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Subject Differential equation

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# Cauchy Euler Method

Q NO: 7 (01)

$$x^3 y''' + 2x^2 y' + 2y = 10x + \frac{10}{x}$$

Solution

$$x^3 y''' + 2x^2 y' + 2y = 10x + \frac{10}{x} \rightarrow \textcircled{1}$$

put  $x = e^t$  then

$$\frac{dx}{dt} = e^t \Rightarrow \frac{dt}{dx} = e^{-t}$$

Now

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot e^{-t}$$

OR

$$y' = \frac{dy}{dx} = e^{-t} Dy \quad \because \frac{d}{dt} \Rightarrow D$$

P.T.O

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Similarly

$$y'' = e^{-2t} [D(D-1)]y$$

$$y''' = e^{-3t} [D(D-1)(D-2)]y$$

Using these values in (1)

$$e^{3t} \cdot e^{-3t} [D(D-1)(D-2)]y + 2e^{2t} \cdot e^{-2t} [D(D-1)]y + 2y = 10e^t + 10e^{-t}$$

$$\Rightarrow (D^3 - 3D^2 + 2D)y + (2D^2 - 2D)y + 2y = 10e^t + 10e^{-t}$$

$$\Rightarrow D^3y - D^2y + 2y = 10e^t + 10e^{-t}$$

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 10e^t + 10e^{-t} \rightarrow (2)$$

The associated homogeneous equation of (2)

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 0$$

$$\text{Say, } \frac{d}{dt} = K^2, \frac{d^3}{dy^3} = K^3$$

$$\Rightarrow (K^3y - K^2y + 2y) = 0$$

$$\Rightarrow (K^3 - K^2 + 2)y = 0$$

For non-trivial sol.,  $y \neq 0 \Rightarrow$   
 $K^3 - K^2 + 2 = 0$

P.T.O

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→ Roots are  $K = -1, 1 \pm i$

$$\Rightarrow y_c(t) = A e^{-t} + (B \cos t + C \sin t) e^t$$

where is Complementary Sol.

Q No: 7 (03)

$$x^2 y'' + 2xy' - 6y = 10x^2 = y(1) = 1$$

$$y'(1) = -6$$

Solution

$$x^2 y'' + 2xy' - 6y = 10x^2 \rightarrow \textcircled{1}$$

let

$$x = e^t \text{ i.e.t. } \log x$$

$$y(1) = 1$$

$$y'(1) = -6$$

Now

$$xy' = \Delta y \Rightarrow x^2 y'' = \Delta(\Delta - 1)y$$

$$\text{where } \Delta = \frac{d}{dt}$$

$$\text{Then equation } \textcircled{1} \Rightarrow [\Delta(\Delta - 1) + 2\Delta - 6]y = 10e^{2t}$$

$$[\Delta^2 - \Delta + 2\Delta - 6]y = 10e^{2t}$$

P.T.O

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$$\Rightarrow [\Delta^2 + \Delta - 6]y = 10e^{2t}$$

Char form Equation  $\Delta^2 + \Delta - 6 = 0$

$$\Delta + 3\Delta - 2\Delta - 6 = 0$$

$$\Delta = -3, \Delta = 2$$

Complementary function

$$C.F = C_1 e^{-3t} + C_2 e^{2t}$$

Also P. Integrat

$$P.I = \frac{1}{\Delta^2 + \Delta - 6} 10e^{2t}$$

$$= 10 \frac{1}{(2)^2 + 2 - 6} e^{2t} \text{ replace } \Delta \text{ by } 2$$

Case of failure

$$P.I = 10t \frac{1}{2\Delta + 1} e^{2t} = 10t \frac{1}{2(2) + 1} e^{2t}$$

$$= 10t \frac{1}{5} e^{2t} = 2t e^{2t}$$

Hence general Sol.  $y = C.F + P.I$

$$y = C_1 e^{-3t} + C_2 e^{2t} + 2t e^{2t}$$

$$C_1 x^3 + C_2 x^2 + 2(\log x) x^2$$

P.I.O



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Apply Initial Condition

$$y(1) = 1 \text{ we get}$$

$$1 = c_1 + c_2 + 0 \rightarrow \textcircled{A}$$

$$\textcircled{4}. y'(1) = -6$$

$$y' = -3c_1 x^{-4} + 2c_2 x + 2x + 4x \log x$$

$$-6 = 3c_1 + 2c_2 + 2 + 0$$

$$-3c_1 + 2c_2 = -8 \rightarrow \textcircled{B}$$

eq  $\textcircled{A}$   $\times 3$  and add with  $\textcircled{B}$

$$3 = 3c_1 + 3c_2$$

$$-8 = -3c_1 + 2c_2$$

$$5c_2 = -5$$

$$\boxed{c_2 = -1}$$

$$\text{eq } \textcircled{A} \rightarrow 1 = c_1 - 1$$

$$\boxed{c_1 = 2}$$

$$\text{thus } \Rightarrow \boxed{y = 3x^{-3} - x^2 + 2x^2 \log x}$$

P.T.O

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Q NO: → (04)

$$x^2 y'' + 7xy' + 5y = x^5;$$

$$y(0) = 2 \neq y'(1) = 2$$

Solution

Let

$$x = e^t \Rightarrow t = \log x, \quad \Delta = \frac{d}{dt}$$

$$\text{Now } xy' = \Delta y \Rightarrow x^2 y'' = \Delta(\Delta - 1)y$$

then

$$(\Delta(\Delta - 1) + 7\Delta + 5)y = e^{5t}$$

$$(\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$$

$$(\Delta^2 + 6\Delta + 5)y = e^{5t}$$

$$\text{Char eq/ is } \Delta^2 + 6\Delta + 5 = 0$$

$$\Delta^2 + 5\Delta + \Delta + 5 = 0$$

$$\Delta = -5, -1$$

Complementary eq/ is

$$C.F = C_1 e^{-5t} + C_2 e^{-t}$$

P.T.O

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→ P. Integral

$$P.I. = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}$$

$$= \frac{1}{s^2 + 6(s) + 5} \quad \text{Replacing } \Delta \text{ by } s$$

$$= \frac{1}{60} e^{5t}$$

Thus

$$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5$$

$$y' = -5C_1 x^{-6} - C_2 x^{-2} + \frac{1}{12} x^4$$

$$y(0) = 2 \quad x=0, y=2$$

$$2 = C_1 + C_2 + \frac{1}{60}$$

$$C_1 + C_2 = \frac{119}{60} \rightarrow \textcircled{A}$$

$$y'(1) = 2 \quad x=1, y'=2$$

$$2 = -5C_1 - C_2 + \frac{1}{12}$$

$$-5C_1 - C_2 = \frac{23}{12} \rightarrow \textcircled{B}$$

$$A+B \quad -4C_1 = \frac{239}{60} \Rightarrow 4C_1 = -\frac{117}{120}$$

P.T.O



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Now

$$y = \frac{-117}{120} x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5$$

$$C_1 = \frac{-117}{120} \text{ put in (A)} \quad \frac{-117}{120} + C_2 = \frac{119}{60}$$

$$C_2 = \frac{119}{60} + \frac{117}{120}$$

$$= \frac{238}{120} + \frac{117}{120} = \frac{355}{120}$$

Q No: 7 (05)

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

Solution

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2 \rightarrow (1)$$

Let

$$x+1 = e^t \Rightarrow x = e^t - 1$$

$$\text{Diff } \log(x+1) = t$$

$$\text{Also } (x+1)y' = Dy, \quad \left\{ \begin{array}{l} \frac{d}{dt} = D \\ \text{and } D = \frac{d}{dx} \end{array} \right\}$$

P.I.O

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$$\text{Then eq (1)} \Rightarrow (\Delta(\Delta-1) - 3\Delta + 4)y = (e^t - 1)^2$$

$$(\Delta^2 - 4\Delta + 4)y = e^{2t} - 2e^t + 1$$

$$\text{Char eq is } \Delta^2 - 4\Delta + 4 = 0$$

$$(\Delta - 2)^2 = 0$$

$$\Delta = 2, 2$$

This Complementary function is

$$C.F = (C_1 + C_2 t) e^{2t}$$

Also particular Integral is

$$P.I = \frac{1}{(\Delta - 2)^2} (e^{2t} - 2e^t + 1)$$

$$= \frac{1}{(\Delta - 2)^2} e^{2t} - 2 \frac{1}{(\Delta - 2)^2} e^t + \frac{1}{(\Delta - 2)^2} \rightarrow (2)$$

Now

$$\frac{1}{(\Delta - 2)^2} e^{2t} = \frac{1}{(2-2)^2} e^{2t} = \frac{1}{0} e^{2t}$$

Case of failure

$$\frac{1}{(\Delta - 2)^2} e^{2t} = t \frac{1}{2(1-2)^2} e^t = \frac{t^2 e^t}{2}$$

$$\text{and } 2 \frac{1}{(\Delta - 2)^2} e^t = 2 \frac{1}{(1-2)^2} e^t = 2 e^t$$

P.I. = 0

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$$\text{and } \frac{1}{(\Delta-2)^2}(1) = \frac{1}{(\Delta-2)^2} e^{0t} = \frac{1}{4}$$

$$\text{eq (2)} \Rightarrow P.I = \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}$$

Hence Complete Solution is

$$y = C.F + P.I$$

$$y = (C_1 + C_2 t) e^{2t} + \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}$$

Repeat Value of  $e^t$

$$y = (C_1 + C_2 \log(x+1))(x+1)^2 + \frac{1}{2} [(\log(x+1))^2 (x+1)^2] - 2(x+1) + \frac{1}{4}$$

OR

$$y = C_1 + C_2 \log(x+1)(x+1)^2 + \frac{1}{2} (\log(x+1))^2 (x+1)^2 - 2x - \frac{1}{4}$$

which is the Required.

P.T.O



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Q NO: (02)

$$x^3 y''' + 4x^2 y'' - 5xy' - 15y = x^4$$

Solution

$$x^3 y''' + 4x^2 y'' - 5xy' - 15y = x^4 \rightarrow (1)$$

let

$$xy = e^t \rightarrow x = e^{t-1}$$

$$Diff \log(xy) = t$$

Also

$$(xy) = \Delta y, \quad \left. \begin{array}{l} \frac{d}{dt} \\ \text{and } D = \frac{d}{dx} \end{array} \right\}$$

$$\text{then eq (1)} \quad (\Delta(\Delta-1) - 15\Delta + 5)y = (e^t - 1)^2$$

$$(\Delta^2 - 15\Delta + 5)y = e^{2t} - 2e^t + 1$$

$$\text{Char eq is } (\Delta^2 - 15\Delta + 5)y = 0$$

$$(\Delta - 5)^2 = 0$$

$$\Delta = 5, 5$$

P.T.O



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This Complementary function is

$$C.F = (C_1 + C_2 t) e^{2t}$$

Also particular Integral is

$$P.I = \frac{1}{(\Delta - 5)^2} (e^{2t} - 2e^t + 1)$$

$$= \frac{1}{(\Delta - 2)^2} e^{2t} - \frac{2}{(\Delta - 5)^2} e^t + \frac{1}{(\Delta - 5)^2} \rightarrow (2)$$

$$\text{Now } \frac{1}{(\Delta - 5)^2} e^{2t} = \frac{1}{(5 - 5)^2} e^{2t} = \frac{1}{0} e^{2t}$$

Case of failure

$$\frac{1}{(\Delta - 5)^2} e^{2t} = t \frac{1}{5(1 - 5)^2} e^t = \frac{t^2 e^t}{5}$$

$$\text{and } \frac{5}{(\Delta - 5)^2} e^t = \frac{5}{(1 - 5)^2} e^t = 5e^t$$

$$\text{and } \frac{1}{(\Delta - 5)^2} (1) = \frac{1}{(\Delta - 5)^2} e^{0t} = \frac{1}{15}$$

$$\text{eq (2)} \Rightarrow P.I = \frac{1}{5} t^2 e^{2t} - 2e^t + \frac{1}{15}$$

Hence Complete Solution is

P.T.O

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$$\Rightarrow y = C.F + P.I$$

$$y = (C_1 + C_2 t) e^{2t} + \frac{1}{5} t^2 e^{2t} - 5e^t + \frac{1}{15}$$

$$y = C.F + P.I$$

$$y = (C_1 + C_2 t) e^{2t} + \frac{1}{5} t^2 e^{2t} - 5e^t + \frac{1}{15}$$

Repeat Value of  $e^t$

$$y = (C_1 + C_2 \log(x+1)) (x+1)^2 + \frac{1}{2} \left[ (\log(x+1))^2 (x+1)^2 \right] - 5(x+1) + \frac{1}{15}$$

