x|$

$y_2 = u y_1 = y_1 \ln(x) = x^2 \ln x$

$y_1 = y_2 \in R$

General Solution is

$y = c_1 y_1 + c_2 y_2$

$c_1 x^2 + x^2 \ln x$

$x^2 (c_1 + c_2 \ln x)$

$y' = (x^2)'(c_1 + c_2 \ln x) + x^2 (c_1 + c_2 \ln x)'$

$= 2x(c_1 + c_2 \ln x) + c_2 x^2 = 1/x$

$$= 2c_1x + 2c_2x\ln x + c_2x$$

$$= 2c_1x + c_2x(2\ln x + 1)$$

$$\begin{cases} -\pi = y(1) = 1^2(c_1 + c_2\ln 1) \\ 2\pi = y'(1) = 2\cdot c_1 + c_2(2\ln 1 + 1) \end{cases}$$

$$\begin{cases} -\pi = c_1 \\ 2\pi = 2c_1 + c_2 \end{cases}$$

$$\begin{cases} -\pi = c_1 \\ 4\pi = c_2 \end{cases}$$

Particular Solution is

$$y = x^2(-\pi + 4\pi\ln x)$$

Ans

17) $(x^2D^2 + xD + I)y = 0$, $y(1) = 1$, $y'(1) = 1$

First Applying given operations to the function

$$x^2D^2y + xDy + Iy = x^2D(Dy) + xDy + y$$

$$= x^2y'' + xy' + y$$

Now

$$x^2y'' + xy' + y = 0$$

$$y = x^m, \quad y' = mx^{m}, \quad y'' = m(m-1)$$

$$x^2m(m-1)x^{m-2} + xmx^{m-1} + x^m = 0$$

$$\cancel{x^2}m(m-1)\cancel{x^m}\cdot\cancel{x^2}\ \cancel{x}mx^m x^{-1} + x^m = 0$$

Dropping the common factor x^m

$$m(m-1) + m + 1 = 0 \Rightarrow m^2 - \cancel{m} + \cancel{m} + 1 = 0$$

$$= m^2 + 1 = 0$$

Now finding the roots.

$$m^2 + 1 = 0, \quad m^2 + i^2 = 0, \quad (m-i)(m+i) = 0$$

$$m_1 = i \wedge m_2 = -i$$

$$x = e^{\ln x}$$

$$x^{m_1} = x^i = (e^{\ln x})^i = e^{i\ln x}$$

$$u^{m_1} = u^{i} = (e^{\ln u})^{i} = e^{i\ln u}$$

$$e^{z} = e^{a+ib} = e^{a}(\cos b + i\sin b),\ z \in \mathbb{C}$$

$$e^{i\ln u} = (e^{0}(\cos(\ln u) + i\sin(\ln u)))$$

$$= \cos(\ln u) + i\sin(\ln u)$$

$$e^{-i\ln u} = e^{0}(\cos(\ln u) - i\sin(\ln u))$$

$$u^{m_1} = \cos(\ln u) + i\sin(\ln u)$$

$$u^{m_2} = \cos(\ln u) - i\sin(\ln u)$$

$$u^{m_1} + u^{m_2} = \cos(\ln u) + i\sin(\ln u) + \cos(\ln u) - i\sin(\ln u)$$

$$= 2\cos(\ln u)$$

$$\frac{u^{m_1} + u^{m_2}}{2} = \cos(\ln(u))$$

$$u^{m_1} - u^{m_2} = \cos(\ln u) + i\sin(\ln u) - \cos(\ln u) + i\sin(\ln u)$$

$$\frac{u^{m_1} - u^{m_2}}{2i} = \sin(\ln u)$$

$$y_1 = \cos(\ln u) \wedge y_2 = \sin(\ln u)$$

$$y' = -c_1 \sin(\ln x) \cdot (\ln x)' + c_2 \cos(\ln x) \cdot (\ln x)'$$

$$= \frac{-c_1}{x} \sin(\ln x) + \frac{c_2}{x} \cos(\ln x)$$

To determine c_1 and c_2

$$\begin{cases} 1 = y(1) = c_1 \cos(\ln 1) + c_2 \sin(\ln 1) + c_2 \sin(\ln 1) \\ 1 = y'(1) = c_1 \sin(\ln 1) + 3c_2 \cos(\ln 1) \end{cases}$$

$$\begin{cases} 1 = c_1 \cos(0) + c_2 \sin(0) \\ 1 = -c_1 \sin(0) + c_2 \cos(0) \end{cases}$$

$$\begin{cases} 1 = c_1 \\ 1 = c_2 \end{cases}$$

$$y = \sin(\ln x) + \cos(\ln x)$$

Ans

(18) $(9x^2D^2 + 3xD + I)y = 0$ $\quad y(1) = 1$

$y'(1) = 0$

Apply given operation to the equation

$$9x^2D^2y + 3xDy + Iy = 9x^2D(Dy) + 3xDy$$

$$= 9x^2y'' + 3xy' + y$$

$$9x^2y'' + 3xy' + y = 0$$

let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$$9x^2m(m-1)x^{m-2} + 3xmx^{m-1} + x^m = 0$$

Drop x^m

$$9x^2m(m-1)x^m \cdot x^{-2} + 3xmx^m x^{-1} + x^m = 0$$

$9m(m-1) + 3m + 1 = 0$, $\quad 9m^2 - 9m + 3m + 1 = 0$

$= 9m^2 - 6m + 1 = 0$

finding the root of equation

$m^2 - 4m + 4 = 0$, $\quad (m-2)^2 = 0$

$9m^2 - 6m + 1 = 0$ $\quad m_{1/2} = \dfrac{6 \pm \sqrt{6^2 - 4 \cdot 9}}{18}$

$m_{1/2} = 6/18$

$m_{1/2} = 1/3$

$$y'' + \frac{1}{3x} \cdot y' + \frac{1}{9x^2} \cdot y = 0$$

$$P(x) = \frac{1}{3} \cdot \frac{1}{x} \Rightarrow \int P\,dx = \tfrac{1}{3} \ln(x)$$

$$y_2 = v y_1$$

$$v = \int U\,dx \quad \wedge \quad U = \frac{1}{y_1^2} e^{\int P\,dx}$$

finding U

$$e^{-\int P\,dx} = e^{-1/3 \ln|x|} = \left(e^{\ln|x|}\right)^{-1/3} = x^{-1/3}$$

$$U = x^{-1/3} \cdot \frac{1}{(x^{1/3})^2} = x^{-1/3 - 2/3} = x^{-1} = 1/x$$

$$v = \int \frac{dx}{x} = \ln|x|$$

$$y_2 = v y_1 = y_1 \ln|x| = x^{1/3} \ln x$$

here

$$y = C_1 y_1 + C_2 y_2$$

$$c_1 x^{1/3} + x^{1/3}\ln x$$

$$x^{1/3}(c_1 + c_2 \ln x)$$

$$y' = (x^{1/3})(c_1 + c_2\ln x) + x^{1/3}(c_1 + c_2 \ln x)$$

$$= \tfrac{1}{3} \cdot x^{-2/3}(c_1 + c_2 \ln x) + x^{1/3}\, c_2 \cancel{x} \cdot \tfrac{1}{\cancel{x}}$$

$$= \frac{1}{3} \cdot x^{-2/3}(c_1 + c_2 \ln x) + x^{1/3} c_2$$

$$\left\{ \begin{array}{l} 1 = y(1) = 1^{1/3}(c_1 + c_2 \ln 1) \\ 0 = y'(1) = \tfrac{1}{3} \cdot 1^{-2/3}(c_1 + c_2 \ln 1) + 1^{1/3} c_2 \end{array} \right\}$$

$$\left\{ \begin{array}{l} 1 = c_1 \\ 0 = \dfrac{c_1}{3} + c_2 \end{array} \right\}$$

$$\left\{ \begin{array}{l} 1 = c_1 \\ -\tfrac{1}{3} = c_2 \end{array} \right\}$$

$$y = x^{1/3}\left(1 - \tfrac{1}{3}\ln x\right)$$

Ans

Q2 :— 4(e) :—

$$(2u+1)u - (t+1) = 0$$

Sol :—

$$(2u+1)\frac{du}{dt} = (t+1)$$

$$(2u+1)du = (t+1)dt$$

$$\int(2u+1)du = \int(t+1)dt$$

$$u^2+u = \frac{t^2}{2}+t+c$$

$$\frac{2(u^2+u)}{t^2} - t = c$$

$$c = \frac{2(u^2+u)}{t^2} - t$$

(4d) :— $R' = (t+1)(R^2+1)$

$$\frac{dR}{dt} = (t+1)(R^2+1)$$

$$\frac{dR}{R^2+1} = (t+1)dt$$

$$\int\frac{dR}{R^2+1} = \int(t+1)dt$$

$$\cot(R) = \frac{t^2}{2} + t + c$$

$$C = \frac{2\cot(R)}{t^2} - t$$