NAME = MUHAMMAD ATIF KHAN

ID = 7901

SECTION = A

Miss Shomaila

Assignment :: 02

Date 14 June 2021

Cauchy Euler Method

Question - 1

The couse by eulor equation

① $x^3 y''' + 2x^2 y' + 2y = 10x + 10/x$

Solution:

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$$

$$(x^3 D^3 + 2x^2 D + 2) y = 10x + 10x^{-1} \quad - ①$$

let $x = e^t \Rightarrow t = \ln x$.

$$xD = \Delta$$

$$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2)$$

Substituting into eq (1)

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2) y = 10^x + 10$$

$(D^3 - D^2 + 2)y = 10x + 10x^{-1}$

$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{e^t}$

using synthetic division

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$D^2 - 2D + 2 = 0$

using Q-formula,

$a = 1, \quad b = -2, \quad c = 2.$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$$D = \frac{2 \pm \sqrt{4-8}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1} + \sqrt{4}}{2}$$

$$D = \frac{2 + 2i}{2}$$

$$D = \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$D = 1 \pm i$$

Since roots are complex.

$$y_c = e^{-x}(c_1 \cos t + c_2 \sin t)$$

Now particular integration

$$y_p = \frac{1}{D^3 - D^2 + 2} 10e^t + \frac{1}{D^3 - D^2 + 2} 10 e^t$$

$$= \frac{\cancel{10}^{5} e^t}{\cancel{2}} + \frac{\cancel{10}^{5} e^{-t}}{\cancel{2}}$$

$$= 5e^t + 5e^t$$

$$y_p = 5e^t + 5e^t$$

General solution.

$y = y_c + y_p$ (1)

$y = e^{-x}(c_1\cos t + c_2\sin t) + 5e^t + 5e^t$

put $e^t = x$ and $t = \ln x$.

$y = e^{-x}(c_1 \ln x + c_2 \sin \ln x) + 5e^x + 5e^{-x}$

Ans.

Question # 2.

(a) $x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x\frac{dy}{dx} - 15y = x^4.$

Solution:

let $\frac{d}{dx} = D$

$x^3 D^3 y + 4x^2 D^2 y - 5xDy - 15y = x^4$

$(x^3 D^3 + 4x^2 D^2 - 5xD - 15)y = x^4$

let $x = e^t \Rightarrow t = \ln x$

$xD = \Delta$

$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x^3 D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

$(D^3 - 3D^2 + 2D + 4(D^2 - D) - 5(D) - 15)y = e^{4t}$ (5)

$(D^3 + D^2 - 7D - 15)y = e^{4t}$

Synthetic division

$$\begin{array}{c|cccc} 5 & 1 & +1 & -7 & -15 \\ & & 3 & 12 & 15 \\ \hline & 1 & 4 & 5 & \boxed{0} \end{array}$$

$D^2 + 4D + 5 = 0$

Quadratic formula

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= -4 \pm 2i$$

$$D = \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$y_c = e^{3n}(c_1 \cos t + c_2 \sin t)$

for $y_p = ?$

⑥

$$y_p = \frac{1}{D^3 + D^2 - 7D - 15} e^{4t}$$

$$= \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} e^{4t}$$

$$= \frac{1}{64 + 16 - 28 - 15} e^{4t}$$

$$= \frac{1}{80 - 43} e^{4t}$$

$$y_p = \frac{1}{37} e^{4t}$$

Hence

$$y = y_c + y_p$$

$$y = (c_1 \cos t + c_2 \sin t) + \frac{1}{37} e^{4t}$$

again put $t = \ln x$ and $x = \ln x$

$$y = e^{3x}(c_1 \cos \ln x + c_2 \sin \log x) + \frac{1}{37} e^{4x}$$ Ans

Question # 3

$x^2 y'' + 2xy' - 6y = 10x^2$

Solution:

$y(1) = 1$ and $y'(1) = -6$

$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2$

$\Rightarrow (x^2 d^2/dx^2 + 2x\, d/dx - 6)y = 10x^2$

put $xD = \Delta \Rightarrow x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t$ and $\log x = t$

$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$

$(\Delta^2 + \Delta - 6)y = 10e^{2t}$

The characteristic equation

$\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - \Delta 2 - 6 = 0$

$\Rightarrow \Delta(\Delta + 3) - 2(\Delta + 3) = 0$

$\Rightarrow (\Delta + 3)(\Delta - 2) = 0$

(8)

$D+3=0, \quad D-2=0$

$D=-3, \quad D=2$

Since roots are real and distinct

for $y_c = ?$

$y_c = c_1 e^{-3t} + c_2 e^{2t}$

for $y_p = ?$

$$y_p = \frac{1}{D^2-D-6} \cdot 10^{2t}$$

$$= \frac{10}{D^2-D-6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t}$$

Now

$$10 \frac{1}{d[dD(D^2+D-6)} e^{2t}$$

$$\Rightarrow 10 \frac{t}{2D+1} e^{2t}$$

$$= 10 \frac{1 \cdot t}{4+1} e^{2t}$$

$$y_p = 2te^{2t}$$

General Solution

$y = y_c + y_p$

$= c_1 e^{3t} + c_2 e^{2t} + 2te^{2t}$

$y = c_1 x^{-3} + c_2 x^2 + 2(\log x)x^2$ — (B)

put $y(1) = 1$ i.e $x = 1$, $y = 1$ in (B)

$1 = c_1 (1)^{-3} + c_2 (1)^2 + 2\log(1)$

$1 = c_1 + c_2 \rightarrow$ (C)

Now diff eq (B) w.r.t x.

$y' = -3c_1 x^{-4} + 2c_2 x + \frac{2}{x}(x^2) + 4x\log x$

Now put $y'(1) = -6$ i.e $y' = 6$ and $x = -6$

$-6 = -3c_1 + 2c_2 + 2 + 0$

$\Rightarrow -6 = -3c_1 + 2c_2 + 2$

$\Rightarrow -6-2 = -3c_1 + 2c_2 + 2$

$-8 = -3c_1 + 2c_2$ — (D)

xing eq (i) with (2) and ÷ing from (D)

$$2c_1 + \cancel{2c_2} = 2$$
$$\mp 3c_1 \pm \cancel{2c_2} = \mp 8$$

$$5c_1 = 10$$

$$c_1 = \frac{\cancel{10}\,2}{\cancel{5}} \quad \boxed{c_1 = 2}$$

$$-8 = -3(2) + 2c_2$$
$$-8 = -6 + 2c_2$$
$$2c_2 = -8+6$$
$$2c_2 = -2$$
$$c_2 = \frac{-\cancel{2}\,1}{\cancel{2}}$$

$$\boxed{c_2 = -1}$$

Now put the value of c_1 and c_2 in eq (B)

$$y = 2x^{-3} - x^2 + 2 \ln x \, x(x^2)$$
$$y = \frac{2}{x^3} - x^2 + 2x^2 \log x \quad \text{Ans}$$

Question # 04 (11)

$x^2y'' + 7xy' + 5y = x^5$

$y(0) = 2$ and $y'(1) = 2$

Solution:

$$x^2\frac{dy^2}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 5y\right) = x^5 \quad \text{— (A)}$$

put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = e^t \Rightarrow \log_x = t$ in eq (A)

$\Rightarrow (\Delta^2 - \Delta + 7\Delta + 5)\cdot y = 5e^t$

$\Rightarrow (\Delta^2 + 6\Delta + 5)\, y = e^{5t}$

By quadratic formula

$$\Delta = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{4^2}}{2}$$

$$= \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$D = -3 \pm 2$ since roots are real and distinct

$$y_c = c_1 e^{-5t} + c_2 e^{-t}$$

for y_p -- ?

$$y_p = \frac{1}{D^2 + 6D + 15} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

Now General Solution is

(3)

$y = y_c + y_p$

$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{60} e^{5t}$

$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5 \rightarrow (B)$

$x = 0$ put in this equation

No in eq (B) $e^0 = 1$

put $y(0) = 2$ i-e $y = 2$ & $x = 2$

$2 = c_1 (2)^{-5} + c_2 (2)^{-1} + \frac{1}{60} (2)^5$

$2 = -32 c_1 - 2 c_2 + \frac{1}{60_{15}} (32)^8$

$2 = -32 c_1 - 2 c_2 + \frac{8}{15}$

$2 - \frac{8}{15} = -32 c_1 - 2 c_2$

$\frac{22}{15} = -32 c_1 - 2 c_2 \rightarrow (C)$

Now diff eq (B) w.r.t x,

$y' = -5c_1 x^{-6} - c_2 x^{-2} + \frac{1}{12} x^4 \rightarrow$

put $y'(1) = 2$ i-e $y' = 2$ and $x = 2$ in above equation

(14)

$2 = -5c_1 (2)^{-6} - c_2 (2)^{-2} + \frac{1}{12} (2)^4$

$2 = -5c_1(-64) - c_2(4) + \frac{1}{12}(16)$

$2 = 320c_1 + 4c_2 + 4/3$

$\Rightarrow 2 - 4/3 = 320c_1 + 4c_2$

$\Rightarrow 2/3 = 320c_1 + 4c_2 \rightarrow$ (D)

xing eq (C) with 2 and then -ing eq (C)

from D

$-\frac{44}{15} = 64c_1 + 4c_2$

$-\frac{44}{15} = 64c_1 + 4c_1$

$+2/3 = \pm 320c_1 \pm 4c_2$

$+$

$\frac{34}{15} = -256c_1$

$c_1 = \frac{34}{15} \times 256$

$c_1 = 580$

put the value of c_1 in eq (C)

$2^3/15 = -18560 - 2c_2$

16

$\Rightarrow \frac{22}{15} + 18560 = -2c_2$

$\Rightarrow \frac{18561}{-2} = c_2$

$-9280 = c_2$

Now put the value of c_1 and c_2

in eq (B)

$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$

$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$

Ans

Question # 05

$(x-1)^2 y'' - 3(x+1)' y' + 4y = x^2$

Solution:

$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$

$\Rightarrow \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right)y = x^2$

$\Rightarrow [(x+1)^2 D^2 - 3(x+1)D + 4]y = x^2 + B$ (B)

Put $(x+1)D = \Delta \Rightarrow (x+1)^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t$ in eq (B)

$\Rightarrow [\Delta^2 - \Delta - 3\Delta + 4]y = e^{2t}$

$\Rightarrow [\Delta^2 - 4\Delta + 4]y = e^{2t}$

$\Rightarrow (\Delta^2 - 4\Delta + 4)2 = e^{2t}$

for y_c we find the roots

$\Delta^2 - 4\Delta + 4 = 0$

$\Delta^2 - 2\Delta - 2\Delta + 4 = 0$

$\Delta(\Delta - 2) - 2(\Delta - 2) = 0$

$\Delta - 2 = 0, \quad \Delta = 2$

$\Delta - 2 = 0, \quad \Delta = 2$

So the roots are real and repeat the general solution are.

$y = (c_1 + c_2 x)^{2x}$

$y = (c_1 + c_3 x)^{2x}$

(17)

$y = (c_1 + c_2 x) e^{2x}$

for $y_p = ?$

$$y_p = \frac{1}{D^2 - 4D + 4} \quad \Big| \quad (2)^2 - 4(2) + 4 \Rightarrow 0$$

$$y_p = \frac{2}{2D-4} e^{2t}$$

if we put 2"

$2D - 4 \Rightarrow 2(2) - 4 = 0$

we take again derivative

$$y_p = \frac{\cancel{2}}{\cancel{2}} e^{2t}$$

$y = (c_1 + c_2 x) e^{2t} + e^{2t}$ Ans.

General solution