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Midterm Assignment –Spring 2020

Course Title: Differential Equations

Instructor: Engr. Latif Jan

Program: BS (CS-SE-EE) **Total Marks: 30** **Time Allowed: 6 days**

Note: Attempt all Questions:

Q 1: a) Define differential equation along with 2 examples? **(1+1 Marks)**

b) Define a Separable Differential Equation (DE)? **(1+4+3 Marks)**

i. Solve the following **Initial Value Problem (IVP) using separable DE** and find the interval of validity of the solution.

$$\text{(a)}\ \ y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

$$\text{(b)}\ \ y' = e^{-y}(2x-4) \qquad y(5) = 0$$

Q 2: a) Solve the following IVP using Linear Differential method **(2+5+3 Marks)**

(i) **Explain the steps for solving Linear Differential Equation.**

(ii) $cos(x)y' + sin(x)y = 2cos^3(x)sin(x) - 1 \quad y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \le x \le \frac{\pi}{2}$

(iii) $x' + 2x = sint$

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Question # 01

Part A:-

Differential Equations:- A differential equation is an equation which involves derivatives of an unknown function & possibly the function itself as well as the Independent variable

Example:- ① $y' = \sin(x)$ — 1st order

② $y'' + y^3 + x = 0$ — 2nd order

⇒ The order of the differential equation is the highest order of the derivatives of the unknown function appearing in the equation.

Examples:- ① $y' = \sin(x)$

$y = -\cos(x) + C$

② $y'' = 6x + e^x$

$y' = 3x^2 + e^x + C_1$

$y = x^3 + e^x + C_2$

The first order equation has one parameter while 2nd order equation depends on two parameters.

Part b:- Separable differential Equation:-

Any ordinary differential equation that can be manipulated into the form where all x's & dx's are on one side of the equation while all y's & dy's are on the other side is called separable differential equation & can be Integrated directly.

⇒ Examples:-

① $$\frac{dy}{dx} = x^2 (y+1)$$

$$\Rightarrow \frac{1}{y+1}\, dy = x^2\, dx$$

② $$\frac{dy}{dx} = y^2 \sin(x)$$

$$\Rightarrow \frac{1}{y^2}\, dy = \sin(x)\, dx$$

(i)

(a) $y' = \dfrac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$

Solve:- $\dfrac{dy}{dx} = \dfrac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$

Now 1st separate the variables

$$y^{-3}\,dy = \frac{x}{\sqrt{1+x^2}}\,dx$$

Taking Integration of L.H.S & R.H.S

$$\int y^{-3}\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx$$

To Integrate R.H.S let $U = x^2+1$ integrating we get

$$-1/2\, y^{-2} = \sqrt{x^2+1} + C$$

$$\Rightarrow 1/y^2 = C_2 - 2\sqrt{x^2+1}$$

we can solve for the constant now.

$1 = C_2 - 2$, So $C = 3$ Solve for y

$$y(x) = \frac{1}{\sqrt{3-\sqrt{x^2+1}}}$$

we can take the positive root since the initial condition was positive. The solution will exist as long as the denominator is not zero

$$3-2\sqrt{x^2+1} = 0 \;, \quad \sqrt{x^2+1} = 3/2$$

$$x = \pm\sqrt{5/2}$$

The solution is valid for

$$-\sqrt{5/2} < x < \sqrt{5/2}$$

(b) $y' = e^{-y}(2x-4) \;, \quad y(5) = 0$

Solutions

$$\frac{dy}{dx} = e^{-y}(2x-4) \;, \quad y(5) = 0$$

$$\frac{dy}{dx} = \frac{(2x-4)}{e^{y}}$$

$$e^{y}dy = (2x-4)\,dx$$

Integrating both sides

$$\int e^{y}dy = \int (2x-4)\,dx$$

$$\Rightarrow e^{y} = x^2 - 4x + C \qquad \because \frac{\cancel{2}x^2}{\cancel{2}} - 4x + C$$

by taking Natural log

$$\Rightarrow y = \ln(x^2 - 4x + C)$$

Now we need to find C using $y(5) = 0$

$$y(5) = \ln\left((5)^2 - 4(5) + c\right)$$

$$= \ln(5 + C) = 0$$

$$\Rightarrow 5 + C = 1$$

$$c = -4$$

Hence Solution is:

$$\boxed{y = \ln(x^2 - 4x - 4)}$$

Question # 02

(i) Steps for Linear differential Equation:-

An equation of the form

$\frac{dy}{dx} + P(x)y = Q(x)$ is called linear-differential equation

* Steps / Working Rule:

1. Convert the given equation into standard linear differential equation.
2. Identify "P" & "Q" in the given equation (where P is the coefficient of "y" & Q is the coefficient of "x")
3. Find the Integrating factor as:

$$I.F = e^{\int P(x)\,dx}$$

④ Complete Solution is

$$y \times I.F = \int (Q \times I.F)\, dx$$

(ii) $(\cos x)y' + y\sin x = 2\cos^3 x \sin x - 1$

$$y(\pi/4) = 3\sqrt{2} \quad , \quad 0 \le x \le \pi/2$$

Solution:-

$$\cos x \frac{dy}{dx} + (\sin x)y = 2\cos^3 x \sin x - 1 \quad \text{—— (A)}$$

Convert into standard linear diff eq

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{\cancel{\cos x}}{\cancel{\cos x}}\frac{dy}{dx} + \frac{\sin x}{\cos x}y = 2\frac{\cancel{\cos^3 x}\sin x}{\cancel{\cos x}} - \frac{1}{\cos x}$$

$$\frac{dy}{dx} + \underbrace{\overset{P(x)}{\tan x}\, y} = \underbrace{2\cos^2 x \sin x - \sec x}_{Q(x)} \quad \text{—— (1)}$$

Now $I.F = e^{\int P(x)\,dx}$

$$I.F = e^{\int (\tan x)\,dx} = e^{\int \left(\frac{\sin x}{\cos x}\right)dx}$$

using formula

$$I.F = e^{\ln(\cos x)} = \cancel{e}^{\cancel{\ln}(\cos x)^{-1}}$$

$$I.F = (\cos x)^{-1} = \frac{1}{\cos x} = \sec x$$

$$\boxed{I.F = \sec x}$$

Now for complete Solution

$$y \times I.F = \int (Q \times I.F)\, dx$$

$$y \times (\sec x) = \int \sec x \left(2\cos^2 x \sin x - \sec x\right) dx$$

$$y \sec x = \int (2 \sec x \cos^2 x \sin x - \sec^2 x)\, dx$$

$$y \sec x = \int \left(2 \frac{1}{\cancel{\cos x}} \cancel{\cos^2} x \sin x\right) dx - \int \sec^2 x\, dx$$

$$y \sec x = 2 \int \sin x \cos x\, dx - \int \sec^2 x\, dx$$

$$= 2 \frac{(\sin x)^{1+1}}{1+1} - \tan x + C$$

$$= \cancel{2} \frac{\sin^2 x}{\cancel{2}} - \tan x + C$$

$$y \sec x = \sin^2 x - \tan x + C$$

dividing both sides by $\sec x$

$$y = \frac{\sin^2 x}{\sec x} - \frac{\tan x}{\sec x} + C$$

$$y = \sin^2 x \cdot \cos x - \frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x} + C$$

$$y = \sin^2 x \cos x - \sin x + C$$

OR

$$y = \sin x (\sin x \cos x - 1) + C$$

is Required general Solution

Condition Given

$$y(\pi/4) = 3\sqrt{2}$$

Put in general Solution

$$y = 3\sqrt{2} \quad \& \quad x = \pi/4$$

$$y = \sin x (\sin x \cdot \cos x - 1)$$

$$3\sqrt{2} = \sin(\pi/4)\left(\sin(\pi/4) \cos(\pi/4) - 1\right) + C$$

$$3\sqrt{2} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) - 1\right) + C$$

$$3\sqrt{2} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} - 1\right) + C$$

$$3\sqrt{2} = \frac{1}{\sqrt{2}}(-1/2) + C$$

$$3\sqrt{2} = \frac{-1}{2\sqrt{2}} + C$$

$$3\sqrt{2} + \frac{1}{2\sqrt{2}} = C$$

$$\boxed{\frac{13\sqrt{2}}{4} = C}$$

Put value of C in general eq

$$y = \sin x(\sin x \cos x - 1) + \frac{13\sqrt{2}}{4}$$

(iii) $x' + 2x = \sin t \qquad 0 \le x \le \pi/2$

Solution:-

$$\frac{dx}{dt} + 2x = \sin t$$

linear in x

$$P = 2x \quad , \quad Q = \sin t$$

$$I.F = e^{\int 2dt} = e^{2t}$$

Now for complete solution

$$x \times I.F = \int (Q \times I.F)\, dt$$

$$x e^{2t} = \int e^{2t} \sin t \, dt$$

$$\boxed{x e^{2t} \int e^{2t} \sin t \, dt}$$ Ans

Question # 03

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$

$y(0) = -3$

Solution:-

$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$ —— (*)

which is in the form

$M(x,y)\,dx + N(x,y)\,dy = 0$

Now

$M = 2xy - 9x^2$

$N = 2y + x^2 + 1$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\, 2xy - 9x^2$, $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2y + x^2 + 1)$

$\frac{\partial M}{\partial y} = 2x - 0$, $\frac{\partial N}{\partial x} = 0 + 2x + 0$

$\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 2x$

$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$

So the given eq ⓐ is exact to find the general solution we consider or we check the following conditions.

① $I_1 = \int_{y \text{ Constant}} M\,dx$

$I_1 = \int_{y \text{ constant}} (2xy - 9x^2)\,dx$

$I_1 = 2y\int x\,dx - 9\int x^2\,dx$

$I_1 = \frac{\cancel{2}yx^2}{\cancel{2}} - \frac{\overset{3}{\cancel{9}}x^3}{\cancel{3}}$

$\boxed{I_1 = x^2y - 3x^3}$

② $I_2 = \int N(\text{terms free from } x)\,dy$

$I_2 = \int (2y + x^2 + 1)\,dy$

$I_2 = \int (2y + 1)\,dy$

$I_2 = \frac{\cancel{2}y^2}{\cancel{2}} + y$

$I_2 = y^2 + y$

★ Complete Solution / General Solution :-

$I_1 + I_2 = C$

Putting values of I_1 & I_2 in general Solution

$x^2y - 3x^3 + y^2 + y = C$

★ Given Condition :-

$y(0) = -3$

$y = -3$ & $x = 0$

Put in general solution

$x^2y - 3x^3 + y^2 + y = C$

$(0)(-3) - 3(0) + (-3)^2 + (-3) = C$

$9 - 3 = C$

$\boxed{6 = C}$

Put value of C in general solution

$x^2y - 3x^3 + y^2 + y = 6$

is the required Particular Solution of IVP

(ii) $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))\, y' = 0 \quad , \quad y(5) = 0$

$\Rightarrow$ Solution:-

$$\frac{2ty}{t^2+1} - 2t - \{2 - \ln(t^2+1)\}\, y' = 0$$

$$\frac{2ty}{t^2+1} - 2t - \{2 - \ln(t^2+1)\} \frac{dy}{dt} = 0$$

$$\left(\frac{2ty}{t^2+1} - 2t\right) dt - [2 - \ln(t^2+1)]\, dy = 0 \quad \text{—} \ (1)$$

Here,

$$M = \frac{2ty}{t^2+1} - 2t \quad , \quad N = -[2 - \ln(t^2+1)]$$

$$\frac{\partial M}{\partial y} = \frac{2t}{t^2+1} - 0 \quad , \quad \frac{\partial N}{\partial t} = \left(\frac{1}{t^2+1}\right) \frac{\partial}{\partial t}(t^2+1)$$

$$\frac{\partial M}{\partial y} = \frac{2t}{t^2+1} \quad , \quad \frac{\partial N}{\partial t} = \frac{2t}{t^2+1}$$

$$\frac{\partial M}{\partial y} = \frac{2t}{t^2+1} = \frac{\partial N}{\partial t}$$

So the given eq ① is exact
Now to find its general solution
let consider,

① $I_1 = \int_{y\ \text{constant}} M\,dt$

$$I_1 = \int_{y\ \text{constant}} \left(\frac{2ty}{t^2+1} - 2t\right) dt$$

$$I_1 = 2y \int \left(\frac{t}{t^2+1}\right) dt - 2\int t\,dt$$

(ftn)

$$I_1 = y \int \left(\frac{2t}{t^2+1}\right) dt - 2\int t\,dt$$

(ftn)

$$I_1 = y \ln (t^2+1) - \frac{2t^2}{2}$$

$$\boxed{I_1 = y \ln (t^2+1) - t^2}$$

② $I_2 = \int N \,(\text{free of terms})\, dy$

$$I_2 = \int (-2 + \ln (t^2+1)\, dy$$

$$I_2 = -\int 2\,dy$$

$$\boxed{I_2 = -2y}$$

* Complete Solution:-

$I_1 + I_2 = c$

$y \ln(t^2+1) - t^2 - 2y = c$

* Given Condition:-

$y(5) = 0$

$y = 0 , \quad t = 5$

put in general solution

$(0) \ln(5^2+1) - (5)^2 - 2(0) = c$

$\boxed{-25 = c}$

Put the value of c in general solution

$y \ln(t^2+1) - t^2 - 2y = -25$

OR

$\boxed{y \ln(t^2+1) - t^2 - 2y + 25 = 0}$

is the required particular solution of initial value problem.

"End of Paper"